International Journal of Approximate Reasoning 54 (2013) 1130–1148

Contents lists available at SciVerse ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

# Two basic double-quantitative rough set models of precision and grade and their investigation using granular computing

Xianyong Zhang [a,b,c,*], Duoqian Miao [b,c]

[a] College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610068, PR China
[b] Department of Computer Science and Technology, Tongji University, Shanghai 201804, PR China
[c] The Key Laboratory of Embedded System and Service Computing, Ministry of Education, Shanghai 201804, PR China

ARTICLE INFO

*Article history:*
Received 29 March 2012
Received in revised form 8 February 2013
Accepted 12 February 2013
Available online 19 April 2013

*Keywords:*
Rough set theory
Granular computing
Variable precision rough set
Graded rough set
Double quantification
Model region

ABSTRACT

The precision and grade of the approximate space are two fundamental quantitative indexes that measure the relative and absolute quantitative information, respectively. The double quantification of the precision and grade is a relatively new subject, and its effective implementation remains an open problem. This paper approaches the double quantification problem using basic rough set models. The Cartesian product is a natural operator for combining the two indexes given their completeness and complementary natures, and we construct two new models using this strategy. The fundamental items (i.e., the complete system, quantitative semantics and optimal computing) of the model regions are studied using granular computing. First, the model regions (MR granules) and basic model regions (BMR granules) are defined in the traditional fashion using logical double-quantitative semantics; basic semantics (BS) is provided for the double-semantic description, and the semantic extraction of the MR and BMR granules is realized within the BS framework. Computing granules (BMRC granules) are then proposed for the basic model regions to optimize the computation, and a two-dimensional plane and granular hierarchical structure are provided. Two basic algorithms for computing the MR and BMR granules are proposed and analyzed, and the BMRC-granules algorithm generally exhibits superior performance in terms of the temporal and spatial complexity. We also explore the properties of the approximation operators and the notions of attribute approximate dependence and reduction. Finally, we provide an example application from the medical field. The two models provide a basic double quantification of the precision and grade and have concrete double-quantitative semantics; they also represent a quantitatively complete expansion of the Pawlak model.



## 1. Introduction

### 1.1. The Pawlak, VPRS and GRS models

Rough set theory is a mathematical tool for handling vague and incomplete information. This relatively new soft computing methodology has received great attention in recent years, and its effectiveness has been confirmed through successful applications in many science and engineering fields. Rough set theory is currently one of the most promising research directions in artificial intelligence theory and applications.

Let $U$ be a finite and non-empty universe, and let $R$ be an equivalence relation on $U$. Then, $(U, R)$ constitutes an approximate space, and $R$ represents the available knowledge; $[x]_R$ and $U/R$ denote the equivalence class (i.e., the knowledge granule)

* Corresponding author at: College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610068, PR China.
*E-mail addresses:* xianyongzh@sina.com.cn (X. Zhang), miaoduoqian@163.com (D. Miao).

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http://dx.doi.org/10.1016/j.ijar.2013.02.005

and knowledge quotient set, respectively. The basic concept $X \subseteq U$. In the Pawlak Model [1], the following approximations are made:

$$\overline{R}X = \cup\{[x]_R : [x]_R \cap X \neq \phi\}, \quad \underline{R}X = \cup\{[x]_R : [x]_R \subseteq X\}. \tag{1}$$

Here,

$$posRX = \underline{R}X, \ \ negRX = \sim \overline{R}X, \ \ bnRX = \overline{R}X - \underline{R}X \tag{2}$$

are the positive region, negative region and boundary region, respectively. These regions have qualitative semantics and reflect the positive certainty, negative certainty and uncertainty, respectively; they therefore act as fundamental notions for knowledge discovery. Attribute reduction is performed primarily using the positive region [2,3], while the boundary region determines the uncertainty in the rough sets. Previous studies [4,5] have examined feature selection or attribute reduction based on the boundary region. However, the Pawlak model has certain limitations. The relationships between the basic concepts and knowledge granules are so strict that there are no fault-tolerance mechanisms, and the quantitative information on the approximate space is not considered. Therefore, neither wider relationships nor quantitative information can be utilized. However, inclusion relationships usually occur in practice, and the extent of overlap among the sets provides important information. Improving the Pawlak model is therefore a promising direction, and expansions of the model that include quantification are of particular value.

Probability is an important tool for describing uncertainty and has been introduced into rough set theory. The probabilistic rough set (PRS) model [6–9] has many merits, such as the measurability of the probabilistic information, the generality and flexibility of the model and its insensitivity to noise. The PRS model has been investigated extensively, and many concrete realizations of the model are available, including the decision-theoretic rough sets (DTRS) [10,11], game-theoretic rough sets (GTRS) [12,13], variable precision rough sets (VPRS) [14,15], 0.5-probabilistic rough sets [16], parameterized rough set models [17,18] and Bayesian rough sets [19,20]. The graded rough set (GRS) model [21–23] has many features in common with the PRS model and functions as a typical expansion model by including quantification. The VPRS and GRS models are two fundamental expansion models that achieve strong fault tolerance capabilities by utilizing quantitative descriptions. This paper focuses primarily on these two models.

The VPRS model was proposed by Ziarko [14]. This model allows for a relative degree of misclassification $c([x]_R, X) = 1 - |[x]_R \cap X|/|[x]_R|$, and the following approximations are made, with $\beta \in [0, 0.5)$:

$$\overline{R}_\beta X = \cup\{[x]_R : c([x]_R, X) < 1 - \beta\}, \underline{R}_\beta X = \cup\{[x]_R : c([x]_R, X) \leq \beta\}. \tag{3}$$

Here,

$$posR_\beta X = \underline{R}_\beta X, negR_\beta X = \sim \overline{R}_\beta X, bnR_\beta X = \overline{R}_\beta X - \underline{R}_\beta X \tag{4}$$

are the positive region, negative region and boundary region, respectively. The symbol $\beta$ was also used to denote the proportion of correct classifications in previous studies [24–26], where $\beta \in (0.5, 1]$. In this paper,

$$p([x]_R, X) = |[x]_R \cap X|/|[x]_R| \tag{5}$$

is referred to as the precision of $[x]_R$ with respect to $X$ and $\beta \in [0, 0.5)$ and $1 - \beta$ is referred to as the precision parameter. Because the problem of noisy data is substantially mitigated, the VPRS model is highly useful in data acquisition and analysis; moreover, this model is an expansion of the Pawlak model. The VPRS model has become increasingly popular in a variety of theoretical and practical fields, producing many thorough results. Previous researchers [25–30] have studied knowledge reduction and rule extraction in the VPRS model, and some studies [31–37] have applied the model in the medical, geological, management and psychological fields, among others. Both the relative degree of misclassification $c([x]_R, X)$ and the precision $p([x]_R, X)$ are related to the rough membership [38], and the latter has been used in many studies [39,40]. In particular, Greco et al. [39] presented a generalized VPRS model using the absolute and relative rough membership. The inclusion degree, as a generalization of the rough membership, has been used extensively in the study of measures, reasoning, applications of uncertainty [29,41,42], and approximate spaces [43,44].

Yao and Lin [21] explored the relationships between rough sets and modal logics and proposed the GRS model based on graded modal logics. The GRS model primarily considers the absolute quantitative information regarding the basic concept and knowledge granules and is a generalization of the Pawlak model. The studies focused on the model construction, as in [22,23,45]. In the GRS model, $k \in \mathbf{N}$ and the following approximations are made:

$$\overline{R}_k X = \cup\{[x]_R : |[x]_R \cap X| > k\}, \underline{R}_k X = \cup\{[x]_R : |[x]_R| - |[x]_R \cap X| \leq k\}. \tag{6}$$

Note that $\overline{R}_k X$ is the union of the equivalence classes *whose numbers of elements inside X exceed k*; $\underline{R}_k X$ is the union of the equivalence classes *whose numbers of elements outside X are at most k*. In this paper,

$$\overline{g}([x]_R, X) = |[x]_R \cap X|, \underline{g}([x]_R, X) = |[x]_R| - |[x]_R \cap X| \tag{7}$$

are referred to as the internal and external grades of $[x]_R$ with respect to $X$, respectively, and $k$ is referred to as the grade parameter.

### 1.2. Precision and grade

The basic notions of the relative and absolute errors have been employed previously, for example, in the measurement field. Here, the relative degree of misclassification $c([x]_R, X)$ and the external grade $\underline{g}([x]_R, X)$ are referred to as the relative and absolute errors regarding the basic concept $X$ and knowledge $R$, respectively. The precision $p([x]_R, X)$ and the internal grade $\overline{g}([x]_R, X)$ are referred to as the relative overlap ratio and absolute overlap number. The precision and grade therefore reflect the relative and absolute quantitative information of the approximate space, respectively, and become two fundamental quantitative indexes. Through these two indexes, the VPRS and GRS models provide relative and absolute quantifications, extend the Pawlak model and achieve strong relative and absolute fault tolerance capabilities. The relative and absolute quantitative information represent two distinct objective descriptors of the approximate space, and each descriptor has its own merits; therefore, both indexes are needed to describe different application environments. Here, an example [46] is provided to illustrate the importance of the absolute quantitative information.

Example (i). Let A and B be two universities with 40 and 20 proposed projects, respectively. Only 30 projects will be implemented in total. The universities must make a final decision on which projects to implement. In this problem, the universe consists of the 60 proposed projects, and the projects proposed for A and B constitute two equivalence classes; thus, a simple approximate space is formed. If only the relative quantitative information is considered, then A and B should receive 20 and 10 implemented projects, respectively. However, is this division fair in reality? If the two universities have nearly the same research levels, then this proportionate division may be reasonable. However, if the research level of A is much higher than that of B, then A should receive more than 20 projects and B should receive fewer than 10 projects. The number of implemented projects becomes a pivotal index. In practice, the research evaluation of a given university depends primarily on the total number of implemented projects (i.e., the absolute quantitative information) rather than the implementation ratio of the proposed projects (i.e., the relative quantitative information). An ideal evaluation must employ rational combinations of the two evaluation indexes.

Elements of different equivalence classes may exhibit large differences in an important attribute in the approximate space; therefore, different equivalence classes have distinct degrees of the relevant information. In practice, there are usually large information gaps between the equivalence classes, and this situation must be emphasized and utilized. The above example also demonstrates (in one specific case) how the grade index can be used to improve the information provided by the relative quantitative index. Moreover, the relative ratio must be handled in discriminative fashions.

Examples demonstrating the importance of the relative quantitative information are even more readily constructed, and we conclude that both the absolute and relative quantitative information are fundamental. The combination of the two types of information is therefore novel and valuable. Next, another example is provided for illustration by the role of measurement errors.

Example (ii). Suppose that $r$ and $m$ are the real and measured values in a measurement process, respectively. It is known that $\varepsilon_R = (m - r)/r$ and $\varepsilon_A = m - r$ represent the relative and absolute errors, respectively, and both of these quantities reflect the reliability of the measured result, $m$. In a measurement evaluation, the relative and absolute errors can be utilized individually. For example, the statement that $\varepsilon_R = 10\%$ reflects certain measurement information but provides only a partial evaluation of the result. Suppose that $\varepsilon_{R1} = 10\%$ and $\varepsilon_{R2} = 30\%$. It cannot be established that the first measurement is more precise or superior compared to the latter. Moreover, consider the case in which $\varepsilon_{R1} = \varepsilon_{R2}$. However, a more definite determination can be made by considering both errors. Clearly, the measurement system $(r, m)$ is two-dimensional. The case $\varepsilon_R = 0$ (i.e., $m = r$) is so specific that it is not discussed here. In the usual case in which $\varepsilon_R \neq 0$,

$$\begin{cases} \varepsilon_R = \dfrac{m - r}{r} \\ \varepsilon_A = m - r \end{cases} \tag{8}$$

$$\begin{cases} r = \dfrac{\varepsilon_A}{\varepsilon_R} \\ m = \dfrac{1 + \varepsilon_R}{\varepsilon_R}\varepsilon_A \end{cases} \tag{9}$$

Formulas (8) and (9) are therefore equivalent. Formula (9) shows that $r$ and $m$ can be represented by the errors, and the two systems $(r, m)$ and $(\varepsilon_R, \varepsilon_A)$ are therefore equivalent. Thus, only $\varepsilon_R$ and $\varepsilon_A$ can completely determine the system $(r, m)$. In other words, $\varepsilon_R$ and $\varepsilon_A$ have quantitative completeness so can reflect the exact measurement information. For example, suppose that there are two possible projects with different costs, $\varepsilon_{R1} = 10\%$, $\varepsilon_{A1} = 1000$(Meters) and $\varepsilon_{R2} = 30\%$, $\varepsilon_{A2} = 30$(Meters). Then, the following results can be obtained using formula (9): $r_1 = 10000$(Meters), $m_1 = 11000$(Meters) and $r_2 = 100$(Meters), $m_2 = 130$(Meters). Although $\varepsilon_{R2} = 30\% > 10\% = \varepsilon_{R1}$, measurement 2 may provide a superior evaluation because its absolute loss (30 Meters) is considerably lower than measurement 1's (1000 Meters). In practice, a rational strategy is usually adopted: if $r$ (or $m$) is higher, then $\varepsilon_R$ is more highly emphasized and controlled because $\varepsilon_R$

will determine a more effective change – the change in $\varepsilon_A$ on $r$ (or $m$). Therefore, the change in the relative error is neither balanced nor linear. Another case is as follows. If $r_1 = 10000$, $r_2 = 100$, $\triangle\varepsilon_{A_1} = 10 = \triangle\varepsilon_{A_2}$, then $\triangle\varepsilon_{R_1} \approx 0.1\%$ while $\triangle\varepsilon_{R_2} \approx 10\%$. In fact,

$$\varepsilon_R = \frac{\varepsilon_A}{r},\ \frac{\partial\varepsilon_R}{\partial r} = -\frac{\varepsilon_A}{r^2},\ \frac{\partial\varepsilon_R}{\partial\varepsilon_A} = \frac{1}{r},\ d\varepsilon_R = -\frac{\varepsilon_A}{r^2}dr + \frac{1}{r}d\varepsilon_A. \tag{10}$$

Formula (10) shows that the relationship between $\varepsilon_R$ and $\varepsilon_A$ is nonlinear, and the change in $\varepsilon_R$ depends on $\varepsilon_A$. Moreover, the reverse holds as well, i.e., the change in $\varepsilon_A$ also depends on $\varepsilon_R$. Therefore, $\varepsilon_R$ and $\varepsilon_A$ are closely related.

For the approximate space $(U, R)$ and the set $X$, $|[x]_R|$ and $|[x]_R \cap X|$ function as two core quantitative variables. The system $(|[x]_R|, |[x]_R \cap X|)$ is therefore a basic quantitative system for $(U, R, X)$ and is also two-dimensional. We have the following formulas:

$$\begin{cases} p([x]_R, X) = \dfrac{|[x]_R \cap X|}{|[x]_R|} \\ \underline{g}([x]_R, X) = |[x]_R| - |[x]_R \cap X| \end{cases} \tag{11}$$

$$\begin{cases} |[x]_R| = \dfrac{\underline{g}([x]_R, X)}{1 - p([x]_R, X)} \\ |[x]_R \cap X| = \dfrac{p([x]_R, X)}{1 - p([x]_R, X)}\underline{g}([x]_R, X) \end{cases} \tag{12}$$

$$p([x]_R, X) = 1 - \frac{\underline{g}([x]_R, X)}{|[x]_R|}, \tag{13}$$

$$\frac{\partial p([x]_R, X)}{\partial |[x]_R|} = \frac{\underline{g}([x]_R, X)}{|[x]_R|^2},\ \frac{\partial p([x]_R, X)}{\partial \underline{g}([x]_R, X)} = -\frac{1}{|[x]_R|}, \tag{14}$$

$$dp([x]_R, X) = \frac{\underline{g}([x]_R, X)}{|[x]_R|^2}d|[x]_R| - \frac{1}{|[x]_R|}d\underline{g}([x]_R, X). \tag{15}$$

According to formulas (11)–(15), the system $(|[x]_R|, |[x]_R \cap X|)$ is similar to the measurement system $(r, m)$, and the relevant results can be achieved. Therefore, $p([x]_R, X)$ and $\underline{g}([x]_R, X)$ are usually complete for quantification, and the case is the same for the other pair, $p([x]_R, X)$ and $\overline{g}([x]_R, X)$. Unfortunately, most PRS models utilize only the conditional probability related to $c([x]_R, X)$ or $p([x]_R, X)$, and there is no information regarding $\underline{g}([x]_R, X)$ or $\overline{g}([x]_R, X)$. The relative information has the merits of being macroscopic, statistical, and convenient for applications. However, without the absolute information, the relative information is incomplete for quantification. The quantitative completeness determines the degrees of accuracy and certainty and is a primary and fundamental factor in many cases. The complexity of the model is another important factor to consider. In practice, the computational complexity of the composite system (evaluating both the precision and grade) has the same feasibility level as that of the individual systems to within a factor of approximately two. The combined quantification is therefore necessary and valuable. Moreover, the mapping shows that the precision and grade are mutually dependent and that the relationship between them is nonlinear. However, most PRS models have the balanced or linear feature, and the relative measurement is unified for all cases. In other words, in performing the relative extraction, the relative single quantification does not consider the absolute information environment and also roughens and neglects the objective environment (related to $|[x]_R|$). For example, for two knowledge granules $[x]_R^1 \neq [x]_R^2$, if $p([x]_R^1, X) = p([x]_R^2, X)$, then $[x]_R^1$ and $[x]_R^2$ are indiscernible or equal in most PRS models. However, $|[x]_R^1| \neq |[x]_R^2|$ or $|[x]_R^1 \cap X| \neq |[x]_R^2 \cap X|$, and $[x]_R^1$ and $[x]_R^2$ can be discerned by introducing the absolute information, $|[x]_R|$ or $|[x]_R \cap X|$. The change in precision can be further analyzed using the total differential formula (15). Examples (i) and (ii) and formulas (11)–(15) show that the introduction of the absolute information is rational and necessary. The above analysis demonstrates the complementarity of the precision and grade. Therefore, the double quantification formed by adding the absolute quantitative information can improve the descriptive abilities of PRS models and expand their range of applicability. The relative information similarly complements the absolute description and can be used to improve the GRS model. Based on the above examples and analysis, the sharp contrast between the precision and grade environments is typical of double quantification applications. For example, if the precision varies over a small range while the grade changes significantly, then the double quantification can play an effective role.

In summary, the precision and grade are related to the relative and absolute quantitative information, respectively. Using these two indexes, single quantification models have been extended to capture more general relationships and complex situations. However, the two indexes are not equivalent, and the relationship between them is often close, complementary and dialectical. Therefore, given the completeness and complementarity of the precision and grade, their double quantification has substantial value. This double quantification method can provide a thorough description of the approximate space, yield new models with strong double fault tolerance capabilities to adapt to increasingly complex environments, accelerate the development of both VPRS and GRS models, and promote knowledge discovery based on double-quantitative information. The VPRS and GRS models, two basic quantitative models of the precision and grade, form the basis of our subject. They are

closely interrelated; for example, the variable precision and grade approximation operators have similar basic properties. We have performed a comparative study of the two models and provided both the relationship and formal transformations between them [46]. The mutual transformations based on more general relations were also studied [45,47]. Therefore, the two models provide an ideal foundation for the double quantification and render the subject feasible.

### 1.3. Granular computing

The term granular computing first appeared in the academic literature in the paper by Zadeh in 1997 [48]. Granular computing is often loosely defined as an umbrella term covering all theories, methodologies, techniques, and tools that make use of granules in complex problem solving [49]. Granular computing has emerged as one of the fastest growing information processing paradigms in computational intelligence and human-centric systems. In recent years, many special issues of journals, international conferences and books have been devoted to the topic of granular computing. Yao first proposed the triarchic model of granular computing (i.e., the philosophy of structured thinking, methodology of structured problem solving, and computation of structured information processing), and the basic theory was developed in several subsequent papers, such as [49,50]. The notion of granular computing primarily reflects the use of multiple granules, multiple levels and multiple views and provides a concrete methodology for information processing and problem solving.

Rough set theory provides a concrete foundation for granular computing, and certain aspects of rough-granular computing have been thoroughly studied. For example, Pawlak and Skowron [17] took a rough set approach to granular computing; Skowron and Stepaniuk [43] proposed the formation of granules based on various rough computing criteria; Skowron et al. [44] investigated certain important issues related to rough granular computing models based on approximation spaces; Yao [51] studied both information granulation and rough set approximations; Zhu [52] explored covering-based rough sets from the topological viewpoint of granular computing; Liu et al. [53] explored granular computing from a rough logic perspective. The dominance-based rough set approach provides an alternative granular computing methodology based on rough sets. In particular, Pal et al. [54] investigated image object extraction within the framework of both rough sets and granular computing.

### 1.4. Approach and outline

Given the completeness and complementarity of the precision and grade, the double quantification of these two indexes is a novel, necessary, valuable and feasible objective. However, the implementation of this subject remains an open problem. In contrast to a simple fusion, the Cartesian product uses multi-dimensional constructs to store data and information and has the advantages of information losslessness and data recovery ability. The Cartesian product is therefore a natural operator for the combination of the precision and grade. We therefore construct the double quantification based on the Cartesian product and study basic double-quantitative rough set models. The transition from the single quantification to the double quantification is emphasized, and our development is based on the basic concepts of both rough set theory and granular computing. The model is constructed using a Cartesian product and investigated using granular computing. The complete system, quantitative semantics and optimal computing of macroscopic regions are fundamental issues in applications of the rough set model. These issues will be resolved in detail using granular computing in the constructed models.

The remainder of the paper is organized as follows. In Section 2.1, we naturally construct two basic models using the Cartesian product: $(U, \overline{R}_\beta, \underline{R}_k)$ and $(U, \overline{R}_k, \underline{R}_\beta)$. Both the model regions (MR granules) and basic model regions (BMR granules) are defined and studied; the quantitative semantics of the MR and BMR granules are then explored, and basic semantics (BS) is provided for the granule-semantic extraction. Section 2.2 explores the expansion and completeness of the two models. Section 3.1 proposes and investigates computing granules for the basic model regions (i.e., BMRC granules), and Section 3.2 explores the two-dimensional plane and granular hierarchical structure. Section 4 proposes and analyzes two basic algorithms for computing the MR and BMR granules. We discuss the properties of the approximation operators and the notions of attribute approximate dependence and reduction in Section 5. Sections 3–5 are primarily concerned with $(U, \overline{R}_\beta, \underline{R}_k)$; Section 6 provides the corresponding results for the latter model, $(U, \overline{R}_k, \underline{R}_\beta)$. In Section 7, we present an illustrative example from the medical field. Finally, the paper is concluded in Section 8.

## 2. Two basic double-quantitative rough set models of precision and grade

### 2.1. Modeling, model regions and their semantics

Based on multiple arguments, it is necessary to implement the double quantification using a Cartesian product of the precision and grade. Based on the relative and absolute quantitative information, the VPRS and GRS models perform a basic single quantification of the precision and grade, respectively. If the upper and lower approximations quantify both the precision and grade, then two fundamental double-quantitative models can be naturally constructed.

**Definition 2.1.1**

$$\overline{R}_\beta : 2^U \to 2^U, \underline{R}_k : 2^U \to 2^U, \forall X \in 2^U,$$
$$\overline{R}_\beta X = \cup\{[x]_R : p([x]_R, X) > \beta\}, \underline{R}_k X = \cup\{[x]_R : \underline{g}([x]_R, X) \leq k\}. \tag{16}$$

Then, $\overline{R}_\beta$ and $\underline{R}_k$ are the approximation operators. The construct $(\overline{R}_\beta, \underline{R}_k)$ is referred to as the Cartesian product of $\overline{R}_\beta$ and $\underline{R}_k$, and can determine a rough set model, denoted by $(U, \overline{R}_\beta, \underline{R}_k)$. Similarly, the other model $(U, \overline{R}_k, \underline{R}_\beta)$ is defined using the dual approximation operators $\overline{R}_k$ and $\underline{R}_\beta$, where the core mappings are given by the following approximations:

$$\overline{R}_k X = \cup\{[x]_R : \overline{g}([x]_R, X) > k\}, \underline{R}_\beta X = \cup\{[x]_R : p([x]_R, X) \geq 1 - \beta\}. \tag{17}$$

The model is constructed using a Cartesian product, which rigorously organizes the double-quantitative information contained in the precision and grade; therefore, $(U, \overline{R}_\beta, \underline{R}_k)$ and $(U, \overline{R}_k, \underline{R}_\beta)$ are basic double-quantitative models. Concretely, the two new models are related to the crossed combinations of the approximations in the VPRS and GRS models; the threshold ranges on $\beta$ and $k$ in the two models are equated with the corresponding ranges in the VPRS and GRS models. The new models therefore inherit many of the merits of the VPRS and GRS models and apply primarily to the basic double-quantitative environment. Furthermore, the combined model has concrete double-quantitative semantics and can meet specific practical requirements in complex situations.

**Definition 2.1.2.** Precision-semantics refers to the precise description provided by the precision parameter, while grade-semantics refers to the grade description provided by the grade parameter.

Here, the approximations inherit the original precision-semantics or grade-semantics. In $(U, \overline{R}_\beta, \underline{R}_k)$, $\overline{R}_\beta X$ is the union of the equivalence classes *whose precision with respect to X exceeds* $\beta$, and $\underline{R}_k X$ is the union of the equivalence classes *whose external grade with respect to X does not exceed k*. In $(U, \overline{R}_k, \underline{R}_\beta)$, $\overline{R}_k X$ is the union of the equivalence classes *whose internal grade with respect to X exceeds k*, and $\underline{R}_\beta X$ is the union of the equivalence classes *whose precision with respect to X is at least* $1 - \beta$. Therefore, the two models have double-quantitative semantics, which determines the double fault description and fault tolerance capabilities.

The positive region, negative region and boundary region act as fundamental notions for knowledge discovery, including attribute reduction and rule extraction. These traditional regions are first extended to the new double-quantitative model environments.

**Definition 2.1.3.** (1) In $(U, \overline{R}_\beta, \underline{R}_k)$,

$$\begin{aligned} posR_{\overline{\beta},\underline{k}}X &= \overline{R}_\beta X \cap \underline{R}_k X, \\ negR_{\overline{\beta},\underline{k}}X &= \sim (\overline{R}_\beta X \cup \underline{R}_k X), \\ ubnR_{\overline{\beta},\underline{k}}X &= \overline{R}_\beta X - \underline{R}_k X, \\ lbnR_{\overline{\beta},\underline{k}}X &= \underline{R}_k X - \overline{R}_\beta X, \\ bnR_{\overline{\beta},\underline{k}}X &= ubnR_{\overline{\beta},\underline{k}}X \cup lbnR_{\overline{\beta},\underline{k}}X \end{aligned} \tag{18}$$

are referred to as the $R$-positive region, negative region, upper boundary region, lower boundary region and boundary region of $X$, respectively. There are now a total of seven regions, including the five regions listed and the upper and lower approximations, and each region is referred to as a model-region (MR granule). In other words, the term MR granule is only a general designation for the seven regions. The MR granules are similarly defined for $(U, \overline{R}_k, \underline{R}_\beta)$, and except the approximations, the symbols for the additional MR granules are as follows:

$$posR_{\overline{k},\underline{\beta}}X,\ negR_{\overline{k},\underline{\beta}}X,\ ubnR_{\overline{k},\underline{\beta}}X,\ lbnR_{\overline{k},\underline{\beta}}X,\ bnR_{\overline{k},\underline{\beta}}X.$$

**Theorem 2.1.4**

$$(1)\ \overline{R}_\beta X = posR_{\overline{\beta},\underline{k}}X \cup ubnR_{\overline{\beta},\underline{k}}X, \underline{R}_k X = posR_{\overline{\beta},\underline{k}}X \cup lbnR_{\overline{\beta},\underline{k}}X. \tag{19}$$

$$(2)\ \overline{R}_k X = posR_{\overline{k},\underline{\beta}}X \cup ubnR_{\overline{k},\underline{\beta}}X, \underline{R}_\beta X = posR_{\overline{k},\underline{\beta}}X \cup lbnR_{\overline{k},\underline{\beta}}X. \tag{20}$$

In the new models, because the relationships between the approximations are no longer single or simple, the additional positive and negative regions, upper and lower boundary regions, and boundary region are proposed as generalizations of

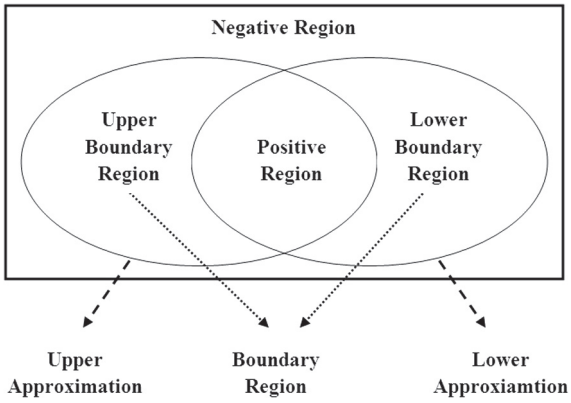


**Fig. 1**. BMR and MR granules.

the traditional regions in the Pawlak model. In the Pawlak model, $posRX = \underline{R}X = \overline{R}X \cap \underline{R}X$, and the positive region therefore becomes the intersection set of the approximations. As for the negative and boundary regions, we have $negRX = \sim \overline{R}X = \sim (\overline{R}X \cup \underline{R}X)$ and $bnRX = \overline{R}X - \underline{R}X = (\overline{R}X - \underline{R}X) \cup (\underline{R}X - \overline{R}X)$, and similar results are obtained. Similar to the qualitative semantics of the traditional regions, each MR granule has its own quantitative semantics, which is provided below.

In knowledge discovery, the complete and classified regions act as fundamental notions. A macroscopic granulation is therefore performed based on the MR granules.

**Definition 2.1.5.** The basic model regions (BMR granules) refer to the following four regions: the positive, negative, upper boundary and lower boundary regions.

**Theorem 2.1.6**

(1) *The BMR granules are complete and disjoint, i.e., they form a classification of the universe.*
(2) *The BMR granules are a decomposition of the MR granules, while the MR granules are a construction of the BMR granules.*

The four BMR granules form a complete partition of the universe. In the Pawlak model, there are only three traditional application regions (the positive, negative and boundary regions), which have been extended to the BMR granules in the new models. The double-quantitative system has the distinctive feature of four macroscopic regions; this fact is referred to as *the four-region principle*. In the double-quantitative environment, the upper and lower boundary regions can more accurately describe the boundary region; therefore, the completeness of the double-quantification leads to a more refined description of the uncertainty, and the uncertainty is thereby decreased to some extent. The four-region system therefore has an advantage over the classical three-region system. Definition 2.1.3 and Theorem 2.1.4 provide the basic relationships and structures of the MR granules and BMR granules, completely specifying the new system. Fig. 1 clearly illustrates these results. The approximations are the core notions that generate all of the MR granules; however, the MR granules can also be constructed from the BMR granules. These basic properties underlie the later algorithms for computing the MR granules. In the two models, the BMR and MR granules are the basic notions leading to knowledge discovery; these granules are therefore investigated thoroughly in this paper, particularly in relation to their semantic extraction and concrete computation.

Similar to the approximations of the new models, all BMR granules have concrete quantitative semantics, which are first analyzed in $(U, \overline{R}_\beta, \underline{R}_k)$. The concrete results are as follows:

(i) the positive region, $posR_{\overline{\beta},\underline{k}}X$, is the union of the equivalence classes *whose precision with respect to the set X exceeds* $\beta$ *and whose external grade with respect to X does not exceed k*;
(ii) the negative region, $negR_{\overline{\beta},\underline{k}}X$, is the union of the equivalence classes *whose precision with respect to the set X does not exceed* $\beta$ *and whose external grade with respect to X exceeds k*;
(iii) the upper boundary region, $ubnR_{\overline{\beta},\underline{k}}X$, is the union of the equivalence classes *whose precision with respect to the set X exceeds* $\beta$ *and whose external grade with respect to X exceeds k*;
(vi) the lower boundary region, $lbnR_{\overline{\beta},\underline{k}}X$, is the union of the equivalence classes *whose external grade with respect to X does not exceed k and whose precision with respect to the set X does not exceed* $\beta$.

The BMR-granular semantics becomes double-quantitative and reflects the special logics of the precision-semantics and grade-semantics. In particular, the BMR-granular semantics is related to the double fault description and fault tolerance features. Furthermore, the semantics of the boundary region can be logically constructed using the semantics of both the upper and lower boundary regions; the semantics of all of the BMR and MR granules have therefore been provided. The

granular semantics arises from both the semantics and logical relationships of the approximations. Next, we present the granular semantics in a simple and clear form using a new semantics-based technology.

**Definition 2.1.7.** In $(U, \overline{R}_\beta, \underline{R}_k)$, there are only two complete precision intervals and external grade intervals, i.e.,
$$0 \le p \le \beta, \beta < p \le 1, 0 \le \underline{g} \le k, \underline{g} > k, \text{where } p = p([x]_R, X), \underline{g} = \underline{g}([x]_R, X).$$
Therefore, based on the complete classification, only four types of double semantics occur for the precision and grade; these four classes are referred to as the basic semantics (BS), and the results are as follows:

(1) $p > \beta, \underline{g} \le k$;
(2) $p \le \beta, \underline{g} > k$;
(3) $p > \beta, \underline{g} > k$;
(4) $p \le \beta, \underline{g} \le k$.

**Theorem 2.1.8** (BS features)

(1) *The BS is complete and disjoint, i.e., it constructs a classification of the quantitative semantics.*
(2) *The BMR granule serves as the bearing granule of the BS, and BMR-granular semantics is the corresponding BS.*
(3) *The MR-granular semantics can be constructed from the BS; the BS in fact forms a basis for the double-quantitative semantics or quantitative semantics in the model* $(U, \overline{R}_\beta, \underline{R}_k)$.

Theorem 2.1.8 demonstrates the importance of the BS and the relationships between the BS and the BMR and MR granules. Therefore, all of the BMR and MR granules have semantics that are constructible from the BS. In fact, the semantics of the positive region, negative region, upper boundary region and lower boundary region are given by BS (1),(2),(3) and (4), respectively, reflecting the corresponding relationships between the BMR granules and BS. Furthermore, the semantics of the upper and lower approximations and boundary region are constructed from BS (1) and (3), BS (1) and (4), BS (3) and (4), respectively. The double semantics used to construct an approximation is equivalent to the original single semantics of the approximation. In summary, the BS is equivalent to the BMR granular semantics and therefore has carried great knowledge discovery power; moreover, the BS forms the basic units of the model and provides a unified description of the quantitative semantics, which is especially important for the more complex double semantics.

We have now proposed two systems for the quantitative semantics; one originates from the original semantics of the approximations, and the other is based on the BS (BMR-granular semantics). Both systems show that all of the BMR and MR granules have concrete quantitative semantics; the BMR and MR granules therefore serve as the basic granules for model applications.

The semantics of the BMR and MR granules can be obtained similarly for $(U, \overline{R}_k, \underline{R}_\beta)$. (i) The four types of BS in this model are as follows:

(1) $\overline{g} > k, p \ge 1 - \beta$;
(2) $\overline{g} \le k, p < 1 - \beta$;
(3) $\overline{g} > k, p < 1 - \beta$;
(4) $\overline{g} \le k, p \ge 1 - \beta$.

Here, $\overline{g} = \overline{g}([x]_R, X)$. (ii) The BMR-granular semantics is equivalent to the BS. (iii) The double semantics of the additional MR granules (i.e., the upper and lower approximations and boundary region) are constructed from BS (1) and (3), BS (1) and (4), BS (3) and (4), respectively.

### *2.2. Model expansion and completeness*

The model generalization is an important issue [55], and the expansions represented by the new models are investigated in this section. Both the VPRS and GRS models are expansions of the Pawlak model. The nature of the expansions is first analyzed, and this analysis leads to a formulation of the basic model expansions.

In the VPRS model, $(U, \overline{R}_\beta, \underline{R}_\beta)$,

$$\beta \ge 0 \Leftrightarrow \overline{R}_\beta X \subseteq \overline{R}_0 X, \underline{R}_\beta X \supseteq \underline{R}_0 X, \tag{21}$$

$$\overline{R}_\beta X - \underline{R}_\beta X \subseteq \overline{R}X - \underline{R}X, \text{ i.e., } bnR_\beta X \subseteq bnRX. \tag{22}$$

The VPRS model degenerates to the Pawlak model when $\beta = 0$. Therefore, the VPRS model extends the Pawlak model by lessening the upper approximation and enlarging the lower approximation, and the boundary region in the VPRS model becomes a subset of the boundary region in the Pawlak model. An expansion with a subset boundary is preferable because the size of the boundary region determines the roughness degree to a great extent. Similarly, when the Pawlak model is extended to the GRS model, the upper approximation becomes smaller, while the lower approximation becomes larger. The

relevant formula is as follows:

$$k \geq 0 \Leftrightarrow \overline{R}_k X \subseteq \overline{R}_0 X, \underline{R}_k X \supseteq \underline{R}_0 X. \tag{23}$$

**Definition 2.2.1.** A directional expansion refers to a model expansion with a reduced upper approximation and magnified lower approximation.

**Proposition 2.2.2.** *The VPRS and GRS models are both directional expansions of the Pawlak model.*

The directional expansion defines a particular expansion direction for the approximations. For the VPRS model, the directional expansion guides the relative reduction of the boundary region. The boundary region is related to the uncertainty, so the directional expansion has practical significance. As for the GRS model, the directional expansion also provides a clear framework for the change in the region. The directional expansion therefore functions as an important model expansion. The two new models will be evaluated using this criterion.

**Theorem 2.2.3** (Model expansion)

(1) *In* $(U, \overline{R}_\beta, \underline{R}_k)$, *if* $\beta = 0$ *and* $k = 0$, *then* $(U, \overline{R}_\beta, \underline{R}_k) = (U, \overline{R}, \underline{R})$;
*moreover,* $\overline{R}_\beta X \subseteq \overline{R}X$, $\underline{R}_k X \supseteq \underline{R}X$.
*In other words,* $(U, \overline{R}_\beta, \underline{R}_k)$ *is a directional expansion of the Pawlak model.*
(2) *In* $(U, \overline{R}_k, \underline{R}_\beta)$, *if* $\beta = 0$ *and* $k = 0$, *then* $(U, \overline{R}_k, \underline{R}_\beta) = (U, \overline{R}, \underline{R})$;
*moreover,* $\overline{R}_k X \subseteq \overline{R}X$, $\underline{R}_\beta X \supseteq \underline{R}X$.
*In other words,* $(U, \overline{R}_k, \underline{R}_\beta)$ *is also a directional expansion of the Pawlak model.*

According to Theorem 2.2.3, the two models with their thresholds exhibit favorable directional expansion properties, which originate from the basic expansions of both the VPRS and GRS models. In contrast, the Pawlak model is merely a special case of the two models, with $k = 0$ and $\beta = 0$. The generalizations represented by the new models therefore have favorable theoretical properties.

**Proposition 2.2.4.** *For the approximate space* $(U, R)$ *and concept* $X$, *the basic quantitative system* $(|[x]_R|, |[x]_R \cap X|)$ *is two-dimensional.*

**Theorem 2.2.5** (Model completeness). *If* $p([x]_R, X) = 1$ *or* $\underline{g}([x]_R, X) = 0$, *then* $[x]_R \subseteq X$, *and this case is well-defined for both new models. Otherwise,* $p([x]_R, X)$ *and* $\underline{g}([x]_R, X)$ *can induce* $|[x]_R|, |[x]_R \cap X|$. *In other words, the applied quantitative system* $(p([x]_R, X), \underline{g}([x]_R, X))$ *is usually two-dimensional. Furthermore, the relevant model* $(U, \overline{R}_\beta, \underline{R}_k)$ *is complete with respect to the basic quantification. Similarly, the other model,* $(U, \overline{R}_k, \underline{R}_\beta)$, *is also complete with respect to the quantitative approximate space.*

The relevant and extensive contents were analyzed in the Introduction (Section 1.2). Proposition 2.2.4 demonstrates that the quantitative approximate space is two-dimensional. Therefore, a one-dimensional model with quantitative incompleteness may lead to some information loss in the quantitative description. Within this framework, the model completeness theorem (Theorem 2.2.5) demonstrates that the two proposed models exhibit quantitative completeness and are therefore of great value. Moreover, based on this novel quantitative criterion, the GRS model is also complete, while most PRS models are not complete. Thus, using pairings related to the Cartesian product, the two models have improved the previous models not only in terms of their fault tolerance capabilities but also in terms of their quantitative completeness. Moreover, the temporal and spatial complexity of the new models is nearly *twice* that of the one-dimensional models with $c([x]_R, X)$ or $p([x]_R, X)$. The new models therefore share the computational feasibility of most PRS models.

In summary $(U, \overline{R}_\beta, \underline{R}_k)$ and $(U, \overline{R}_k, \underline{R}_\beta)$ perform basic double quantification of the precision and grade. They have concrete quantitative semantics and therefore thoroughly describe the approximate space. They also exhibit strong double fault tolerance capabilities (in terms of both relative and absolute fault tolerance) and can therefore adapt to complex environments. Moreover, they correspond to positive expansions of the Pawlak model and therefore exhibit a more encompassing theoretical structure, and they have the further advantage of completeness. The new models are therefore promising for both theoretical studies and practical applications involving double quantification. In addition, the two models are similar, parallel and symmetric; however, their upper and lower approximations have directivity, and their different Cartesian product distributions for the relative and absolute quantification will therefore lead to concrete results, such as their specific quantitative semantics. In the following sections, we focus on the former model, $(U, \overline{R}_\beta, \underline{R}_k)$. Section 6 provides the corresponding results for the latter model, $(U, \overline{R}_k, \underline{R}_\beta)$.

## 3. Computing granules of the basic model regions (BMRC granules)

### 3.1. Notion and properties

In Section 2.1, the MR and BMR granules were proposed, and their quantitative semantics were obtained. Next, we cover the computation of the granules using a new type of computing granules.

**Proposition 3.1.1**

$$\begin{aligned} &\overline{R}_\beta X = \cup\{[x]_R : |[x]_R \cap X| > \beta|[x]_R|\}, \\ &\underline{R}_k X = \cup\{[x]_R : |[x]_R \cap X| \geq |[x]_R| - k\}. \end{aligned} \tag{24}$$

**Proposition 3.1.2**

(1) *If* $\beta = 0$, *then* $\overline{R}_\beta X = \overline{R}X$, $\underline{R}_k X = \cup\{[x]_R : |[x]_R \cap X| \geq |[x]_R| - k\}$, $posR_{\overline{\beta},\underline{k}}X = \overline{R}X \cap \underline{R}_k X$, $negR_{\overline{\beta},\underline{k}}X = \sim(\overline{R}X \cup \underline{R}_k X)$, $ubnR_{\overline{\beta},\underline{k}}X = \overline{R}X - \underline{R}_k X$, $lbnR_{\overline{\beta},\underline{k}}X = \underline{R}_k X - \overline{R}X$, $bnR_{\overline{\beta},\underline{k}}X = (\overline{R}X - \underline{R}_k X) \cup (\underline{R}_k X - \overline{R}X)$.
*In particular, if* $\beta = 0$ *and* $k = 0$, *then* $\overline{R}_\beta X = \overline{R}X$, $\underline{R}_k X = \underline{R}X$, $posR_{\overline{\beta},\underline{k}}X = posRX$, $negR_{\overline{\beta},\underline{k}}X = negRX$, $ubnR_{\overline{\beta},\underline{k}}X = bnR_{\overline{\beta},\underline{k}}X = bnRX$, $lbnR_{\overline{\beta},\underline{k}}X = \phi$.

(2) *If* $k = 0$ *and* $\beta \in (0, 0.5)$, *then* $\overline{R}_\beta X = \cup\{[x]_R : |[x]_R \cap X| > \beta|[x]_R|\}$, $\underline{R}_k X = \underline{R}X$, $posR_{\overline{\beta},\underline{k}}X = \underline{R}X$, $negR_{\overline{\beta},\underline{k}}X = \sim \overline{R}_\beta X$, $ubnR_{\overline{\beta},\underline{k}}X = bnR_{\overline{\beta},\underline{k}}X = \overline{R}_\beta X - \underline{R}X$, $lbnR_{\overline{\beta},\underline{k}}X = \phi$.

Proposition 3.1.1 provides the equivalent formulas for the approximations in $(U, \overline{R}_\beta, \underline{R}_k)$, and Proposition 3.1.2 provides the results for MR granules in the special cases where $\beta = 0$ or $k = 0$, which are straightforward. We therefore explicitly consider only the usual case: $\beta \in (0, 0.5)$ and $k \neq 0$. In Proposition 3.1.1, only two parameters are considered: $\beta|[x]_R|$, $|[x]_R| - k$. Full descriptions of the four BMR granules are therefore provided in a complete discussion of the parameter relationships. Theorem 3.1.3 presents the related results.

**Theorem 3.1.3.** *If* $0 < \beta < 0.5$ *and* $k \neq 0$, *then*

(1) $posR_{\overline{\beta},\underline{k}}X = (\cup\{[x]_R : |[x]_R| > k/(1-\beta), |[x]_R \cap X| \geq |[x]_R| - k\})$
$\cup(\cup\{[x]_R : |[x]_R| \leq k/(1-\beta), |[x]_R \cap X| > \beta|[x]_R|\})$;
(2) $negR_{\overline{\beta},\underline{k}}X = (\cup\{[x]_R : |[x]_R| > k/(1-\beta), |[x]_R \cap X| \leq \beta|[x]_R|\})$
$\cup(\cup\{[x]_R : |[x]_R| \leq k/(1-\beta), |[x]_R \cap X| < |[x]_R| - k\})$;
(3) $ubnR_{\overline{\beta},\underline{k}}X = \cup\{[x]_R : |[x]_R| > k/(1-\beta), \beta|[x]_R| < |[x]_R \cap X| < |[x]_R| - k\}$;
(4) $lbnR_{\overline{\beta},\underline{k}}X = \cup\{[x]_R : |[x]_R| \leq k/(1-\beta), |[x]_R| - k \leq |[x]_R \cap X| \leq \beta|[x]_R|\}$.

**Note.** According to Definition 2.1.3, the BMR granules in $(U, \overline{R}_\beta, \underline{R}_k)$ originate primarily from four set operations on the approximations: $\overline{R}_\beta X$, $\underline{R}_k X$, and for example, $posR_{\overline{\beta},\underline{k}}X = \overline{R}_\beta X \cap \underline{R}_k X$. Furthermore, Proposition 3.1.1 provides the equivalent formulas for the approximations. Therefore, we need only combine the two inequalities:

$$|[x]_R \cap X| > \beta|[x]_R|, |[x]_R \cap X| \geq |[x]_R| - k. \tag{25}$$

In other words, the relationship between $\beta|[x]_R|$ and $|[x]_R| - k$ must be discussed. For a knowledge granule $[x]_R$, there are only two cases:

$$\begin{aligned} &\beta|[x]_R| < |[x]_R| - k, \text{ or } \beta|[x]_R| \geq |[x]_R| - k; \\ &\text{i.e., } |[x]_R| > k/(1-\beta), \text{ or } |[x]_R| \leq k/(1-\beta). \end{aligned} \tag{26}$$

For each case, the results are easily obtained using Theorem 3.1.3. In particular, the common factor $k/(1-\beta)$ emerges. In fact, $|[x]_R|$, $|[x]_R \cap X|$ act as two basic core variables for quantification in the approximate space; it is therefore important to represent granular forms in terms of the two core variables. Clearly, the results of Theorem 3.1.3 on the BMR granules have the necessary form for representation on $|[x]_R|$, $|[x]_R \cap X|$. Therefore, $k/(1-\beta)$ is necessary within the framework of $|[x]_R|$, $|[x]_R \cap X|$, and this factor originates from the basic qualitative classification on the BMR granules. In other words, the BMR granules cannot be completely identified without $k/(1-\beta)$, and the factor of $k/(1-\beta)$ also appears in other quantitative classification systems for the model $(U, \overline{R}_\beta, \underline{R}_k)$.

**Table 1**
BMRC granules and their properties in $(U, \overline{R}_\beta, \underline{R}_k)$ when $\beta \in (0, 0.5)$ and $k \neq 0$.

| BMRC-granule | No. 1: $\lvert[x]_R\rvert$ | No. 2: $\lvert[x]_R \cap X\rvert$ | BMR-granule | BS | Basic operations | Auxiliary variables |
|---|---|---|---|---|---|---|
| (1) | $> k/(1-\beta)$ | $\leq \beta\lvert[x]_R\rvert$ | $negR_{\overline{\beta},\underline{k}}X$ | (2) | 3 | 1 |
| (2) | $> k/(1-\beta)$ | $(\beta\lvert[x]_R\rvert, \lvert[x]_R\rvert - k)$ | $ubnR_{\overline{\beta},\underline{k}}X$ | (3) | 5 | 2 |
| (3) | $> k/(1-\beta)$ | $\geq \lvert[x]_R\rvert - k$ | $posR_{\overline{\beta},\underline{k}}X$ | (1) | 5 | 2 |
| (4) | $\leq k/(1-\beta)$ | $< \lvert[x]_R\rvert - k$ | $negR_{\overline{\beta},\underline{k}}X$ | (2) | 3 | 1 |
| (5) | $\leq k/(1-\beta)$ | $[\lvert[x]_R\rvert - k, \beta\lvert[x]_R\rvert]$ | $lbnR_{\overline{\beta},\underline{k}}X$ | (4) | 5 | 2 |
| (6) | $\leq k/(1-\beta)$ | $> \beta\lvert[x]_R\rvert$ | $posR_{\overline{\beta},\underline{k}}X$ | (1) | 5 | 2 |

**Definition 3.1.4.** The computing granules of the basic model regions (i.e., the BMRC granules) refer to the six types of subsets that are used to construct the BMR granules in Theorem 3.1.3.

**Theorem 3.1.5** (BMRC granule features)

(1) *The BMRC granules are complete and disjoint, i.e., they construct a classification of the universe.*
(2) *The BMRC granules are a decomposition of the BMR and MR granules, while the BMR and MR granules are a construction of the BMRC granules; in fact, the BMRC granules form a basis for the MR and BMR granules.*
(3) *The semantics of a BMRC granule is equivalent to the BS extracted by the BMR granule from which it was constructed.*
(4) *The semantics of the MR and BMR granules is a construction of the semantics of the composed BMRC granules.*

Table 1 shows the BMRC granules and their relationships with the BMR granules and BS. For example, BMRC granule (3) is given by

$$\cup\{[x]_R : \lvert[x]_R\rvert > k/(1-\beta), \lvert[x]_R \cap X\rvert \geq \lvert[x]_R\rvert - k\},$$

which is the union of the equivalence classes *whose cardinal numbers exceed* $k/(1-\beta)$ *and whose absolute overlap numbers with respect to X are at least* $\lvert[x]_R\rvert - k$. BMRC granule (3) belongs to a BMR granule (the positive region) and has the semantics of BS (1). In fact, the positive region is a construction of two BMRC granules, BMRC granules (3) and (6). The information on the basic operations and auxiliary variables (provided in Table 1) is useful primarily for the algorithm analyses in Section 4.

### *3.2. The two-dimensional plane and granular hierarchical structure*

Both the form and structure of the BMRC granules are analyzed in this section; a new mathematical technology is introduced for this purpose. As was shown previously, the cardinal number ($\lvert[x]_R\rvert$) and absolute overlap number ($\lvert[x]_R \cap X\rvert$) of an equivalence class are explicitly determined and therefore act as the core indexes; moreover, the three usual indexes – the precision, internal grade and external grade – can be completely combined using these indexes. Therefore, $\lvert[x]_R\rvert$ and $\lvert[x]_R \cap X\rvert$ determine the basic form of the quantitative approximate space through an internal mechanism and provide an ideal computing platform. Various explanations regarding the basic quantitative system ($\lvert[x]_R\rvert$, $\lvert[x]_R \cap X\rvert$) were provided previously in Sections 1.2 and 2.2.

We first introduce several new symbols.

Let $f : U/R \to \mathbf{N}^2, f([x]_R, X) = (V, Y)$;
here, $V = V([x]_R, X) = \lvert[x]_R\rvert, Y = \overline{g}([x]_R, X) = \lvert[x]_R \cap X\rvert$,
and let $X$ serve as a mapping parameter.

Then, the equivalence classes are mapped into a two-dimensional plane by the mapping $f$. This plane is referred to as *the two-dimensional plane of the cardinal number and internal grade* (or simply *the two-dimensional plane*). The equivalence classes are therefore discretely distributed in a region $0 \leq Y \leq V$ in the two-dimensional plane as atomic granules, and a microscopic two-dimensional mathematical form of the quantitative approximate space is produced, which is related to the Cartesian product of $\lvert[x]_R\rvert$ and $\lvert[x]_R \cap X\rvert$. Moreover, the precision, internal grade and external grade (the approximations in the VPRS and GRS models) have their own mathematical/geometric meanings in the two-dimensional plane. The equation $p([x]_R, X) = Y/V$ means that *the precision of an equivalence class is the slope of the line that passes through the origin and the basic granular point*; $\overline{g}([x]_R, X) = Y$ means that *the internal grade of an equivalence class is the distance from the basic granular point to the V-axis*; and $\underline{g}([x]_R, X) = V - Y$ means that *the external grade of an equivalence class is the difference between the two distances from the basic granular point to the Y-axis and V-axis*. Furthermore, the approximations in the VPRS model become the regions where $Y > \beta V$ and $Y \geq (1-\beta)V$, and the approximations in the GRS model become the regions where $Y > k$ and $Y \geq V - k$.

In the two-dimensional plane, the BMR granules are the complete and classified regions with respect to two lines: $Y = \beta V$ and $Y = V - k$. The relationship for the BMR granules, i.e., that between $\lvert[x]_R\rvert$ and $k/(1-\beta)$, is related to the vertical line $V = k/(1-\beta)$, which passes through the intersection point of $Y = \beta V$ and $Y = V - k$. Therefore, the BMRC granules form the complete and classified regions with respect to the three lines. This qualitative result regarding the BMRC granules is stable and does not depend on the quantitative parameter value. Fig. 2 shows the distribution of the six types of BMR granules (where $\beta = 0.4$ and $k = 10$).

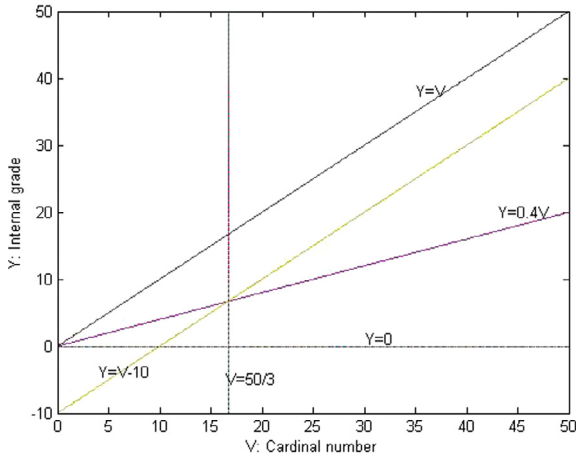


**Fig. 2**. BMR granules in $(U, \overline{R}_\beta, \underline{R}_k)$ in the two-dimensional plane.

The BMRC granules originate from the mathematical decomposition of the BMR granules and are described by both $|[x]_R|$ and $|[x]_R \cap X|$. In other words, both the BMR granules and the initial data determine the granulation of the BMRC granules. The BMRC granules therefore comprise a fundamental granule class in BMR-granular computing (particularly in BMR-granular optimal computing) because of their explicit function based on $|[x]_R|$ and $|[x]_R \cap X|$.

Finally, we provide the granular hierarchical structure in $(U, \overline{R}_\beta, \underline{R}_k)$. The up-bottom strategy (one of the main strategies in granular computing) is adopted throughout our granular computing study of the model. (1) The MR and BMR granules are both based on model applications and therefore lie in a macroscopic layer. (2) The BMRC granules are based on the quantitative parameters ($\beta$ and $k$) and initial data ($|[x]_R|$ and $|[x]_R \cap X|$) and therefore lie in a microscopic layer. (3) The two-dimensional mathematical form of the approximate space in the two-dimensional plane is actually related to the granular structure of the atomic granules (i.e., equivalence classes), which is determined only by the quantitative approximate space, and therefore lies in a basic/atomic layer. The relationships between the MR, BMR and BMRC granules are linear and clear. A figure showing the granules with labeled lines (from the bottom layer to the top layer) is provided as follows:

$$x \to [x]_R \to \text{BMRC-Granule} \to \text{BMR-Granule} \to \text{MR-Granule} \to U.$$

Furthermore, the granular hierarchical structure has provided methods for describing and solving the corresponding granular problems, including the semantic extraction and concrete computation. Moreover, the finest type of granules in $(U, \overline{R}_\beta, \underline{R}_k)$ can be produced by the exhaustive decomposition of the BMRC and BMR granules with respect to $|[x]_R|$ and $|[x]_R \cap X|$. In the two-dimensional plane, the new granules are actually further divided into regions by the horizontal line $Y = \beta k/(1-\beta)$ based on the BMRC granules; these new granules are a decomposition of the BMRC, BMR and MR granules.

## 4. Algorithms and analysis of the MR-granular computing

The MR granules can be constructed from the BMR granules and can therefore be computed using the BMRC granules; however, they can also be obtained using the approximations. Therefore, two basic algorithms are proposed to compute the MR granules, the BMRC-granules algorithm and the approximation algorithm. These two algorithms will be analyzed and compared for optimal computation.

**Algorithm 1 .** The BMRC-granules algorithm

**Input:**

The approximate space $(U, R)$, concept $X$, thresholds $\beta$, $k$;

**Output:**

MR-Granules;

1: Compute BMRC-Granules;
2: Construct BMR-Granules;
3: Obtain other MR-Granules: the approximations and the boundary region.
4: **return** all MR-Granules.

The macroscopic approximation algorithm is conventional and natural, while the microscopic BMRC-granules algorithm originates from the basic structure of the MR granules and the results for both the BMR and BMRC granules. According to Proposition 3.1.2, the MR granules are uniquely determined in the special cases where $\beta = 0$ or $k = 0$. In this section, the two algorithms will be analyzed and compared for optimal computation in the typical case where $\beta \in (0, 0.5)$ and $k \neq 0$.

**Algorithm 2 .** The approximation algorithm

**Input:**

The approximate space $(U, R)$, concept $X$, thresholds $\beta$, $k$;

**Output:**

MR-Granules;

1: Compute the approximations: $\overline{R}_\beta X$ and $\underline{R}_k X$;
2: Obtain other MR granules: BMR granules and the boundary region.
3: **return** all MR-Granules.

The calculation procedure is clear for the approximation algorithm. The computing procedure for the BMRC-granules algorithm will be described in further detail, with a particular focus on the comparison of the orders of the parameters. For the BMRC granules, there are only two steps. Firstly, $|[x]_R|$ is compared with $k/(1-\beta)$. Secondly, (i) if $|[x]_R| > k/(1-\beta)$, then $|[x]_R \cap X|$ is compared with $\beta|[x]_R|$ first and $|[x]_R| - k$ second; (ii) if $|[x]_R| \leq k/(1-\beta)$, then $|[x]_R \cap X|$ is compared with $|[x]_R| - k$ first and $\beta|[x]_R|$ second. The code implementations of the two algorithms are therefore straightforward.

The main task of the two algorithms is to judge whether an equivalence class belongs to a specific set. For each equivalence class, two input data, $|[x]_R|$ and $|[x]_R \cap X|$, are required. Suppose that there are $n$ equivalence classes and $2n$ input data. All of the operations related to $|[x]_R|$, $|[x]_R \cap X|$ and $n$, such as division, reduction, multiplication and comparison are considered to be basic operations.

In the approximation algorithm, $p([x]_R, X)$ and $\underline{g}([x]_R, X)$ are first computed, and then the relationships between $[x]_R$ and $\overline{R}_\beta X$, $\underline{R}_k X$ are determined. For an equivalence class, four basic operations and two auxiliary variables are needed; the additional processes are related only to certain set operations. The temporal and spatial complexity are therefore invariant: $T(n) = 4n, S(n) = 2n$. Moreover, the temporal and spatial complexity of the similar approximation algorithm are $T(n) = 3n$, $S(n) = n$ in the VPRS model. In the BMRC-granules algorithm, there are at most three comparisons and two auxiliary variables ($\beta|[x]_R|$ and $|[x]_R| - k$) in the two computing steps. For BMRC granules, Table 1 (in Section 3.1) also provides the corresponding numbers of basic operations and auxiliary variables. For example, for an equivalence class in BMRC granules (1) or (4), there are only three basic operations and one auxiliary variable. The temporal and spatial complexity become $T(n) = 5n, S(n) = 2n$ and $T(n) = 3n, S(n) = n$ in the worst and best cases, respectively.

The asymptotic analyses of the temporal and spatial complexity of the two algorithms yield the same results: $T(n) = \Theta(n)$, $S(n) = \Theta(n)$; both algorithms are therefore feasible and effective. The BMRC-granules algorithm has an important advantage in terms of the spatial complexity; the spatial complexity of the approximation algorithm actually provides an upper bound for that of the BMRC-granules algorithm. The complexity of the approximation algorithm is invariant, whereas the complexity of the BMRC-granules algorithm is lower in two cases in which the equivalence class belongs to BMRC granules (1) or (4). BMRC granules (1) and (4) belong to the negative region, and the negative region is usually the largest BMR granule; therefore, the BMRC-granules algorithm also has an advantage in terms of the time complexity under most conditions. In fact, according to Theorem 3.1.3, the BMR granules are thoroughly described by the BMRC granules; however, the BMRC granules are determined by the initial data, where $|[x]_R|$ is first described by $k/(1-\beta)$ and then $|[x]_R \cap X|$ is described by $|[x]_R| - k$ and $\beta|[x]_R|$. Therefore, in the BMRC-granules algorithm, the BMR-granular description and computation have been improved by the rational ordering of the BMRC granules.

The MR and BMR granules act as fundamental notions related to the model applications, and the two proposed algorithms can compute all of the MR granules and, in particular, all of the BMR granules. Therefore, a firm foundation for model applications has been constructed based on our algorithms and MR-granular computing analysis. Furthermore, the BMRC-granules algorithm can be used to compute all of the BMRC granules and certain BMR granules explicitly; for example, the positive region can be obtained using a modified BMRC-granules algorithm. Moreover, the MR and BMR granules are typically more accurate in practice because of the concrete distributions of the BMRC granules.

## 5. Properties of the approximation operators and the notions of attribute approximate dependence and reduction

The approximation operators serve as core notions for the rough set models, and their properties, including the idempotence of their actions, are provided for $(U, \overline{R}_\beta, \underline{R}_k)$.

**Proposition 5.1**

(1) $\overline{R}_\beta \phi = \phi, \overline{R}_\beta U = U, \underline{R}_k \phi = \cup\{[x]_R : |[x]_R| \leq k\}, \underline{R}_k U = U.$
(2) $X \subseteq Y \Rightarrow \overline{R}_\beta X \subseteq \overline{R}_\beta Y, X \subseteq Y \Rightarrow \underline{R}_k X \subseteq \underline{R}_k Y.$
(3) $\overline{R}_\beta (X \cup Y) \supseteq \overline{R}_\beta X \cup \overline{R}_\beta Y, \underline{R}_k (X \cup Y) \supseteq \underline{R}_k X \cup \underline{R}_k Y.$
(4) $\overline{R}_\beta (X \cap Y) \subseteq \overline{R}_\beta X \cap \overline{R}_\beta Y, \underline{R}_k (X \cap Y) \subseteq \underline{R}_k X \cap \underline{R}_k Y.$
(5) $\overline{R}_\beta (\sim X) = \sim \underline{R}_\beta X, \underline{R}_k (\sim X) = \sim \overline{R}_k X.$
(6) $\beta \geq \alpha \Rightarrow \overline{R}_\beta X \subseteq \overline{R}_\alpha X, k \geq l \Rightarrow \underline{R}_k X \supseteq \underline{R}_l X.$
(7) $\overline{R}_0 X = \overline{R} X, \underline{R}_0 X = \underline{R} X.$

**Proposition 5.2**

(1) $\overline{R}_\beta(\overline{R}_\beta X) = \overline{R}_\beta X \subseteq \underline{R}_k(\overline{R}_\beta X)$.
(2) $\underline{R}_k(\overline{R}_\beta X) - \overline{R}_\beta X = (\cup\{[x]_R : |[x]_R| \le k\}) - \overline{R}_\beta X$.
(3) $\overline{R}_\beta(\underline{R}_k X) = \underline{R}_k X = \underline{R}_k(\underline{R}_k X)$.

**Corollary 5.3**

$$\overline{R}(\overline{R}X) = \underline{R}(\overline{R}X) = \overline{R}X, \overline{R}(\underline{R}X) = \underline{R}(\underline{R}X) = \underline{R}X. \tag{27}$$

The BMR granules comprise basic notions for knowledge discovery and have been obtained using either the BMRC-granules or approximation algorithms, allowing model applications to be explored. Thus, the attribute approximate dependence and reduction are determined by the positive region.

**Definition 5.4.** Let $S = (U, T)$ be a decision table, and let $T = C \cup D$. Let $IND(C)$ denote the equivalence relation on $C$. In $(U, \overline{R}_\beta, \underline{R}_k)$, suppose that

$$pos(C, D, \beta, k) = \bigcup_{Z \in U/D} posIND(C)_{\overline{\beta},\underline{k}} Z;$$

$$\gamma(C, D, \beta, k) = |pos(C, D, \beta, k)|/|U| \tag{28}$$

is referred to as the approximate dependence degree of decision attribute set $D$ and condition attribute set $C$.

This definition is similar to the corresponding notions in the Pawlak and VPRS models, where the positive regions are equated with the lower approximations. This attribute approximate dependence degree is actually a comprehensive evaluation index of the classification capability with respect to the double-quantitative indexes (the precision and grade). A new definition of attribute approximate reduction is also introduced.

**Definition 5.5.** The set $red(C, D, \beta, k)$, a subset of $C$, is referred to as an approximate reduct of condition attribute set $C$ with respect to decision attribute set $D$ if it satisfies two conditions:

(1) $\gamma(C, D, \beta, k) = \gamma(red(C, D, \beta, k), D, \beta, k)$;
(2) if any attribute is deleted from $red(C, D, \beta, k)$, then (1) no longer holds.

Attribute approximate reduction is a well-established notion in data processing, and the attribute approximate reduction in $(U, \overline{R}_\beta, \underline{R}_k)$ (Definition 5.5) therefore has practical value. In the double-quantitative environment, the attribute approximate reduction reflects the data relationships in the database and provides superior approximate reasoning and discovery of decision rules.

According to Theorem 2.2.3, both the attribute approximate dependence and reduction are a natural generalization of the related notions in the Pawlak model; therefore, we have also developed these two important notions in rough set theory.

## 6. Corresponding results in the latter model

**Proposition 6.1**

(1) *If* $\beta = 0$, *then* $\overline{R}_k X = \cup\{[x]_R : |[x]_R \cap X| > k\}$, $\underline{R}_\beta X = \underline{R}X$, $posR_{\overline{k},\underline{\beta}} X = \overline{R}_k X \cap \underline{R}X$, $negR_{\overline{k},\underline{\beta}} X = \sim (\overline{R}_k X \cup \underline{R}X)$, $ubnR_{\overline{k},\underline{\beta}} X = \overline{R}_k X - \underline{R}X$, $lbnR_{\overline{k},\underline{\beta}} X = \underline{R}X - \overline{R}_k X$, $bnR_{\overline{k},\underline{\beta}} X = (\overline{R}_k X - \underline{R}X) \cup (\underline{R}X - \overline{R}_k X)$.
*In particular, if* $\beta = 0$ *and* $k = 0$, *then* $\overline{R}_\beta X = \overline{R}X$, $\underline{R}_k X = \underline{R}X$, $posR_{\overline{k},\underline{\beta}} X = posRX$, $negR_{\overline{k},\underline{\beta}} X = negRX$, $ubnR_{\overline{k},\underline{\beta}} X = bnR_{\overline{k},\underline{\beta}} X = bnRX$, $lbnR_{\overline{k},\underline{\beta}} X = \phi$.
(2) *If* $k = 0$ *and* $\beta \in (0, 0.5)$, *then* $\overline{R}_k X = \overline{R}X$, $\underline{R}_\beta X = \cup\{[x]_R : |[x]_R \cap X| \ge (1 - \beta)|[x]_R|\}$, $posR_{\overline{k},\underline{\beta}} X = \underline{R}_\beta X$, $negR_{\overline{k},\underline{\beta}} X = \sim \overline{R}X$, $ubnR_{\overline{k},\underline{\beta}} X = bnR_{\overline{k},\underline{\beta}} X = \overline{R}X - \underline{R}_\beta X$, $lbnR_{\overline{k},\underline{\beta}} X = \phi$.

Proposition 6.1 shows the results for the MR granules in the special cases in which $\beta = 0$ or $k = 0$. Therefore, the results are provided only for the typical case, $\beta \in (0, 0.5)$ and $k \ne 0$. Theorem 6.2 presents the full BMR-granular description of the BMRC granules, and Table 2 provides the BMRC granules and several related results, such as the corresponding BS and computational analysis.

**Table 2**
The BMRC granules and their properties in $(U, \overline{R}_k, \underline{R}_\beta)$ when $\beta \in (0, 0.5)$ and $k \neq 0$.

| BMRC-granule | No. 1: $\lvert[x]_R\rvert$ | No. 2: $\lvert[x]_R \cap X\rvert$ | BMR-granule | BS | Basic operations | Auxiliary variables |
|---|---|---|---|---|---|---|
| (1) | $> k/(1-\beta)$ | $\leq k$ | $negR_{\overline{k},\underline{\beta}}X$ | (2) | 2 | 0 |
| (2) | $> k/(1-\beta)$ | $(k, (1-\beta)\lvert[x]_R\rvert)$ | $ubnR_{\overline{k},\underline{\beta}}X$ | (3) | 4 | 1 |
| (3) | $> k/(1-\beta)$ | $\geq (1-\beta)\lvert[x]_R\rvert$ | $posR_{\overline{k},\underline{\beta}}X$ | (1) | 4 | 1 |
| (4) | $\leq k/(1-\beta)$ | $< (1-\beta)\lvert[x]_R\rvert$ | $negR_{\overline{k},\underline{\beta}}X$ | (2) | 4 | 1 |
| (5) | $\leq k/(1-\beta)$ | $[(1-\beta)\lvert[x]_R\rvert, k]$ | $lbnR_{\overline{k},\underline{\beta}}X$ | (4) | 4 | 1 |
| (6) | $\leq k/(1-\beta)$ | $> k$ | $posR_{\overline{k},\underline{\beta}}X$ | (1) | 2 | 0 |

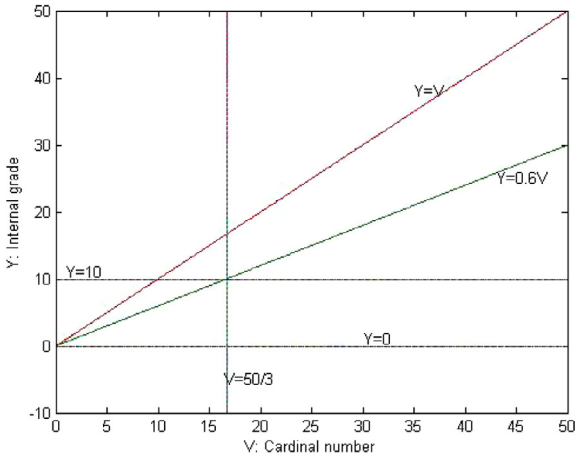


**Fig. 3**. BMR granules for $(U, \overline{R}_k, \underline{R}_\beta)$ in the two-dimensional plane.

**Theorem 6.2.** *If* $0 < \beta < 0.5$ *and* $k \neq 0$, *then*

(1) $posR_{\overline{k},\underline{\beta}}X = (\cup\{[x]_R : \lvert[x]_R\rvert > k/(1-\beta), \lvert[x]_R \cap X\rvert \geq (1-\beta)\lvert[x]_R\rvert\})$
$\cup(\cup\{[x]_R : \lvert[x]_R\rvert \leq k/(1-\beta), \lvert[x]_R \cap X\rvert > k\})$;
(2) $negR_{\overline{k},\underline{\beta}}X = (\cup\{[x]_R : \lvert[x]_R\rvert > k/(1-\beta), \lvert[x]_R \cap X\rvert \leq k\})$
$\cup(\cup\{[x]_R : \lvert[x]_R\rvert \leq k/(1-\beta), \lvert[x]_R \cap X\rvert < (1-\beta)\lvert[x]_R\rvert\})$;
(3) $ubnR_{\overline{k},\underline{\beta}}X = \cup\{[x]_R : \lvert[x]_R\rvert > k/(1-\beta), k < \lvert[x]_R \cap X\rvert < (1-\beta)\lvert[x]_R\rvert\}$;
(4) $lbnR_{\overline{k},\underline{\beta}}X = \cup\{[x]_R : \lvert[x]_R\rvert \leq k/(1-\beta), (1-\beta)\lvert[x]_R\rvert \leq \lvert[x]_R \cap X\rvert \leq k\}$.

In the two-dimensional plane, the BMRC granules are actually the complete and classified regions with respect to three lines: $Y = k$, $Y = (1-\beta)V$ and $V = k/(1-\beta)$. The qualitative result for the BMRC granules also becomes stable, and Fig. 3 shows the distribution of the BMR granules (when $k = 10$ and $\beta = 0.4$). Moreover, the granular hierarchical structure is as same as that in the former model.

The approximation and BMRC-granules algorithms are analyzed and compared in the typical case in which $\beta \in (0, 0.5)$ and $k \neq 0$. As for the second computing step in the BMRC-granules algorithm, $\lvert[x]_R \cap X\rvert$ is first compared with $k$ and then with $(1-\beta)\lvert[x]_R\rvert$. The temporal and spatial complexity of the approximation algorithm are $T(n) = 3n$ and $S(n) = n$. For the BMRC-granules algorithm, Table 2 also provides the corresponding numbers of basic operations and auxiliary variables for the BMRC granules; the temporal and spatial complexity are $T(n) = 4n, S(n) = n$ and $T(n) = 2n, S(n) = c$ in the worst and best cases, respectively. Therefore, the asymptotic values of the complexity of the two algorithms are at most $T(n) = \Theta(n)$ and $S(n) = \Theta(n)$, and the algorithms are both feasible and effective. Furthermore, the BMRC-granules algorithm is necessarily superior in terms of the spatial complexity and usually superior in terms of the temporal complexity. The BMRC-granules algorithm is also advantageous for concrete computing and applications with respect to certain BMR granules.

**Proposition 6.3**

(1) $\overline{R}_k\phi = \phi, \overline{R}_k U = \cup\{[x]_R : \lvert[x]_R\rvert > k\}, \underline{R}_\beta\phi = \phi, \underline{R}_\beta U = U$.
(2) $X \subseteq Y \Rightarrow \overline{R}_k X \subseteq \overline{R}_k Y, X \subseteq Y \Rightarrow \underline{R}_\beta X \subseteq \underline{R}_\beta Y$.
(3) $\overline{R}_k(X \cup Y) \supseteq \overline{R}_k X \cup \overline{R}_k Y, \underline{R}_\beta(X \cup Y) \supseteq \underline{R}_\beta X \cup \underline{R}_\beta Y$.
(4) $\overline{R}_k(X \cap Y) \subseteq \overline{R}_k X \cap \overline{R}_k Y, \underline{R}_\beta(X \cap Y) \subseteq \underline{R}_\beta X \cap \underline{R}_\beta Y$.
(5) $\overline{R}_k(\sim X) = \sim \underline{R}_k X, \underline{R}_\beta(\sim X) = \sim \overline{R}_\beta X$.

**Table 3**
Statistical data on the patient classes.

| $[x]_m$ | $\lvert[x]_m\rvert$ | $\overline{g} = \lvert[x]_m \cap X\rvert$ | $p([x]_m, X)$ | $\underline{g} = \lvert[x]_m\rvert - \lvert[x]_m \cap X\rvert$ |
|---|---|---|---|---|
| $[x]_1$ | 7 | 0 | 0 | 7 |
| $[x]_2$ | 2 | 1 | 0.5 | 1 |
| $[x]_3$ | 3 | 1 | 1/3 | 2 |
| $[x]_4$ | 1 | 1 | 1 | 0 |
| $[x]_5$ | 5 | 2 | 0.4 | 3 |
| $[x]_6$ | 5 | 3 | 0.6 | 2 |
| $[x]_7$ | 4 | 2 | 0.5 | 2 |
| $[x]_8$ | 6 | 4 | 2/3 | 2 |
| $[x]_9$ | 3 | 3 | 1 | 0 |

(6) $k \geq l \Rightarrow \overline{R}_k X \subseteq \overline{R}_l X$, $\beta \geq \alpha \Rightarrow \underline{R}_\beta X \supseteq \underline{R}_\alpha X$.
(7) $\overline{R}_0 X = \overline{R}X$, $\underline{R}_0 X = \underline{R}X$.

**Proposition 6.4**

(1) $\overline{R}_k(\overline{R}_k X) = \overline{R}_k X = \underline{R}_\beta(\overline{R}_k X)$.
(2) $\overline{R}_k(\underline{R}_\beta X) \subseteq \underline{R}_\beta X = \underline{R}_\beta(\underline{R}_\beta X)$.
(3) $\underline{R}_\beta X - \overline{R}_k(\underline{R}_\beta X) = \underline{R}_\beta X \cap (\cup\{[x]_R : \lvert[x]_R\rvert \leq k\})$.

The attribute approximate dependence and reduction in $(U, \overline{R}_k, \underline{R}_\beta)$ are also of practical importance in rough set theory and its applications.

## 7. A medical example

In this section, the medical example [46] is introduced to illustrate the utilization of the new models. Let $S = (U, T, V, f)$ be a decision table, where $U$ is composed of thirty-six patients, and the condition and decision attributes are *fever*, *headache* and *cold*, respectively. Let $R$ denote the equivalence relation on the condition attributes. Based on the measured medical data, Table 3 provides the statistical data on the patient classes, where $[x]_m = (i, j)(m = 1, 2 \ldots 9)$ denote the patient classes on $R$, and $X$ denotes the cold patient set. The BMRC-granular and MR-granular computing results and certain quantitative semantics are provided for each of the two models for the case where $\beta = 0.4$ and $k = 1$.

**The BMRC-granules algorithm**

(I) In $(U, \overline{R}_\beta, \underline{R}_k)$, the computational results are as follows.
  (i) There are only four BMRC granules. BMRC granule (1) is composed of $[x]_1$, $[x]_3$ and $[x]_5$; BMRC granule (2) is composed of $[x]_6$, $[x]_7$ and $[x]_8$; BMRC granule (3) is composed of $[x]_2$ and $[x]_9$; BMRC granule (6) is composed of $[x]_4$.
  (ii) The BMR granules are provided as follows: $posR_{\overline{\beta},\underline{k}}X = [x]_2 \cup [x]_4 \cup [x]_9$, $negR_{\overline{\beta},\underline{k}}X = [x]_1 \cup [x]_3 \cup [x]_5$, $ubnR_{\overline{\beta},\underline{k}}X = [x]_6 \cup [x]_7 \cup [x]_8$, $lbnR_{\overline{\beta},\underline{k}}X = \phi$.
  (iii) $\overline{R}_\beta X = [x]_2 \cup [x]_4 \cup [x]_6 \cup [x]_7 \cup [x]_8 \cup [x]_9$, $\underline{R}_k X = [x]_2 \cup [x]_4 \cup [x]_9$, $bnR_{\overline{\beta},\underline{k}}X = [x]_6 \cup [x]_7 \cup [x]_8$.

(II) In $(U, \overline{R}_k, \underline{R}_\beta)$, the related results are as follows.
  (i) There are only four BMRC granules. BMRC-Granule (1) is composed of $[x]_1$, $[x]_2$ and $[x]_3$; BMRC-Granule (2) is composed of $[x]_5$ and $[x]_7$; BMRC-Granule (3) is composed of $[x]_6$, $[x]_8$ and $[x]_9$; BMRC-Granule (5) is composed of $[x]_4$.
  (ii) $posR_{\overline{k},\underline{\beta}}X = [x]_6 \cup [x]_8 \cup [x]_9$, $negR_{\overline{k},\underline{\beta}}X = [x]_1 \cup [x]_2 \cup [x]_3$, $ubnR_{\overline{k},\underline{\beta}}X = [x]_5 \cup [x]_7$, $lbnR_{\overline{k},\underline{\beta}}X = [x]_4$.
  (iii) $\overline{R}_k X = [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8 \cup [x]_9$, $\underline{R}_\beta X = [x]_4 \cup [x]_6 \cup [x]_8 \cup [x]_9$, $bnR_{\overline{k},\underline{\beta}}X = [x]_4 \cup [x]_5 \cup [x]_7$.

**The approximation algorithm**

(I) In $(U, \overline{R}_\beta, \underline{R}_k)$,

  (i) $\overline{R}_\beta X = [x]_2 \cup [x]_4 \cup [x]_6 \cup [x]_7 \cup [x]_8 \cup [x]_9$, $\underline{R}_k X = [x]_2 \cup [x]_4 \cup [x]_9$;

  (ii) $posR_{\overline{\beta},\underline{k}}X = [x]_2 \cup [x]_4 \cup [x]_9$, $negR_{\overline{\beta},\underline{k}}X = [x]_1 \cup [x]_3 \cup [x]_5$, $ubnR_{\overline{\beta},\underline{k}}X = bnR_{\overline{\beta},\underline{k}}X = [x]_6 \cup [x]_7 \cup [x]_8$, $lbnR_{\overline{\beta},\underline{k}}X = \phi$.

(II) In $(U, \overline{R}_k, \underline{R}_\beta)$,

(i) $\overline{R}_k X = [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8 \cup [x]_9$, $\underline{R}_\beta X = [x]_4 \cup [x]_6 \cup [x]_8 \cup [x]_9$;
(ii) $posR_{\overline{k},\beta} X = [x]_6 \cup [x]_8 \cup [x]_9$, $negR_{\overline{k},\underline{\beta}} X = [x]_1 \cup [x]_2 \cup [x]_3$, $ubnR_{\overline{k},\underline{\beta}} X = [x]_5 \cup [x]_7$, $lbnR_{\overline{k},\underline{\beta}} X = [x]_4$, $bnR_{\overline{k},\underline{\beta}} X = [x]_4 \cup [x]_5 \cup [x]_7$.

Each BMR/MR granule has its own quantitative semantics for the precision and grade. Only two examples are provided here for brevity; the remaining results can be obtained similarly. Based on the condition attributes *fever*, *headache*, the universe is classified into only nine patient classes. In $(U, \overline{R}_\beta, \underline{R}_k)$, $posR_{\overline{\beta},\underline{k}} X = [x]_2 \cup [x]_4 \cup [x]_9$ denotes those patient classes *whose precision with respect to the cold patient set exceeds 0.4 and whose external grade with respect to the cold patient set does not exceed 1*; this quantitative semantics is a kind of double-semantics on the precision and grade and corresponds to BS (1). Equivalently, $posR_{\overline{\beta},\underline{k}} X$ denotes those patient classes *whose relative degree of misclassification with respect to the cold patient set is smaller than 0.6 and whose number of patients outside the cold patient set is at most 1*. In $(U, \overline{R}_k, \underline{R}_\beta)$, $\overline{R}_k X = [x]_5 \cup [x]_6 \cup [x]_7 \cup [x]_8 \cup [x]_9$ denotes those patient classes *whose internal grade with respect to the cold patient set exceeds 1*; equivalently, it denotes those patient classes *whose number of patients inside the cold patient set exceeds 1*. The corresponding double-semantics is a construction of BS (1) and (3), i.e., $\overline{R}_k X$ also denotes those patient classes either *whose internal grade with respect to the cold patient set exceeds 1 and whose precision with respect to the cold patient set is at least 0.6* or *whose internal grade with respect to the cold patient set exceeds 1 and whose precision with respect to the cold patient set is smaller than 0.6*. Clearly, the multiple-semantic descriptions are more diverse and abundant. In contrast to the single-semantics, the double-semantics also exhibits completeness.

This medical example demonstrates how the two new models provide a double-quantification of the precision and grade. They therefore exhibit double fault-description and fault-tolerance features and have substantial practical value. In $(U, \overline{R}_\beta, \underline{R}_k)$, the temporal and spatial complexity of the BMRC-granules and approximation algorithms are $T(9) = 39$, $S(9) = 15$ and $T(9) = 36$, $S(9) = 18$, respectively; in $(U, \overline{R}_k, \underline{R}_\beta)$, the corresponding values are $T(9) = 30$, $S(9) = 6$ and $T(9) = 27$, $S(9) = 9$. The spatial advantage of the BMRC-granules algorithm is clear in this example. As for the time complexity, there is only a small gap between the two algorithms. Moreover, $T(9) = 27$ and $S(9) = 9$ in the VPRS model.

In $(U, \overline{R}_\beta, \underline{R}_k)$, the approximate dependence degree of the decision attribute set *cold* and condition attribute set *fever, headache* is obtained as follows: $\gamma(C, D, 0.4, 1) = 0.44$; where, $posR_{\overline{\beta},\underline{k}} X = [x]_2 \cup [x]_4 \cup [x]_9$, $posR_{\overline{\beta},\underline{k}}(\sim X) = [x]_1 \cup [x]_2 \cup [x]_3$. In $(U, \overline{R}_k, \underline{R}_\beta)$, $posR_{\overline{k},\underline{\beta}} X = [x]_6 \cup [x]_8 \cup [x]_9$, $posR_{\overline{k},\underline{\beta}}(\sim X) = [x]_1 \cup [x]_3 \cup [x]_5$, and $\gamma(C, D, 0.4, 1) = 0.81$.

Finally, we stress a particular point on the advantages of the new models. For $[x]_2$ and $[x]_7$, $p([x]_2, X) = p([x]_7, X) = 0.5$; therefore, $[x]_2$ and $[x]_7$ become indiscernible and equal in the VPRS model. However, they are essentially different; in fact, $|[x]_2| = 2 \neq 4 = |[x]_7|$ and $|[x]_2 \cap X| = 1 \neq 2 = |[x]_7 \cap X|$. For the same threshold value ($\beta = 0.4$), both $[x]_2$ and $[x]_7$ belong to the boundary region in the VPRS model, while they belong to the positive and upper boundary regions, respectively, in $(U, \overline{R}_\beta, \underline{R}_k)$ and belong to the negative and upper boundary regions, respectively, in $(U, \overline{R}_k, \underline{R}_\beta)$. The two knowledge granules have therefore been divided into two different BMR granules and become discernible in the basic double-quantitative models. This discernibility cannot be neglected in the quantification, but cannot be determined using only the precision. Therefore, the grade information is also a fundamental factor, and the double quantification of the precision and grade provides a complete and valuable description. Furthermore, the case in which $p([x]_2, X) = 0.5$ is naturally the case with the highest uncertainty for $[x]_2$. The VPRS model cannot provide a thorough analysis and simply allocates $[x]_2$ to the boundary region. However, in view of both the threshold $k = 1$ and the grade information, $\underline{g}([x]_2, X) = 1$ or $\overline{g}([x]_2, X) = 1$, the two new models allocate $[x]_2$ to the positive and negative regions from two different directions. The certainty of $[x]_2$ has therefore been strengthened and improved based on the model regions. Clearly, the double quantification becomes rational, valuable and applicable in this case. Similarly, we can analyze the results of the GRS model when $k = 1$. For example, knowledge granules $[x]_5$, $[x]_7$ can be compared between the GRS model and the first model. In summary, this example also demonstrates that compared to the VPRS and GRS models, the double quantification and new models exhibit both completeness for quantification and improvements for the applied regions.

## 8. Conclusion

The double-quantification of the precision and grade is a novel, necessary, valuable and feasible technique. Therefore, given the completeness and complementarity of the two indexes, this paper explores their combination using the Cartesian product. The proposed models, i.e., $(U, \overline{R}_\beta, \underline{R}_k)$ and $(U, \overline{R}_k, \underline{R}_\beta)$, perform a basic double quantification of the precision and grade. They therefore have concrete double-quantitative semantics, which thoroughly describe the approximate space, and exhibit strong double fault-tolerance capabilities, enabling them to adapt to complex environments. Moreover, the new models are directional expansions of the Pawlak model and satisfy the quantitative completeness property; the models are therefore promising for practical applications. The new models are therefore suitable as basic double-quantitative models in both theoretical studies and practical applications. Based on our granular computing studies, three types of granules (MR, BMR, and BMRC granules) have been defined to describe the structures and relationships inherent in double-quantification. Furthermore, our semantic extraction and concrete computing methods, especially the optimal BMRC-granules algorithm, have substantial importance for knowledge discovery in double quantification. The fundamental issues in model applications

(the complete system, quantitative semantics and optimal computing of macroscopic regions) have therefore been resolved in the new models. Moreover, the two-dimensional plane on $|[x]_R|$ and $|[x]_R \cap X|$ has provided a novel technology and effective tool for the quantitative study of the approximate space, especially for the basic quantitative system $(|[x]_R|, |[x]_R \cap X|)$.

Several aspects of the new models are worth investigating in further depth, including the uncertainty measures and the properties of the models with respect to the concept and parameter systems. Thorough studies of the new models based on generalized relations and practical applications of the new models (e.g., based on the attribute approximate dependence/reduction) are also interesting directions for future research. Other double-quantitative models should also be extensively explored in future work.

## Acknowledgements

The authors thank both the editors and the anonymous referees for their valuable suggestions, which substantially improved this paper. This work was supported by the National Science Foundation of China (61203285, 61273304, 60970061, 11071178), China Postdoctoral Science Foundation Funded Project (2012M520930), Fundamental Research Funds for the Central Universities of China, and Key Project of the Sichuan Provincial Education Department of China (12ZA138).

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