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Graded rough set model based on two universes and its properties

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ABSTRACT

In recent years, much attention has been given to the rough set models based on two universes of discourse and different kinds of rough set models on two universes have been developed from different points of view. In this paper, a novel model, i.e., the graded rough set model on two distinct but related universes (GRSTU) is proposed from the absolute quantitative point of view. We study the basic properties of approximation operators in GRSTU, and introduce a relation matrix based algorithm to compute the lower and upper approximations of a set of objects in GRSTU. Furthermore, the relationships between classical rough set model and GRSTU are discussed and some conclusions related to the GRSTU are given. Finally, several examples are employed to demonstrate the conceptual arguments of GRSTU, and an application of GRSTU is also illuminated in details.

1. Introduction

Rough set theory, originally proposed by Pawlak [1,2] as an extension of set theory, is an effective approach to dealing with imprecise, uncertain and incomplete information. It has been successfully used in many research areas, such as pattern recognition, machine learning, knowledge acquisition and data mining [3–7,32–35,38,39,41].

As we know, Pawlak's rough set model has a basic hypothesis, that is, whether an object belongs to a class or not is completely certain. However, in practice, allowing some extent of uncertainty in the classification process may lead to a deeper understanding and a better utilization of the data being analyzed. In order to deal with the uncertainty in such cases, a lot of models have been proposed. For example, based on the Bayesian decision procedure with minimum cost (risk), Yao [8,9] proposed a decision-theoretic rough set model (DTRSM) which brings new insights into the probabilistic approaches to rough set model. DTRSM not only has good semantic interpretation, but also be beneficial for rule acquisition in the applications involving cost and risk. And Yao and Lin [10,11] presented a graded rough set model (GRS) from the absolute quantitative point of view. Moreover, many other models have also been proposed, such as rough set models based on arbitrary binary relations [12–14,40], rough set models based on incomplete systems [15–17,43], covering rough sets [18–20], rough fuzzy sets and fuzzy rough sets [21,42,44], variable precision rough sets [22,23], etc. Through loosening the strict definition of the approximations in Pawlak's rough set model, these models enrich the application scope of rough set theory.

In the real world, we often face some situations in which making a decision is difficult. For example, in the process of identifying or determining the nature and cause of a disease, since a certain disease may simultaneously have several symptoms but the same symptom may be shared by diverse diseases, a doctor (or a decision-maker) often finds it is difficult to distinguish whether a person has suffered from the disease or not. In these kinds of situations, more than one universes of discourse are often involved. However, Pawlak's rough set model and its extensions mentioned above are all based on only one universe, therefore these models may be not suitable to deal with the above problem. Hence, it is meaningful to propose a rough set model based on two universes.

The generalization of rough set model from only one universe of discourse to the two distinct but related universes of discourse has attracted much attention [24–29,36]. Wong et al. [24] first proposed a rough set model on two universes from the viewpoint of compatibility. In [26,28,29], the applications and some interesting properties about the rough set model on two universes were discussed. Wu et al. [25] developed a general framework for the study of the fuzzy rough set models on two universes in which both constructive and axiomatic approaches were considered and surveyed. In [27], four types of rough fuzzy approximation operators on two universes have been proposed. Zhang et al. [36] studied the generalized interval-valued fuzzy rough sets on two universes of discourse.

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Although the above models can effectively overcome the limitations of rough set models on one universe, they still lack the adaptability in solving uncertainty problems. To solve this problem, Shen et al. [30] proposed a variable precision rough set model on two universes from the relative quantitative point of view. Ma and Sun [37] introduced a probabilistic rough sets over two universes and used it to deal with the problem of Bayesian risk decision. In this paper, from the absolute quantitative point of view, we propose a graded rough set model defined on two distinct but related universes. Our model is not only an extension of the rough set model on two universes but also an extension of Pawlak's rough set model. To compare with Yao and Lin's graded rough set model, our model may be more appropriate to handle the problems where more than one universe is involved. Paralleling with Pawlak's rough set model, the basic properties of our model are discussed. Meanwhile, a relation matrix based algorithm for computing the lower and upper approximations in GRSTU is proposed.

The remainder of this paper is organized as follows. In the next section, we briefly introduce some notions using in Pawlak's rough set model, graded rough set model on one universe and rough set model on two universes. In Section 3, we define the lower and upper approximation operators in GRSTU and discuss the basic properties of GRSTU. In Section 4, two examples are employed to substantiate the conceptual arguments of GRSTU. An application of GRSTU is discussed in details in Section 5. Finally, Section 6 concludes the paper.

2. Preliminaries

In this section, we outline some basic concepts in rough sets and some current rough set models, such as Pawlak's rough set model [1], graded rough set model on one universe [10] and rough set model on two universes [24]. Throughout this paper, we suppose that the universe U or V is a finite non-empty set.

2.1. Pawlak's rough set model

Let U be a universe of discourse, for any binary relation R on U, we call R an equivalence relation on U, if.

(1) R is reflexive if for all $x \in U$, xRx;
(2) R is symmetric if for all $x,y \in U$, xRy implies yRx;
(3) R is transitive if for all $x,y,z \in U$, xRy and yRz implies xRz.

An equivalence relation is a reflexive, symmetric and transitive relation. The equivalence relation R partitions U into disjoint subsets (or equivalence classes). Let U/R denote the family of all equivalence classes of R. For every object $x \in U$, let $[x]_R$ denote the equivalence class of relation R that contains element x, called the equivalence class of x under relation R.

Let U be a universe of discourse, R an equivalence relation on U, for any $X \subseteq U$, one can describe X by a pair of lower and upper approximations defined as follows.

$$\underline{R}(X) = \{x \in U | [x]_R \subseteq X\}$$
$$\overline{R}(X) = \{x \in U | [x]_R \cap X \neq \emptyset\}$$

$\underline{R}(X)$ is called the lower approximation of X, which is the union of all the equivalence classes which contain in X, and $\overline{R}(X)$ is called the upper approximation of X, which is the union of all equivalence classes which have non-empty intersection with X. Then $(\underline{R}(X), \overline{R}(X))$ is called the rough sets of X. Accordingly, the positive, negative and boundary regions of X on the approximation space (U,R) can be defined as follows: $pos(X) = \underline{R}(X)$, $neg(X) = \sim \overline{R}(X), bnd(X) = \overline{R}(X) - \underline{R}(X)$, where $\sim$ stands for the complement of a set.

2.2. Generalized rough set operators

The Pawlak rough set model may be extended by using an arbitrary binary relation.

Let U be a universe of discourse and R a binary relation on U, the following two operators: $r(x) = \{y \in U | xRy\}, l(x) = \{y \in U | yRx\}$ are called the successor and predecessor neighborhood operator, respectively.

Definition 1 ([12]). Let U be a universe of discourse and R a binary relation on U. For any $X \subseteq U$, its lower and upper approximations based on the successor neighborhood operator are respectively defined as follows:

$$\underline{R}(X) = \{x \in U | r(x) \subseteq X\}$$
$$\overline{R}(X) = \{x \in U | r(x) \cap X \neq \emptyset\}$$

Analogously, for any $X \subseteq U$, one can define the lower and upper approximations based on the predecessor neighborhood operator.

In the remainder of this paper, we shall only take the case of the successor neighborhood operator into consideration.

2.3. Yao and Lin's graded rough set model on one universe [10]

Let U be a universe and R a binary relation on U, $n \in N$, where N is the set of natural numbers. For any subset $A \subseteq U$, the lower and upper approximations of A with respect to n (denoted by $\underline{apr}_n(A)$ and $\overline{apr}_n(A)$, respectively) are defined as follows.

$$\underline{apr}_n(A) = \{x \in U | |r(x)| - |r(x) \cap A| \leqslant n\}$$
$$= \{x \in U | |r(x) - A| \leqslant n\}$$
$$\overline{apr}_n(A) = \{x \in U | |r(x) \cap A| > n\}$$

where $|r(x)|$ denotes the cardinality of set $r(x)$.

An element of U belongs to $\underline{apr}_n(A)$ if at most n of its R-related elements do not belong to A, and belongs to $\overline{apr}_n(A)$ if more than n of its R-related elements belong to A.

2.4. Rough set model on two universes

Next, we shall review some basic concepts and properties of the rough set model on two universes. Detailed description of the model can be found in [12,26,28].

The above model can be generalized to the case of two universes.

Definition 2 ([12,28]). Let U and V be two universes of discourse and R a binary relation from U to V, i.e. $R \subseteq U \times V$. The ordered triple (U,V,R) is called a two-universe approximation space. For any $Y \subseteq V$, the lower and upper approximations of Y can be defined as follows.

$$\underline{R}(Y) = \{x \in U | r(x) \subseteq Y\}$$
$$\overline{R}(Y) = \{x \in U | r(x) \cap Y \neq \emptyset\}$$

$\underline{R}(Y)$ is called the lower approximation of Y and $\overline{R}(Y)$ the upper approximation of Y. $(\underline{R}(Y), \overline{R}(Y))$ is called the rough sets of Y. Accordingly, the positive, negative and boundary regions of Y over the approximation space (U,V,R) are defined as follows: $Pos(Y) = \underline{R}(Y)$, $Neg(Y) = \sim \overline{R}(Y)$, $Bnd(Y) = \overline{R}(Y) - \underline{R}(Y)$.

Proposition 1. *Given a two-universe approximation space (U,V,R), for any Y, Y_1, $Y_2 \subseteq V$, the approximation operators given in Definition 2 have the following properties:*

(1) $\underline{R}(Y) = \sim \overline{R}(\sim Y), \overline{R}(Y) = \sim \underline{R}(\sim Y)$
(2) $\underline{R}(V) = \overline{R}(V) = U, \underline{R}(\emptyset) = \overline{R}(\emptyset) = \emptyset$

$\overline{R}(Y_1 \cup Y_2) = \overline{R}(Y_1) \cup \overline{R}(Y_2)$
(3) $\underline{R}(Y_1 \cap Y_2) = \underline{R}(Y_1) \cap \underline{R}(Y_2)$,
(4) $\underline{R}(Y_1 \cup Y_2) \supseteq \underline{R}(Y_1) \cup \underline{R}(Y_2)$,
$\overline{R}(Y_1 \cap Y_2) \subseteq \overline{R}(Y_1) \cap \overline{R}(Y_2)$
(5) $Y_1 \subseteq Y_2, \underline{R}(Y_1) \subseteq \underline{R}(Y_2), \overline{R}(Y_1) \subseteq \overline{R}(Y_2)$

Proof 1. Since the proofs can be found in [12], we omit them here. □

By now, we have briefly recalled several interesting rough set models. The relationships among them are shown in Fig. 1. In Fig. 1, each node denotes one model, where Model 1 is the Pawlak's rough set model, Model 2 is the model defined by Definition 1, Model 3 is the graded rough set model on one universe and Model 4 is the model defined by Definition 2. Each arrow connects two models, where the model located at the beginning of the arrow is called the first model, and the model located at the end of the arrow is called the second model. For each arrow, the second model is a special case of the first model, and the numbers located on the arrow denotes the conditions that should be satisfied for the first model to degenerate to the second model.

3. Graded rough sets on two universes and their properties

In parallel to graded modal logics, Yao and Lin [10] first introduced graded rough sets on one universe. In this section, based on the work of Yao and Lin, we shall define graded rough sets on two universes and discuss the basic properties of this model.

Definition 3. Let U and V be two universes of discourse, R a binary relation from U to V, i.e. $R \subseteq U \times V$, and $n \in N$, where N is the set of natural numbers. For any $Y \subseteq V$, its lower and upper approximations with respect to the graded n are defined respectively as follows:

$$\underline{R}_n(Y) = \{x \in U \| r(x) - Y| \leqslant n\}$$
$$\overline{R}_n(Y) = \{x \in U \| r(x) \cap Y| > n\}$$

According to Definition 3, an element of U belongs to $\underline{R}_n(Y)$ if and only if at most n of its R-related elements do not belong to Y, and belongs to $\overline{R}_n(Y)$ if and only if more than n of its R-related elements belong to Y. If $\underline{R}_n(Y) \neq \overline{R}_n(Y)$, then Y is called a rough set with respect to the grade n. Otherwise, Y is called a definable set with respect to the grade n. $\underline{R}_n$ and $\overline{R}_n$ are called the lower and upper approximations with respect to the grade n, respectively.

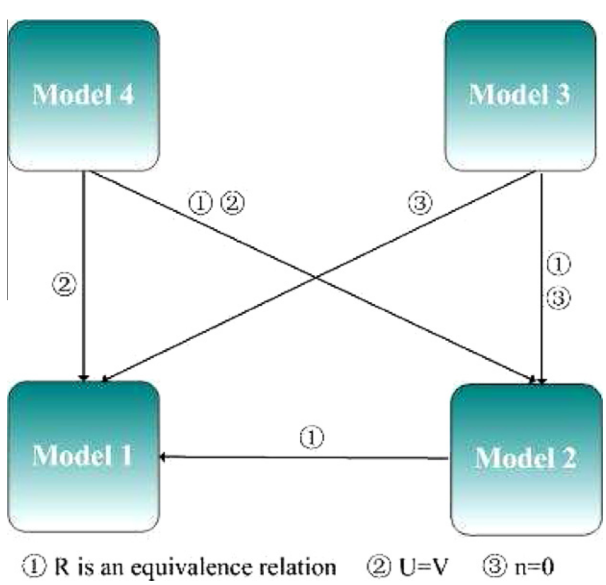

Fig. 1. Relationships of models.

Remark 1.

(1) If $n = 0$, then

$$\begin{aligned}\underline{R}_n(Y) &= \{x \in U \| r(x) - Y| \leqslant 0\} \\ &= \{x \in U | r(x) \subseteq Y\}, \\ \overline{R}_n(Y) &= \{x \in U \| r(x) \cap Y| > 0\} \\ &= \{x \in U | r(x) \cap Y \neq \emptyset\}.\end{aligned}$$

That is, if $n = 0$, then the graded rough set model on two universes degenerates to the rough set model on two universes as defined in Definition 2.

(2) If $U = V$, then the graded rough set model on two universes degenerates into the original graded rough set model proposed by Yao and Lin [10].
(3) If $n = 0$ and $U = V$, then

$$\begin{aligned}\underline{R}_n(Y) &= \{x \in U \| r(x) - Y| \leqslant 0\} \\ &= \{x \in U | [x]_R \subseteq Y\}, \\ \overline{R}_n(Y) &= \{x \in U \| r(x) \cap Y| > 0\} \\ &= \{x \in U | [x]_R \cap Y \neq \emptyset\}\end{aligned}$$

In the current case, the graded rough set model on two universes degenerates into the rough set model based on a binary relation over a given universe.

Remark 2. In the graded rough set model on two universes, the positive, negative and boundary regions of set $Y \subseteq V$ cannot be directly defined as those in Pawlak's rough set model, since the property $\underline{R}_n(Y) \subseteq \overline{R}_n(Y)$ does not always hold (as show in Example 2).

For completeness, we give the definitions of positive, negative and boundary regions of $Y \subseteq V$ in the graded rough set model on two universes.

Definition 4. Let U and V be two universes of discourse, R a binary relation from U to V, i.e. $R \subseteq U \times V$, and $n \in N$, N is the set of natural numbers. For any $Y \subseteq V$, the positive, negative and boundary regions of Y are respectively defined as follows.

$$\begin{aligned}POS_n(Y) &= \{x \in U \| r(x) - Y| \leqslant n \text{ and } |r(x) \cap Y| > n\} \\ &= \underline{R}_n(Y) \cap \overline{R}_n(Y) \\ NEG_n(Y) &= \{x \in U \| r(x) - Y| > n \text{ and } |r(x) \cap Y| \leqslant n\} \\ &= \sim (\underline{R}_n(Y) \cup \overline{R}_n(Y)) \\ BND_n(Y) &= \sim (POS_n(Y) \cup NEG_n(Y))\end{aligned}$$

From Definition 4, we can see that the positive, negative and boundary regions of a set in the graded rough set model on two universes have more complex structure than those in Pawlak's rough set model.

Proposition 2. *Given a two-universe approximation space (U, V, R), for any Y, Y_1, $Y_2 \subseteq V$, the rough set approximation operators with respect to graded n satisfy the following properties.*

(1) $\underline{R}(Y) = \underline{R}_0(Y)$, $\overline{R}(Y) = \overline{R}_0(Y)$

(2) $\underline{R}_n(V) = U$, $\overline{R}_n(\emptyset) = \emptyset$
(3) If $Y_1 \subseteq Y_2$, then $\underline{R}_n(Y_1) \subseteq \underline{R}_n(Y_2)$, $\overline{R}_n(Y_1) \subseteq \overline{R}_n(Y_2)$.
(4) $\underline{R}_n(Y_1 \cap Y_2) \subseteq \underline{R}_n(Y_1) \cap \underline{R}_n(Y_2)$
$\overline{R}_n(Y_1 \cup Y_2) \supseteq \overline{R}_n(Y_1) \cup \overline{R}_n(Y_2)$
(5) $\underline{R}_n(Y_1 \cup Y_2) \supseteq \underline{R}_n(Y_1) \cup \underline{R}_n(Y_2)$
$\overline{R}_n(Y_1 \cap Y_2) \subseteq \overline{R}_n(Y_1) \cap \overline{R}_n(Y_2)$
(6) $\underline{R}_n(Y) = \sim \overline{R}_n(\sim Y)$, $\overline{R}_n(Y) = \sim \underline{R}_n(\sim Y)$
(7) If $n \geqslant m$, then $\underline{R}_n(Y) \supseteq \underline{R}_m(Y)$, $\overline{R}_n(Y) \subseteq \overline{R}_m(Y)$.

Proof 2. Since (1), (2) and (6) can be obtained directly from Definition 3, we omit the proofs of them here.

(3) For any $x \in \underline{R}_n(Y_1)$, we have that $|r(x) - Y_1| \leqslant n$. If $Y_1 \subseteq Y_2$, then $|r(x) - Y_2| \leqslant |r(x) - Y_1| \leqslant n$, that is $x \in \underline{R}_n(Y_2)$. Thus, if $Y_1 \subseteq Y_2$, then $\underline{R}_n(Y_1) \subseteq \underline{R}_n(Y_2)$;

Analogously, for any $x \in \overline{R}_n(Y_1)$, we have that $n < |r(x) \cap Y_1|$. If $Y_1 \subseteq Y_2$ then $n < |r(x) \cap Y_1| \leqslant |r(x) \cap Y_2|$, that is, $x \in \overline{R}_n(Y_2)$. Thus, if $Y_1 \subseteq Y_2$, then $\overline{R}_n(Y_1) \subseteq \overline{R}_n(Y_2)$.

(4) Since $Y_1 \cap Y_2 \subseteq Y_1$ and $Y_1 \cap Y_2 \subseteq Y_2$, from (3), we have that $\underline{R}_n(Y_1 \cap Y_2) \subseteq \underline{R}_n(Y_1)$ and $\underline{R}_n(Y_1 \cap Y_2) \subseteq \underline{R}_n(Y_2)$. Combining the two formulas, we can obtain that $\underline{R}_n(Y_1 \cap Y_2) \subseteq \underline{R}_n(Y_1) \cap \underline{R}_n(Y_2)$.

Another part of (4) and all the parts of (5) can be proved analogously, we omit them here.

(7) For any $x \in \underline{R}_m(Y)$, we have that $|r(x) - Y| \leqslant m$. If $n \geqslant m$, then $|r(x) - Y| \leqslant m \leqslant n$, i.e., $x \in \underline{R}_n(Y)$. Thus, if $n \geqslant m$, then $\underline{R}_n(Y) \supseteq \underline{R}_m(Y)$;

For any $x \in \overline{R}_n(Y)$, we have that $|r(x) \cap Y| > n$, if $n \geqslant m$, then $|r(x) \cap Y| > n \geqslant m$, that is, $x \in \overline{R}_m(Y)$. Thus, if $n \geqslant m$, then $\overline{R}_n(Y) \subseteq \overline{R}_m(Y)$. □

It is well-known that the computation of lower and upper approximations is not an easy task. To deal with this issue in the graded rough set model on two universes, we develop a novel method on the basis of relation matrix defined as following.

Definition 5. Let $U = \{x_1, x_2, \ldots, x_n\}$ and $V = \{y_1, y_2, \ldots, y_m\}$ be two universes of discourse, $R \subseteq U \times V$. The matrix $M_R = (a_{ij})_{n \times m}$ induced by R is called the relation matrix of R, where for any i and j, $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, if $x_i R y_j$ then $a_{ij} = 1$, otherwise $a_{ij} = 0$. We define $sum(i) = \sum_{j=1}^{m} a_{ij}$ as the summation of the ith row of M_R, $1 \leqslant i \leqslant n$.

Definition 6. Let $U = \{x_1, x_2, \ldots, x_n\}$, for any set $X \subseteq U$, the corresponding matrix of set X is defined as $X = (u_1, u_2, \ldots, u_n)^T$($T$ is the transposition of matrix), where for any i, $1 \leqslant i \leqslant n$, if $x_i \in X$ then $u_i = 1$, otherwise $u_i = 0$.

Based on Definitions 5 and 6, we can prove the following proposition.

Proposition 3. *Let $U = \{x_1, x_2, \ldots, x_n\}$ and $V = \{y_1, y_2, \ldots, y_m\}$ be two universes of discourse, and $R \subseteq U \times V$, $M_R = (a_{ij})_{n \times m}$ be the relation matrix of R. Given any $Y \subseteq V$ (let $(u_1, u_2, \ldots, u_m)^T$ be the matrix of set Y) and $k \in N$, where N is the set of natural numbers. Let $Z = M_R \cdot Y = (z_1, z_2, \ldots, z_m)^T$, where $z_i = \sum_{j=1}^{m} a_{ij} \cdot u_j$ and $\cdot$ represents standard multiplication, then the lower and upper approximations of Y with respect to the graded k can be computed as following:*

$$\begin{aligned}\underline{R}_k(Y) &= \{x \in U \,|\, |r(x) - Y| \leqslant k\} \\ &= \{x_i \in U \,|\, sum(i) - z_i \leqslant k\}\\ \overline{R}_k(Y) &= \{x \in U \,|\, |r(x) \cap Y| > k\} \\ &= \{x_i \in U \,|\, z_i > k\}\end{aligned}$$

Proof 3. In fact, according to Definition 5, $sum(i)$ is the cardinality of set $\{y_j\}$, where y_j is related to x_i based on R, i.e. $sum(i) = |r(x_i)|$, $1 \leqslant i \leqslant n$. And z_i is the cardinality of set $\{y_j\}$, where y_j is related to x_i based on R and y_j belongs to Y, i.e. $z_i = |r(x_i) \cap Y|$, $1 \leqslant i \leqslant n$. Proposition 3 is proved. □

As a summary, we can describe the algorithm as Algorithm 1.

Algorithm 1. A matrix-based algorithm for computing approximations

Input: $U = \{x_1, x_2, \ldots, x_n\}$, $V = \{y_1, y_2, \ldots, y_m\}$, $R \subseteq U \times V$
$Y \subseteq V$ and $(u_1, u_2, \ldots, u_m)^T$ is the matrix of Y, $k \in N$;
Output: Approximations of Y with respect to R;
1 Let $\underline{R}_k(Y) = \emptyset$; $\overline{R}_k(Y) = \emptyset$;
2 Compute $M_R = (a_{ij})_{n \times m}$ of R, sum and Z;
3 **for** $i = 1$ to m **do**
4 **if** $sum(i) - z_i \leqslant k$ **then**
5 $\underline{R}_k(Y) = \underline{R}_k(Y) \cup \{x_i\}$;
6 **end**
7 **if** $z_i > k$ **then**
8 $\overline{R}_k(Y) = \overline{R}_k(Y) \cup \{x_i\}$;
9 **end**
10 **end**
11 return $\underline{R}_k(Y), \overline{R}_k(Y)$

Algorithm 1 involves two closely integrated stages. In the first stage (from lines 1 to 2), it computes M_R, sum and y_i according to Definitions 5, 6 and Proposition 3. This stage is the basis for the next stage. In the second stage (from lines 3 to 10), it computes approximations by using Proposition 3.

As shown in Algorithm 1, its major computation lies in the establishment of relation matrix, sum and Z. Assume $|U| = n$ and $|V| = m$, the time complexity of building a relation matrix is $O(n * m)$ and the time complexity for computing sum and Z is also $O(n * m)$. The time complexity for calculating approximations is $O(m)$. Therefore, the total complexity of Algorithm 1 is $O(n * m + m)$, which is approximate to $O(n * m)$.

In order to show the implicit relations between GRSTU and graded rough set model on one universe, we introduce two binary relations: E_U^R and E_V^R, which are induced by $R \subseteq U \times V$, where U and V are two universes of discourse. The two relations are defined based on only one universe U or V.

A binary relation $E_U^R \subseteq U \times U$ induced by R can be defined as $x E_U^R x' \iff r(x) = r(x'), x, x' \in U$. Obviously, E_U^R is an equivalence relation on $U \times U$. Equivalence relation E_U^R partitions the universe U into disjoint subsets. Let U/E_U^R denotes the set of all equivalence classes of E_U^R and $[x]_{E_U^R}$ denotes the equivalence class containing x, where $x \in U$. For convenience, $[x]_{E_U^R}$ is replaced by $[x]_U$ in this paper.

Definition 7. Let U and V be two universes of discourse, $R \subseteq U \times V$, $E_U^R \subseteq U \times U$ be the equivalence relation induced by R, and $n \in N$. For any $X \subseteq U$, its lower and upper approximations with respect to the grade n under E_U^R are defined respectively as follows:

$$\underline{E_{Un}^R}(X) = \{x \in U \,|\, |[x]_U - X| \leqslant n\}$$

$$\overline{E_{Un}^R}(X) = \{x \in U \,|\, |[x]_U \cap X| > n\}$$

Obviously, $\underline{E_{Un}^R}$ and $\overline{E_{Un}^R}$ are two mappings from $P(U)$ to $P(U)$, where $P(U)$ denotes the power set of U. They are the approximation operators with respect to grade n over universe U. Similarly, $\underline{E_{Vn}^R}$, $\overline{E_{Vn}^R} : P(V) \to P(V)$ can also be defined.

Here, it should be noted that graded rough set model defined by Definition 7 is also a special case of the model defined by Yao and Lin in [10].

From Definitions 3 and 7, we can see that $\underline{R}_n, \overline{R}_n : P(V) \to P(U)$, $\underline{E}^R_{Un}$, $\overline{E}^R_{Un} : P(U) \to P(U)$ and $\underline{E}^R_{Vn}$, $\overline{E}^R_{Vn} : P(V) \to P(V)$ are all induced by relation R. Therefore, there must exist some interrelations among them.

Proposition 4. *Let U and V be two universes of discourse, R a binary relation from U to V, and $n \in N$. $E^R_U \subseteq U \times U$ and $E^R_V \subseteq V \times V$ are two equivalence relations induced by R. For any $Y \subseteq V$, the following properties are satisfied.*

(1) $\overline{R}_n\left(\overline{E}^R_{Vn}(Y)\right) \subseteq \overline{E}^R_{Un}(\overline{R}_n(Y)) \subseteq \overline{R}_n(Y)$
(2) $\underline{E}^R_{Un}(\underline{R}_n(Y)) \supseteq \underline{R}_n\left(\underline{E}^R_{Vn}(Y)\right) \supseteq \underline{R}_n(Y)$
(3) $\overline{E}^R_{Un}(\underline{R}_n(Y)) \subseteq \overline{R}_n\left(\underline{E}^R_{Vn}(Y)\right)$
(4) $\underline{R}_n\left(\overline{E}^R_{Vn}(Y)\right) \subseteq \underline{E}^R_{Un}(\overline{R}_n(Y))$

Proof 4.

(2) For any $x \in \overline{R}_n\left(\overline{E}^R_{Vn}(Y)\right)$, from Definition 3, we have that $|r(x) \cap \overline{E}^R_{Vn}(Y)| > n$, that is, there exist at least $n+1$ elements $y_i \in r(x)$ such that $y_i \in r(x) \cap \overline{E}^R_{Vn}(Y)$. Thus, there exist at least $n+1$ elements $y_i \in r(x) \cap Y$, i.e., at least $n+1$ elements $x \in [x]_U \cap \overline{R}_n(Y)$. According to Definition 3, we have that $x \in \overline{E}^R_{Un}(\overline{R}_n(Y))$. Then $\overline{R}_n\left(\overline{E}^R_{Vn}(Y)\right) \subseteq \overline{E}^R_{Un}(\overline{R}_n(Y))$ holds;

For any $x \in \overline{E}^R_{Un}(\overline{R}_n(Y))$, from Definition 3, we have that $|[x]_U \cap \overline{R}_n(Y)| > n$, i.e., there exist at least $n+1$ elements $x \in [x]_U$ such that $x \in \overline{R}_n(Y)$. Therefore, $|r(x) \cap Y| > n$, i.e., $\overline{E}^R_{Un}(\overline{R}_n(Y)) \subseteq \overline{R}_n(Y)$ holds.

Hence, $\overline{R}_n(\overline{E}^R_{Vn}(Y)) \subseteq \overline{E}^R_{Un}(\overline{R}_n(Y)) \subseteq \overline{R}_n(Y)$.

(2) For any $x \in \underline{R}_n\left(\underline{E}^R_{Vn}(Y)\right)$, from Definition 3, $|r(x) - \underline{E}^R_{Vn}(Y)| \leqslant n$, then $r(x)$ and $\underline{E}^R_{Vn}(Y)$ contain at most n different elements, i.e., $r(x)$ and $[y]_V \cap Y$ contain at most n different elements. Thus, we have that $r(x)$ and Y contain at most n different elements. Therefore, $|r(x) - Y| \leqslant n$, and $x \in [x]_U \cap \underline{R}_n(Y)$, i.e., $x \in \underline{E}^R_{Un}(\underline{R}_n(Y))$. Hence, $\underline{E}_{Un}(\underline{R}_n(Y)) \supseteq \underline{R}_n(\underline{E}_{Vn}(Y))$;

Since $Y \subseteq \underline{E}^R_V(Y) = \underline{E}^R_{V0}(Y)$ and according to (7) of Proposition 2, we have that $Y \subseteq \underline{E}^R_V(Y) = \underline{E}^R_{V0}(Y) \subseteq \underline{E}^R_{Vn}(Y)$, where $n \geqslant 0$, i.e. $Y \subseteq \underline{E}^R_{Vn}(Y)$. And from (3) of Proposition 2, $\underline{R}_n(\underline{E}^R_{Vn}(Y)) \supseteq \underline{R}_n(Y)$.

(3) For any $x \in \overline{E}^R_{Un}(\underline{R}_n(Y))$, we have that $|[x]_U \cap \underline{R}_n(Y)| > n$. Hence, there exist at least $n+1$ elements $x \in [x]_U$ such that $x \in \overline{R}_n(Y)$, i.e., $|r(x) \cap Y| > n$. Therefore, $x \in \overline{R}_n(Y)$, i.e., $\overline{E}^R_{Un}(\underline{R}_n(Y)) \subseteq \overline{R}_n(Y)$.

As we have proved in (2), $Y \subseteq \underline{E}^R_{Vn}(Y)$. According to (3) of Proposition 2, we have that $\overline{R}_n(Y) \subseteq \overline{R}_n(\underline{E}^R_{Vn}(Y))$. Thus, $\overline{E}^R_{Un}(\underline{R}_n(Y)) \subseteq \overline{R}_n(\underline{E}^R_{Vn}(Y))$.

(4) For any $x \in \underline{R}_n\left(\overline{E}^R_{Vn}(Y)\right)$, from Definition 3, we have that $|r(x) - \overline{E}^R_{Vn}(Y)| \leqslant n$, i.e., $r(x)$ and $\overline{E}_{Vn}(Y)$ contain at most n different elements, which implies that xRy and $|[y]_V \cap Y| > n$. Hence, $|r(x) \cap Y| > n$, i.e., $x \in \overline{R}_n(Y)$. Therefore, $x \in [x]_U \cap \overline{R}_n(Y)$, i.e., $x \in \underline{E}^R_{Un}(\overline{R}_n(Y))$. Thus, $\underline{R}_n(\overline{E}^R_{Vn}(Y)) \subseteq \underline{E}^R_{Un}(\overline{R}_n(Y))$. □

Given any subset $Y \subseteq V$ and a fixed grade n, if we apply different approximation operators on Y, then we can obtain different results, and there exist some interrelations among these results. Proposition 4 gives a description of such interrelations among the results induced by different approximation operators on Y. As shown in (1) of Proposition 4, for any subset $Y \subseteq V$, the result induced by $\overline{E}^R_{Vn}$ and $\overline{R}_n$ on Y, i.e., $\overline{R}_n(\overline{E}^R_{Vn}(Y))$, is a subset of that induced by $\overline{R}_n$ and $\overline{E}^R_{Vn}$ on Y, and the latter is a subset of the result induced by $\overline{R}_n$ on Y. On the contrary, when the lower approximation operators $\underline{E}^R_{Vn}$, $\underline{E}^R_{Un}$ and $\underline{R}_n$ are concerned, we can obtain a different conclusion, as shown in (2) of Proposition 4. Moreover, when the lower and upper approximation operators are used simultaneously, the outcomes can be found in (3) and (4) of Proposition 4.

Proposition 5. *Let U and V be two universes of discourse, R a binary relation from U to V, and $n, m \in N$, $n \geqslant m$. $E^R_U \subseteq U \times U$ and $E^R_V \subseteq V \times V$ are two equivalence relations induced by R. For any $Y \subseteq V$, the following properties are satisfied.*

(1) $\overline{E}^R_{Un}(\underline{R}_n(Y)) \subseteq \overline{E}^R_{Um}(\underline{R}_m(Y))$
$\overline{R}_n(\underline{R}_n(Y)) \subseteq \overline{R}_m(\underline{R}_m(Y))$
(2) $\underline{E}^R_{Un}(\overline{R}_n(Y)) \supseteq \underline{E}^R_{Um}(\overline{R}_m(Y))$
$\underline{R}_n(\overline{R}_n(Y)) \supseteq \underline{R}_m(\overline{R}_m(Y))$
(3) $\underline{R}_n(\overline{E}^R_{Vn}(Y)) \supseteq \underline{R}_m(\overline{E}^R_{Vm}(Y))$
(4) $\overline{R}_n(\underline{E}^R_{Vn}(Y)) \subseteq \overline{R}_m(\underline{E}^R_{Vm}(Y))$

Proof 5. (1) If $n \geqslant m$, according to (7) of Proposition 2, we have that $\underline{R}_n(Y) \supseteq \underline{R}_m(Y)$. Thus, we have that $\overline{E}^R_{Un}(\underline{R}_n(Y)) \subseteq \overline{E}^R_{Un}(\underline{R}_m(Y))$ and $\overline{R}_n(\underline{R}_n(Y)) \subseteq \overline{R}_n(\underline{R}_m(Y))$. And from (3) of Proposition 2, we can obtain that $\overline{E}^R_{Un}(\underline{R}_m(Y)) \subseteq \overline{E}^R_{Um}(\underline{R}_m(Y))$ and $\overline{R}_n(\underline{R}_m(Y)) \subseteq \overline{R}_m(\underline{R}_m(Y))$. Therefore, $\overline{E}^R_{Un}(\underline{R}_n(Y)) \subseteq \overline{E}^R_{Um}(\underline{R}_m(Y))$ and $\overline{R}_n(\underline{R}_n(Y)) \subseteq \overline{R}_m(\underline{R}_m(Y))$.

Analogously, one can prove (2), (3) and (4), we omit the proofs of them here. □

Differing from Proposition 4, given any subset $Y \subseteq V$, Proposition 5 shows the interrelations between the results induced by the approximation operators on Y with different grades. As shown in (1) of Proposition 5, for any subset $Y \subseteq V$, the result induced by $\underline{R}_n$ and $\overline{E}^R_{Un}$ on Y is a subset of that induced by $\underline{R}_m$ and $\overline{E}^R_{Um}$ on Y, where $n \geqslant m$.

Definition 8 ([7]). Let U be a universe of discourse, P and R are two equivalence relations on U. We say that P is finer than R, denoted by $P \preccurlyeq R$, if each equivalence class in U/R is a union of some equivalence classes in U/P.

Proposition 6. *Let U and V be two universes of discourse, P and R are two binary relations from U to U, and $n \in N$. If $P \preccurlyeq R$, then for any $X \subseteq U$, $\underline{P}_n(X) \supseteq \underline{R}_n(X)$.*

Proof 6. For any $x \in \underline{R}_n(X)$, from Definition 7, $|[x]_U - X| \leqslant n$ holds. If $P \preccurlyeq R$, then $[x]_U = \bigcup_{i=1}^m N_i$, where N_i is the equivalence class with respect to P. Hence, we have that $|[x]_U - X| = \bigcup_{i=1}^m N_i - X| \leqslant n$, i.e., $\sum_{i=1}^m |N_i - X| \leqslant n$. Obviously, $|N_i - X| \geqslant 0$, thus, $|N_i - X| \leqslant n$, i.e., $x \in \underline{P}_n(X)$. Therefore, we can obtain that $\underline{P}_n(X) \supseteq \underline{R}_n(X)$.

Remark 3. $\overline{P}_n(X) \supseteq \overline{R}_n(X)$ does not always hold.

Example 1. Let U be the universe of discourse, P and R two equivalence relations on U. U, U/P and U/R are specified as follows:

$$U = \{x_1, x_2, \ldots, x_6\}$$
$$U/P = \{\{x_1\}, \{x_2, x_4\}, \{x_3\}, \{x_5, x_6\}\}$$
$$U/R = \{\{x_1, x_2, x_3, x_4\}, \{x_5, x_6\}\}$$

Obviously, we have that $P \preccurlyeq R$.

Suppose that $X = \{x_2, x_3, x_4\} \subseteq U$ and $n = 1$.

Then we can obtain that

$$\underline{P}_n(X) = \{x_1, x_2, x_3, x_4\}, \overline{P}_n(X) = \{x_2, x_4\}$$
$$\underline{R}_n(X) = \{x_1, x_2, x_3, x_4\}, \overline{R}_n(X) = \{x_1, x_2, x_3, x_4\}$$

Therefore $\underline{P}_n(X) \supseteq \underline{R}_n(X)$ and $\overline{P}_n(X) \subset \overline{R}_n(X)$.

4. Two illustrative examples

In this section, two illustrative examples are employed to demonstrate the concepts, method and properties which discussed in Section 3.

Example 2. Let U and V be two universes of discourse, R a binary relation from U to V, and $M_R = (a_{ij})_{n \times m}$ be the relation matrix of R, which are respectively given as follows:

$$U = \{x_1, x_2, \ldots, x_6\}$$
$$V = \{y_1, y_2, \ldots, y_7\}$$

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Let $Y = \{y_2, y_3, y_4\}$ be a subset of V, and $n = 2$.

First, we compute the lower and upper approximations of Y with respect to n using Definition 3.

From M_R, we can obtain that

$$r(x_1) = \{y_1, y_4\}, \quad r(x_2) = r(x_4) = \{y_1, y_2, y_3, y_4\}$$
$$r(x_3) = \{y_1, y_4, y_5, y_6, y_7\}, \quad r(x_5) = r(x_6) = \{y_5, y_6, y_7\}.$$

Hence, we can further obtain that

$$r(x_1) \cap Y = \{y_4\}, \quad r(x_2) \cap Y = r(x_4) \cap Y = \{y_2, y_3, y_4\}$$
$$r(x_3) \cap Y = \{y_4\}, \quad r(x_5) \cap Y = r(x_6) \cap Y = \emptyset.$$

Therefore,

$$|r(x_1) - Y| = 1 < 2, \quad |r(x_2) - Y| = |r(x_4) - Y| = 1 < 2$$
$$|r(x_3) - Y| = 4 > 2, \quad |r(x_5) - Y| = |r(x_6) - Y| = 3 > 2$$
$$|r(x_1) \cap Y| = 1 < 2, \quad |r(x_2) \cap Y| = |r(x_4) \cap Y| = 3 > 2$$
$$|r(x_3) \cap Y| = 1 < 2, \quad |r(x_5) \cap Y| = |r(x_6) \cap Y| = 0 < 2$$

According to Definition 3, $\underline{R}_2(Y) = \{x_1, x_2, x_4\}$ and $\overline{R}_2(Y) = \{x_2, x_4\}$.

Second, we compute the lower and upper approximations of Y with respect to n using Proposition 3.

From M_R, we can obtain that

$$\begin{aligned} sum &= (sum(1), sum(2), sum(3), sum(4), sum(5), sum(6))^T \\ &= (2, 4, 5, 4, 3, 3)^T \end{aligned}$$

and

$$Z = M_R \cdot Y = (z_1, z_2, \ldots z_m)^T = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}$$

Therefore,

$$sum(1) - z_1 = 1; \quad sum(2) - z_2 = 1; \quad sum(3) - z_3 = 4;$$
$$sum(4) - z_4 = 1; \quad sum(5) - z_5 = 3; \quad sum(6) - z_6 = 3.$$

Then, we can obtain that $\underline{R}_2(Y) = \{x_i \in U | sum(i) \ - z_i \leqslant 2\} = \{x_1, x_2, x_4\}$ and $\overline{R}_2(Y) = \{x_i \in U | z_i > 2\} = \{x_2, x_4\}$.

In the above example, the two methods produce the same result. And we can conclude that the method based on Proposition 3 is more efficient and convenient than that based on Definition 3.

Example 3. (Continued from Example 2)

From M_R, we can obtain that

$$r(x_1) = \{y_1, y_4\}, \quad r(x_2) = r(x_4) = \{y_1, y_2, y_3, y_4\}$$
$$r(x_3) = \{y_1, y_4, \ldots, y_7\}, \quad r(x_5) = r(x_6) = \{y_5, y_6, y_7\}$$
$$l(y_1) = l(y_4) = \{x_1, x_2, x_3, x_4\}, \quad l(y_2) = l(y_3) = \{x_2, x_4\}$$
$$l(y_5) = l(y_6) = l(y_7) = \{x_3, x_5, x_6\}$$

Then, we have that

$$U/E_U^R = \{\{x_1\}, \{x_2, x_4\}, \{x_3\}, \{x_5, x_6\}\}$$
$$V/E_V^R = \{\{y_1, y_4\}, \{y_2, y_3\}, \{y_5, y_6, y_7\}\}$$

Suppose $Y_1 = \{y_2, y_3, y_4\}$, $Y_2 = \{y_2, y_3\}$ and $Y_3 = \{y_1, y_2, y_3, y_5\}$. Then $Y_2 \subset Y_1$, $Y_1 \cap Y_3 = \{y_2, y_3\}$, $Y_1 \cup Y_3 = \{y_1, y_2, y_3, y_4, y_5\}$.

First, let $n = 2$.

Then, we have that

$$\overline{R}_2(Y_1) = \{x_2, x_4\}, \quad \overline{R}_2(Y_2) = \emptyset$$
$$\overline{R}_2(Y_3) = \{x_2, x_3, x_4\}, \quad \overline{E_{V2}^R}(Y_1) = \emptyset$$
$$\overline{R}_2(Y_1 \cap Y_3) = \emptyset, \quad \overline{R}_2(Y_1 \cup Y_3) = \{x_2, x_3, x_4\}$$
$$\underline{R}_2(Y_1) = \{x_1, x_2, x_4\}, \quad \underline{R}_2(Y_2) = \{x_1, x_2, x_4\}$$
$$\underline{R}_2(Y_3) = \{x_1, x_2, x_4, x_5, x_6\}, \quad \underline{E_{V2}^R}(Y_1) = \{y_1, y_2, y_3, y_4\}$$
$$\underline{R}_2(Y_1 \cap Y_3) = \{x_1, x_2, x_4\},$$
$$\underline{R}_2(Y_1 \cup Y_3) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

From the above definitions, we have that

$$\overline{R}_2(Y_2) = \emptyset \subset \overline{R}_2(Y_1) = \{x_2, x_4\}$$
$$\underline{R}_2(Y_2) = \{x_1, x_2, x_4\} \subseteq \underline{R}_2(Y_1) = \{x_1, x_2, x_4\}$$
$$\overline{R}_2(Y_1 \cap Y_3) = \emptyset \subset \overline{R}_2(Y_1) \cap \overline{R}_2(Y_3) = \{x_2, x_4\}$$
$$\underline{R}_2(Y_1 \cap Y_3) = \{x_1, x_2, x_4\} \subseteq \underline{R}_2(Y_1) \cap \underline{R}_2(Y_3) = \{x_1, x_2, x_4\}$$
$$\overline{R}_2(Y_1 \cup Y_3) = \{x_2, x_3, x_4\} \supseteq \overline{R}_2(Y_1) \cup \overline{R}_2(Y_3) = \{x_2, x_3, x_4\}$$
$$\underline{R}_2(Y_1 \cup Y_3) = \{x_1, x_2, x_3, x_4, x_5, x_6\} \supset \underline{R}_2(Y_1) \cup \underline{R}_2(Y_3) = \{x_1, x_2, x_4, x_5, x_6\}$$
$$\overline{R}_2(\overline{E_{V2}^R}(Y_1)) = \emptyset, \ \overline{E_{U2}^R}(\overline{R}_2(Y_1)) = \emptyset$$
$$\overline{E_{U2}^R}(\underline{R}_2(Y_1)) = \emptyset, \ \underline{R}_2(\overline{E_{V2}^R}(Y_1)) = \{x_1\}$$
$$\underline{E_{U2}^R}(\overline{R}_2(Y_1)) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$
$$\overline{R}_2(\underline{E_{V2}^R}(Y_1)) = \{x_2, x_4\}$$
$$\underline{E_{U2}^R}(\underline{R}_2(Y_1)) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$
$$\underline{R}_2(\underline{E_{V2}^R}(Y_1)) = \{x_1, x_2, x_4\}$$
$$\overline{R}_2(\underline{R}_2(Y_1)) = \{y_1, y_4\}$$
$$\underline{R}_2(\overline{R}_2(Y_1)) = \{y_1, y_2, y_3, y_4\}$$

Then, we can conclude that

$$\overline{R}_2(\overline{E^R_{V2}}(Y_1)) = \emptyset \subseteq \overline{E^R_{U2}}(\overline{R}_2(Y_1)) = \emptyset$$
$$\underline{E^R_{U2}}(\underline{R}_2(Y_1)) = \{x_1,x_2,x_3,x_4,x_5,x_6\} \supseteq \underline{R}_2(\underline{E^R_{V2}}(Y_1)) = \{x_1,x_2,x_4\}$$
$$\overline{E^R_{U2}}(\underline{R}_2(Y_1)) = \emptyset \subseteq \overline{R}_2(\underline{E^R_{V2}}(Y_1)) = \{x_2,x_4\}$$
$$\underline{R}_2(\overline{E^R_{V2}}(Y_1)) = \{x_1\} \subseteq \underline{E^R_{U2}}(\overline{R}_2(Y_1)) = \{x_1,x_2,x_3,x_4,x_5,x_6\}$$

Second, let $n = 1$.
Analogously, we have that

$$\overline{R}_1(Y_1) = \{x_2,x_4\}, \quad \overline{R}_1(Y_2) = \{x_2,x_4\}, \quad \overline{E^R_{V1}}(Y) = \emptyset$$
$$\underline{R}_1(Y_1) = \{x_1,x_2,x_4\}, \quad \underline{R}_1(Y_2) = \emptyset, \quad \underline{E^R_{V1}}(Y_1) = \{y_1,y_2,y_3,y_4\}$$

From the above definitions, we have that

$$\overline{R}_1(Y_2) = \{x_2,x_4\} \supset \overline{R}_2(Y_2) = \emptyset$$
$$\underline{R}_1(Y_2) = \emptyset \subset \underline{R}_2(Y_2) = \{x_1,x_2,x_4\}$$
$$\overline{R}_1(\overline{E^R_{V1}}(Y_1)) = \emptyset, \quad \overline{E^R_{U1}}(\overline{R}_1(Y_1)) = \{x_2,x_4\}$$
$$\overline{E^R_{U1}}(\underline{R}_1(Y_1)) = \{x_1,x_2,x_4\}\underline{R}_1(\overline{E^R_{V1}}(Y_1)) = \emptyset$$
$$\underline{E^R_{U1}}(\overline{R}_1(Y_1)) = \{x_1,x_2,x_3,x_4\}$$
$$\overline{R}_1(\underline{E^R_{V1}}(Y_1)) = \{x_1,x_2,x_3,x_4\}$$
$$\underline{E^R_{U1}}(\underline{R}_1(Y_1)) = \{x_1,x_2,x_3,x_4\}$$
$$\underline{R}_1(\underline{E^R_{V1}}(Y_1)) = \{x_1,x_2,x_4\}$$
$$\overline{R}_1(\underline{R}_1(Y_1)) = \{y_1,y_2,y_3,y_4\}$$
$$\underline{R}_1(\overline{R}_1(Y_1)) = \{y_2,y_3\}$$

Then, we can conclude that

$$\overline{R}_1(\overline{E^R_{V1}}(Y_1)) = \emptyset \subseteq \overline{E^R_{U1}}(\overline{R}_1(Y_1)) = \{x_2,x_4\}$$
$$\underline{E^R_{U1}}(\underline{R}_1(Y_1)) = \{x_1,x_2,x_3,x_4\} \supseteq \underline{R}_1(\underline{E^R_{V1}}(Y_1)) = \{x_1,x_2,x_4\}$$
$$\overline{E^R_{U1}}(\underline{R}_1(Y_1)) = \{x_1,x_2,x_4\} \subseteq \overline{R}_1(\underline{E^R_{V1}}(Y_1)) = \{x_1,x_2,x_3,x_4\}$$
$$\underline{R}_1(\overline{E^R_{V1}}(Y_1)) = \emptyset \subseteq \underline{E^R_{U1}}(\overline{R}_1(Y_1)) = \{x_1,x_2,x_3,x_4\}$$

Similarly, we can obtain that

$$\overline{E^R_{U2}}(\underline{R}_2(Y_1)) = \emptyset \subseteq \overline{E^R_{U1}}(\underline{R}_1(Y_1)) = \{x_1,x_2,x_4\}$$
$$\overline{R}_2(\underline{R}_2(Y_1)) = \{y_1,y_4\} \subseteq \overline{R}_1(\underline{R}_1(Y_1)) = \{y_1,y_2,y_3,y_4\}$$
$$\underline{E^R_{U2}}(\overline{R}_2(Y_1)) = \{x_1,x_2,x_3,x_4,x_5,x_6\} \supseteq \underline{E^R_{U1}}(\overline{R}_1(Y_1)) = \{x_1,x_2,x_3,x_4\}$$
$$\underline{R}_2(\overline{R}_2(Y_1)) = \{y_1,y_2,y_3,y_4\} \supseteq \underline{R}_1(\overline{R}_1(Y_1)) = \{y_2,y_3\}$$
$$\underline{R}_2(\overline{E^R_{V2}}(Y_1)) = \{x_1\} \supseteq \underline{R}_1(\overline{E^R_{V1}}(Y_1)) = \emptyset$$
$$\overline{R}_2(\underline{E^R_{V2}}(Y_1)) = \{x_2,x_4\} \subseteq \overline{R}_1(\underline{E^R_{V1}}(Y_1)) = \{x_1,x_2,x_3,x_4\}$$

5. The application of GRSTU

In this section, we shall discuss the application of the graded rough set model based on two universes given in Section 3. In fact, we can also find some existing applications of the rough set model based on two universes from the existing references [31].

Assume that we are considering a certain group of patients in a medical diagnosis system. A patient may show several symptoms at the same time and a disease may be accompanied by several symptoms, which will establish a relation between a patient and a disease. In the current situation, how a doctor could decide what kind of treatments that a patient should be taken according to the existing symptoms? This question can be answered by applying the graded rough set model based on two universes to the medical diagnosis system.

Let U denote the set of patients and V the set of symptoms. Then for any patient $u \in U$, there exist some symptoms in V correspond to u. For any $Y \subseteq V$, Y is a certain disease which contains some basic symptoms in V. Then, given a patient, if he or she belongs to set $POS(Y)$, then he or she is certainly suffered from the disease denoted by Y. Therefore, all of the patients belonging to $POS(Y)$ are suitable for the remedy to Y immediately. On the other hand, if he or she belongs to set $BND(Y)$, then his or her disease is connected with Y, but the connection is not certain. Therefore, the doctor should further analyze the pathogeny for the current patient and adopt an appropriate treatment. Furthermore, if he or she belongs to set $NEG(Y)$, then his or her disease has no connection with Y, and the doctor should adopt other strategies.

We will show the above discussion by the following example.

Example 4. Let $U = \{x_1,x_2,x_3\}$ be the set of patients, $V = \{y_1,y_2,y_3,y_4\}$ the set of symptoms, and $n \in N$. Assume that $R \subseteq U \times V$ is a binary relation between U and V, which can be described by the following relation matrix $M_R = (a_{ij})$ (where if patient i has symptom j then $a_{ij} = 1$ else $a_{ij} = 0$):

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

From matrix M_R, we can obtain that

$$r(x_1) = \{y_1,y_4\}, \quad r(x_2) = \{y_1,y_2\}, \quad r(x_3) = \{y_2,y_3,y_4\}.$$

Let $Y = \{y_2,y_4\} \subseteq V$ denote a certain disease, that is, Y shows two symptoms in the clinic. Meanwhile, we suppose that $n = 1$.

From Definitions 3 and 4, we can calculate the lower approximation, the upper approximation, the positive region, the negative region and the boundary region of Y as follows.

$$\underline{R}_1(Y) = \{x_1,x_2,x_3\},$$
$$\overline{R}_1(Y) = \{x_3\},$$
$$POS_1(Y) = \underline{R}_1(Y) \cap \overline{R}_1(Y) = \{x_3\},$$
$$NEG_1(Y) = \sim (\underline{R}_1(Y) \cup \overline{R}_1(Y)) = \emptyset,$$
$$BND_1(Y) = \sim (pos_1(Y) \cup neg_1(Y)) = \{x_1,x_2\}$$

Then, we can obtain the following conclusions:

(1) It is certain that patient x_3 has disease Y and the doctor should take the corresponding treatment immediately.
(2) The doctor cannot decide whether patients x_1 and x_2 have disease Y or not according to the symptoms at present. The patients should be examined further.
(3) None of the three patients is healthy after diagnosis.

6. Conclusions

Rough set theory based on two universes is a generalization of Pawlak's rough set theory. In this paper, the graded rough set model based on two distinct but related universes was proposed. We gave some interesting properties and conclusions about the graded rough set model on two universes, which can help us understand the structure of GRSTU. Moreover, an efficient method for calculating the lower and upper approximations of a given set in GRSTU was proposed and several examples were also given. The outcomes of these examples demonstrated that GRSTU is more suitable to the decision of clinical diagnosis than the traditional rough set model.

In the future work, we shall further discuss other aspects of GRSTU, for instance, the issue of attribute reduction or rules extraction in GRSTU.

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