International Journal of Approximate Reasoning 55 (2014) 1404–1418

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

www.elsevier.com/locate/ijar

On multi-granulation covering rough sets

Caihui Liu [a,b,*], Duoqian Miao [b], Jin Qian [c]

[a] Department of Mathematics and Computer Science, Gannan Normal University, 341000 Ganzhou, China
[b] Department of Computer Science and Technology, Tongji University, 201804 Shanghai, China
[c] College of Computer Engineering, Jiangsu University of Technology, 213015 Changzhou, China

ARTICLE INFO

Article history:
Received 4 September 2012
Received in revised form 25 October 2013
Accepted 7 January 2014
Available online 21 January 2014

Keywords:
Covering
Lower and upper approximations
Multi-granulation
Reduct
Rough sets

ABSTRACT

Recently, much attention has been given to multi-granulation rough sets (MGRS) and different kinds of multi-granulation rough set models have been developed from various viewpoints. In this paper, we propose four types of multi-granulation covering rough set (MGCRS) models under covering approximation space, where a target concept is approximated by employing the maximal or minimal descriptors of objects in a given universe of discourse U. And then, we investigate a number of basic properties of the four types of MGCRS models, and discuss the relationships and differences among the classical MGRS model and our MGCRS models. Moreover, the conditions for two distinct MGCRS models which produce identical lower and upper approximations of a target concept in a covering approximation space are also studied. Finally, the relationships among the four types of MGCRS models are explored. We find that for any subset $X \subseteq U$, the lower approximations of X and the upper approximations of X under the four types of MGCRS models can construct a lattice, if we consider the binary relation of inclusion.

1. Introduction

To enlarge the usage scope of the Pawlak's rough set model [1], many meaningful extensions have been proposed for various requirements in the past decades, such as the graded rough sets [2], arbitrary binary relations based rough sets [9,29], tolerance or similarity relations based rough sets [30–32], rough fuzzy sets and fuzzy rough sets [33], and variable precision rough sets [34,35], etc. As one of the extension models, covering rough sets, which was first proposed by Zakowski [5], has attracted much attention and induced lots of interesting results [6–9,36,37,40–42]. Recently, Yao et al. [4] proposed a framework for the study of covering based rough set approximations, which enables us to reproduce many existing approximation operators and introduce some new approximation operators.

In the viewpoint of Granular Computing [3,10], Pawlak's rough set model and most of its extensions are constructed based on only one granular structure, which is induced by a binary relation (a partition or a covering). Thus, one may call those models the single-granulation rough sets. As Qian et al. [12] have mentioned that, in many cases, a target concept is needed to describe concurrently from some independent environments, that is, multi-granulation spaces are needed. Therefore, Qian and Liang [11,12] introduced the concept of multi-granulation rough sets (MGRS), where the approximations of a set of objects are defined by using multi-equivalence relations. The main difference between single-granulation rough sets and multi-granulation ones lies in that the approximations of a target concept in multi-granulation rough sets are constructed by using multi-distinct sets of information granules. When two attribute subsets in an information system contradict each other or possess an inconsistent relationship, MGRS will show its advantages for knowledge discovery [12].

* Corresponding author at: Department of Mathematics and Computer Science, Gannan Normal University, 341000 Ganzhou, China.
E-mail addresses: liu_caihui@163.com (C. Liu), miaoduoqian@163.com (D. Miao), qjqjlqyf@163.com (J. Qian).

0888-613X/$ – see front matter
http://dx.doi.org/10.1016/j.ijar.2014.01.002

By now, much attention has been given to the multi-granulation rough sets. For instance, Qian et al. [13] presented a multi-granulation rough set model based on multiple tolerance relations in incomplete information systems. In [14], Qian et al. introduced the concept of positive approximation. The positive approximation can be regarded as a special kind of multi-granulation rough set model, in which the granulation structure of a rough set is characterized by using a granulation order, that is, the granulation structure is characterized by a sequence of attribute sets with granulations from coarse to fine. Based on the positive approximation, Qian et al. developed a common accelerator for improving the time efficiency of a heuristic attribute reduction, which provides a vehicle of making algorithms of rough set based feature selection techniques faster. Based on Ref. [13], Yang et al. [15] further discussed the properties of multi-granulation rough sets in incomplete information systems. Moreover, Yang et al. [16] and Xu et al. [21,24,26] constructed multi-granulation rough sets based on the fuzzy approximation space. Yang et al. [17] discussed the hierarchical structures of multi-granulation rough sets and She et al. [28] investigated the topological and lattice-theoretic properties of multi-granulation rough sets. Liang et al. [39] introduced an efficient rough feature selection algorithm with a multi-granulation view. Liu et al. [18,19] introduced the concepts of multi-granulation covering rough sets and multi-granulation covering fuzzy rough sets in a covering approximation space. Lin et al. [20] discussed two kinds of neighborhood-based multi-granulation rough sets and three kinds of covering based multi-granulation rough sets [43]. Xu et al. [22,25,27] constructed three kinds of multi-granulation rough sets based on the tolerance, ordered and generalized relations respectively. Qian et al. [44] developed the multi-granulation decision-theoretic rough set and proved that many existing multi-granulation rough set models can be derived from the multi-granulation decision-theoretic rough set framework.

As an extension of our previous work [18], the purpose of this paper is to further generalize the classical multi-granulation rough sets to covering environment. Through combining the multi-granulation rough sets with covering rough sets, we aim to solve the limitation of classical multi-granulation rough sets induced by equivalence relations. To define the approximations of a target concept in the multi-granulation covering environment, we respectively employ the maximal and minimal descriptors of objects in a given universe of discourse U. For any subset $X \subseteq U$ and object $x \in U$, we first respectively define the type-1 and type-2 approximations (including low and upper approximations) of X by using the intersection and union of all elements in the minimal descriptor of x. Second, the type-3 and type-4 approximations of X are respectively defined based on the intersection and union of all elements in the maximal descriptor of x. Correspondingly, we can obtain four types of multi-granulation covering rough set (MGCRS) models under a covering approximation space. The basic properties of the four types of MGCRS models, and the relationships among the classical MGRS model and our MGCRS models are discussed. Moreover, the conditions for two distinct MGCRS models which produce identical lower and upper approximations of a target concept in a covering approximation space are also studied. Finally, the relationships among the four types of MGCRS models are explored.

The remainder of this paper is organized as follows. The next section deals with some preliminary concepts and properties regarding the Pawlak's rough sets, covering rough sets and multi-granulation rough sets. In Section 3, we introduce four types of multi-granulation covering rough set models and investigate the basic properties of them. Moreover, based on the concept of reduct of a covering, we give the sufficient conditions for two different MGCRS models to produce identical lower and upper approximations of a target concept. In Section 4, we investigate the relationships among the four types of MGCRS models. Finally, Section 5 concludes the paper.

2. Preliminaries

In this section, we review some basic concepts about the Pawlak's rough sets, covering rough sets and multi-granulation rough sets. The detailed descriptions can be found in [1,2,5,6,11,12].

2.1. The Pawlak's rough sets

Let U be a universe of discourse, and R an equivalence relation on U. R partitions the universe U into disjoint subsets, each subset is called an equivalence class or equivalence granule, the family of all equivalence classes is denoted by U/R. For any $X \subseteq U$, one can describe X by a pair of lower and upper approximations defined as follows.

$$\underline{R}(X) = \{x \in U \mid [x]_R \subseteq X\};$$

$$\overline{R}(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\}.$$

$\underline{R}(X)$ is called the lower approximation of X, which is the union of all the equivalence classes which are subsets of X, and $\overline{R}(X)$ is called the upper approximation of X, which is the union of all equivalence classes which have non-empty intersection with X. Then $(\underline{R}(X), \overline{R}(X))$ is called the rough sets of X.

Let $\sim X = U - X$, we have the following basic properties of Pawlak's rough sets.

$(L1)$ $\underline{R}(X) \subseteq X$	$(U1)$ $X \subseteq \overline{R}(X)$
$(L2)$ $\underline{R}(\emptyset) = \emptyset$	$(U2)$ $\overline{R}(\emptyset) = \emptyset$
$(L3)$ $\underline{R}(U) = U$	$(U3)$ $\overline{R}(U) = U$
$(L4)$ $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$	$(U4)$ $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$
$(L5)$ $X \subseteq Y \Rightarrow \underline{R}(X) \subseteq \underline{R}(Y)$	$(U5)$ $X \subseteq Y \Rightarrow \overline{R}(X) \subseteq \overline{R}(Y)$
$(L6)$ $\underline{R}(X \cup Y) \supseteq \underline{R}(X) \cup \underline{R}(Y)$	$(U6)$ $\overline{R}(X \cap Y) \subseteq \overline{R}(X) \cap \overline{R}(Y)$
$(L7)$ $\underline{R}(\sim X) = \sim \overline{R}(X)$	$(U7)$ $\overline{R}(\sim X) = \sim \underline{R}(X)$
$(L8)$ $\underline{R}(\underline{R}(X)) = \underline{R}(X)$	$(U8)$ $\overline{R}(\overline{R}(X)) = \overline{R}(X)$
$(L9)$ $\forall K \in U/R \Rightarrow \underline{R}(K) = K$	$(U9)$ $\forall K \in U/R \Rightarrow \overline{R}(K) = K$

2.2. Basic concepts of covering rough sets

Let U be a universe of discourse and C a family of non-empty subsets of U. If $\cup C = U$, then C is called a covering of U. Obviously, a partition of U is also a covering of U.

Definition 1. (See [5].) The pair $\langle U, C\rangle$ is called a covering approximation space, where U is a universe of discourse and C a covering of U.

Definition 2. (See [5].) Let $\langle U, C\rangle$ be a covering approximation space, where $C = \{C_1, C_2, \ldots, C_p\}$. For any $X \subseteq U$, the covering lower and upper approximations of X with respect to C can be respectively defined as follows.

$$\underline{C}(X) = \cup\big\{C_i \subseteq X, i \in \{1, 2, \ldots, p\}\big\}, \qquad \overline{C}(X) = \cup\big\{C_i \cap X \neq \emptyset, i \in \{1, 2, \ldots, p\}\big\}.$$

Definition 3. Given a covering approximation space $\langle U, C\rangle$, for any $x \in U$, sets $md_C(x)$ and $MD_C(x)$ are respectively called the minimal and maximal descriptors of x with respect to C, where $md_C(x) = \{K \in C | x \in K \wedge (\forall S \in C \wedge x \in S \wedge S \subseteq K \Rightarrow K = S)\}$, $MD_C(x) = \{K \in C | x \in K \wedge (\forall S \in C \wedge x \in S \wedge S \supseteq K \Rightarrow K = S)\}$.

Based on the minimal descriptor of x, Xu et al. [23] have introduced a kind of covering rough set, which is the basis of Definition 7.

$$C^*(X) = \big\{x \in U | \big(\cap md(x)\big) \subseteq X\big\}, \qquad C_*(X) = \big\{x \in U | \big(\cap md(x)\big) \cap X \neq \emptyset\big\}.$$

Definition 4. Let C be a covering of U and $K \in C$. If K is a union of some elements in $C - \{K\}$, then we call K a reducible element of C, otherwise we call K an irreducible element of C. If all the reducible elements have been removed from C, then the obtained subset is called the **reduct** of C on U, denoted by $reduct(C)$.

From Definitions 3 and 4, it is not difficult to obtain that for any $x \in U$, the minimal descriptor of x with respect to C is the same as that of x with respect to $reduct(C)$, but the maximal descriptor of x with respect to C may be different from that of x with respect to $reduct(C)$.

2.3. Multi-granulation rough sets

In recent years, Qian et al. [11,12] have proposed a new rough set model called multi-granulation rough set model. In this model, a target concept is approximated by multiple binary relations. Next, we briefly outline two definitions of multi-granulation rough set models, i.e., optimistic and pessimistic multi-granulation rough sets respectively. Detailed descriptions could be found in [11,12].

Definition 5. Let $K = (U, \mathbf{R})$ be a knowledge base, where $\mathbf{R}$ is a family of equivalence relations on the universe U. Let $A_1, A_2, \ldots, A_m \in \mathbf{R}$, where m is a natural number. For any $X \subseteq U$, its optimistic lower and upper approximations with respect to $A_1, A_2, \ldots, A_m$ are respectively defined as follows.

$$\underline{\sum_{i=1}^{m} A_i}(X) = \big\{x \in U | [x]_{A_1} \subseteq X \text{ or } [x]_{A_2} \subseteq X \text{ or } \ldots \text{ or } [x]_{A_m} \subseteq X\big\};$$

$$\overline{\sum_{i=1}^{m} A_i}(X) = \sim \underline{\sum_{i=1}^{m} A_i}(\sim X).$$

$(\underline{\sum_{i=1}^{m} A_i}(X), \overline{\sum_{i=1}^{m} A_i}(X))$ is called the optimistic multi-granulation rough sets of X. Here, the word "optimistic" means that only one granular structure is needed to satisfy with the inclusion condition between an equivalence class and a target concept when multiple independent granular structures are available in problem processing.

Definition 6. Let $K=(U,\mathbf{R})$ be a knowledge base, where $\mathbf{R}$ is a family of equivalence relations on the universe U. Let $A_1, A_2, \ldots, A_m \in \mathbf{R}$, where m is a natural number. For any $X \subseteq U$, its pessimistic lower and upper approximations with respect to $A_1, A_2, \ldots, A_m$ are respectively defined as follows.

$$\underline{\sum_{i=1}^{m} A_i}(X)=\left\{x \in U \mid [x]_{A_1} \subseteq X \text{ and } [x]_{A_2} \subseteq X \text{ and } \ldots \text{ and } [x]_{A_m} \subseteq X\right\};$$

$$\overline{\sum_{i=1}^{m} A_i}(X)=\sim \underline{\sum_{i=1}^{m} A_i}(\sim X).$$

$(\underline{\sum_{i=1}^{m} A_i}(X), \overline{\sum_{i=1}^{m} A_i}(X))$ is called the pessimistic multi-granulation rough sets of X. Here, the word "pessimistic" means that all granular structures are needed to satisfy with the inclusion condition between an equivalence class and a target concept when multiple independent granular structures are available.

3. Multi-granulation covering rough sets (MGCRS)

In this section, we extend the classical multi-granulation rough sets to a multi-granulation covering approximation space. First, we introduce four types of multi-granulation covering rough set models by employing the maximal or minimal descriptors of objects in a given universe of discourse. Second, we investigate the basic properties of the four types of MGCRS models. Finally, we give the conditions for two distinct MGCRS models which produce identical lower and upper approximations of a target concept.

It should be noted that in the remainder of this paper, differing from the definition of covering approximation space introduced by Zakowski [5], we define the pair $\langle U,\mathbf{C}\rangle$ as a multi-granulation covering approximation space, where U is a universe of discourse and $\mathbf{C}$ is a family of coverings on the universe U.

Given an approximation space $\langle U,\mathbf{C}\rangle$, for any object $x \in U$, the minimal descriptor of x contains the core objects in the approximation space that are related to x, and the minimal descriptor may provide a simple and key description for x when we discuss the issue of set approximations in $\langle U,\mathbf{C}\rangle$. By virtue of the minimal descriptors of objects in U, we first give the following two types of MGCRS models, which are respectively called type-1 and type-2 multi-granulation covering rough sets.

Definition 7 *(Type-1 MGCRS).* (See [18].) Let $\langle U,\mathbf{C}\rangle$ be a multi-granulation covering approximation space, and $C_1, C_2, \ldots, C_m \in \mathbf{C}$, where m is a natural number. For any $X \subseteq U$, its type-1 lower and upper approximations with respect to $C_1, C_2, \ldots, C_m$ are respectively defined as follows.

$$\underline{FR_{\sum_{i=1}^{m} C_i}}(X)=\left\{x \in U \mid \cap md_{C_1}(x) \subseteq X \text{ or } \cap md_{C_2}(x) \subseteq X \text{ or } \ldots \cap md_{C_m}(x) \subseteq X\right\};$$

$$\overline{FR_{\sum_{i=1}^{m} C_i}}(X)=\left\{x \in U \mid \left(\cap md_{C_1}(x)\right) \cap X \neq \emptyset \text{ and } \left(\cap md_{C_2}(x)\right) \cap X \neq \emptyset \text{ and } \ldots \text{ and } \left(\cap md_{C_m}(x)\right) \cap X \neq \emptyset\right\}.$$

If $\underline{FR_{\sum_{i=1}^{m} C_i}}(X) \neq \overline{FR_{\sum_{i=1}^{m} C_i}}(X)$, then X is called a type-1 multi-granulation covering rough set with respect to $C_1, C_2, \ldots, C_m$, else X is called a definable set.

From Definition 7, we can see that for any $X \subseteq U$ and $x \in U$, the intersection of all elements in the minimal descriptor of x is used to calculate the type-1 lower and upper approximations of X. Moreover, since the above approximations of X are defined in the multi-granulation environment, the minimal descriptors of x with respect to coverings $C_1, C_2, \ldots, C_m$ are simultaneously used. In a word, if at least one of $\{\cap md_{C_i}(x), 1 \leqslant i \leqslant m\}$ is a subset of X, then object x belongs to the lower approximation of X, and if all the intersections between $\cap md_{C_i}(x)$ with X are not empty, then x belongs to the upper approximation of X, where $md_{C_i}(x)$ denotes the minimal descriptor of x with respect to covering C_i, $1 \leqslant i \leqslant m$.

Definition 8 *(Type-2 MGCRS).* Let $\langle U,\mathbf{C}\rangle$ be a multi-granulation covering approximation space, and $C_1, C_2, \ldots, C_m \in \mathbf{C}$, where m is a natural number. For any $X \subseteq U$, its type-2 lower and upper approximations with respect to $C_1, C_2, \ldots, C_m$ are respectively defined as follows.

$$\underline{SR_{\sum_{i=1}^{m} C_i}}(X)=\left\{x \in U \mid \cup md_{C_1}(x) \subseteq X \text{ or } \cup md_{C_2}(x) \subseteq X \text{ or } \ldots \text{ or } \cup md_{C_m}(x) \subseteq X\right\};$$

$$\overline{SR_{\sum_{i=1}^{m} C_i}}(X)=\left\{x \in U \mid \left(\cup md_{C_1}(x)\right) \cap X \neq \emptyset \text{ and } \left(\cup md_{C_2}(x)\right) \cap X \neq \emptyset \text{ and } \ldots \text{ and } \left(\cup md_{C_m}(x)\right) \cap X \neq \emptyset\right\}.$$

If $\underline{SR_{\sum_{i=1}^{m} C_i}}(X) \neq \overline{SR_{\sum_{i=1}^{m} C_i}}(X)$, then X is called a type-2 multi-granulation covering rough set with respect to $C_1, C_2, \ldots, C_m$, else X is called a definable set.

From Definition 8, we can see that for any $X \subseteq U$ and $x \in U$, the union of all elements in the minimal descriptor of x is used to calculate the type-2 lower and upper approximations of X. Moreover, since the above approximations of X are defined in the multi-granulation environment, the minimal descriptors of x with respect to coverings $C_1, C_2, \ldots, C_m$ are simultaneously used.

The maximal descriptor of x contains all objects in the approximation space that are related to x, and the maximal descriptor may provide a detailed and comprehensive description for x when we discuss the issue of set approximations in $\langle U, \mathbf{C} \rangle$. As Yao et al. [4] have pointed out that the utilization of the maximal descriptors of objects is equally reasonable as the utilization of the minimal ones in a covering approximation space. Therefore, we also give the following two types of MGCRS models by virtue of the maximal descriptors of objects in U, which are respectively called type-3 and type-4 multi-granulation covering rough sets.

Definition 9 *(Type-3 MGCRS).* Let $\langle U, \mathbf{C} \rangle$ be a multi-granulation covering approximation space, and $C_1, C_2, \ldots, C_m \in \mathbf{C}$, where m is a natural number. For any $X \subseteq U$, its type-3 lower and upper approximations with respect to $C_1, C_2, \ldots, C_m$ are respectively defined as follows.

$$\underline{TR_{\sum_{i=1}^m C_i}}(X) = \{x \in U \mid \cap MD_{C_1}(x) \subseteq X \text{ or } \cap MD_{C_2}(x) \subseteq X \text{ or } \ldots \cap MD_{C_m}(x) \subseteq X\};$$

$$\overline{TR_{\sum_{i=1}^m C_i}}(X) = \{x \in U \mid (\cap MD_{C_1}(x)) \cap X \neq \emptyset \text{ and } (\cap MD_{C_2}(x)) \cap X \neq \emptyset \text{ and } \ldots \text{ and } (\cap MD_{C_m}(x)) \cap X \neq \emptyset\}.$$

If $\underline{TR_{\sum_{i=1}^m C_i}}(X) \neq \overline{TR_{\sum_{i=1}^m C_i}}(X)$, then X is called a type-1 multi-granulation covering rough set with respect to $C_1, C_2, \ldots, C_m$, else X is called a definable set, then X is called a type-3 multi-granulation covering rough set with respect to $C_1, C_2, \ldots, C_m$, else X is called a definable set.

From Definition 9, we can see that for any $X \subseteq U$ and $x \in U$, the intersection of all elements in the maximal descriptor of x is used to calculate the type-3 lower and upper approximations of X. Moreover, since the above approximations of X are defined in the multi-granulation environment, the maximal descriptors of x with respect to coverings $C_1, C_2, \ldots, C_m$ are simultaneously used.

Definition 10 (*Type-4 MGCRS*). Let $\langle U, \mathbf{C} \rangle$ be a multi-granulation covering approximation space, and $C_1, C_2, \ldots, C_m \in \mathbf{C}$, where m is a natural number. For any $X \subseteq U$, its type-4 lower and upper approximations with respect to $C_1, C_2, \ldots, C_m$ are respectively defined as follows.

$$\underline{LR_{\sum_{i=1}^m C_i}}(X) = \{x \in U \mid \cup MD_{C_1}(x) \subseteq X \text{ or } \cup MD_{C_2}(x) \subseteq X \text{ or } \ldots \text{ or } \cup MD_{C_m}(x) \subseteq X\};$$

$$\overline{LR_{\sum_{i=1}^m C_i}}(X) = \{x \in U \mid (\cup MD_{C_1}(x)) \cap X \neq \emptyset \text{ and } (\cup MD_{C_2}(x)) \cap X \neq \emptyset\} \text{ and } \ldots \text{ and } (\cup MD_{C_m}(x)) \cap X \neq \emptyset\}.$$

If $\underline{LR_{\sum_{i=1}^m C_i}}(X) \neq \overline{LR_{\sum_{i=1}^m C_i}}(X)$, then X is called a type-4 multi-granulation covering rough set with respect to $C_1, C_2, \ldots, C_m$, then X is called a type-4 multi-granulation covering rough set with respect to $C_1, C_2, \ldots, C_m$, else X is called a definable set.

From Definition 10, we can see that for any $X \subseteq U$ and $x \in U$, the union of all elements in the maximal descriptor of x is used to calculate the type-4 lower and upper approximations of X. Moreover, since the above approximations of X are defined in the multi-granulation environment, the maximal descriptors of x with respect to coverings $C_1, C_2, \ldots, C_m$ are simultaneously used.

It is obvious that Definitions 5, 7–10 will be the same if $\mathbf{C}$ is a family of partitions on the universe U, i.e. the above four multi-granulation covering rough sets will degenerate into the optimistic multi-granulation rough sets when $\mathbf{C}$ is a family of partitions. Hence, each of the above four MGCRS model is an extensional model of the classical MGRS model. It is easy to prove that the above four MGCRS models are also extensions of the Pawlak's model and covering rough set model.

Next, we give an example to show the differences among the four MGCRS models defined above.

Example 1. Let $\langle U, \mathbf{C} \rangle$ be a multi-granulation covering approximation space, and $C_1, C_2 \in \mathbf{C}$, where $U = \{a, b, c, d\}$, $C_1 = \{\{a, b\}, \{b, c, d\}, \{c, d\}\}$, and $C_2 = \{\{a, c\}, \{b, d\}, \{a, b, d\}, \{d\}\}$.

Given a subset $X = \{a, d\}$ of U, from Definitions 7–10, we can obtain the following results.

$\underline{FR_{C_1+C_2}}(X) = \{a, d\}$, $\overline{FR_{C_1+C_2}}(X) = \{a, c, d\}$;
$\underline{SR_{C_1+C_2}}(X) = \{d\}$, $\overline{SR_{C_1+C_2}}(X) = \{a, b, c, d\}$;
$\underline{TR_{C_1+C_2}}(X) = \{a\}$, $\overline{TR_{C_1+C_2}}(X) = \{a, c, d\}$;
$\underline{LR_{C_1+C_2}}(X) = \emptyset$, $\overline{LR_{C_1+C_2}}(X) = \{a, b, c, d\}$.

Table 1
Properties of the eight approximation operators.

	$\underline{FR}$	$\underline{SR}$	$\underline{TR}$	$\underline{LR}$		$\overline{FR}$	$\overline{SR}$	$\overline{TR}$	$\overline{LR}$
$(L1)$	y	y	y	y	$(U1)$	y	y	y	y
$(L2)$	y	y	y	y	$(U2)$	y	y	y	y
$(L3)$	y	y	y	y	$(U3)$	y	y	y	y
$(L4)$	n	n	n	n	$(U4)$	n	n	n	n
$(L5)$	y	y	y	y	$(U5)$	y	y	y	y
$(L6)$	y	y	y	y	$(U6)$	y	y	y	y
$(L7)$	y	y	y	y	$(U7)$	y	y	y	y
$(L8)$	y	y	y	y	$(U8)$	y	y	y	y
$(L9)$	y	y	y	y	$(U9)$	n	n	n	n

From Example 1, we can see that for a given subset $X = \{a, d\}$ of U, the four types of approximations of X are different with each other, which means that the above four types of MGCRS models are also different with each other.

Theorem 1. *Let $\langle U, \mathbf{C}\rangle$ be a multi-granulation covering approximation space, and $C_1, C_2, \ldots, C_m \in \mathbf{C}$, for any $X \subseteq U$, we have that*

(1) $\underline{FR}_{\sum_{i=1}^m C_i}(\sim X) = \sim \overline{FR}_{\sum_{i=1}^m C_i}(X)$, $\overline{FR}_{\sum_{i=1}^m C_i}(\sim X) = \sim \underline{FR}_{\sum_{i=1}^m C_i}(X)$;
(2) $\underline{SR}_{\sum_{i=1}^m C_i}(\sim X) = \sim \overline{SR}_{\sum_{i=1}^m C_i}(X)$, $\overline{SR}_{\sum_{i=1}^m C_i}(\sim X) = \sim \underline{SR}_{\sum_{i=1}^m C_i}(X)$;
(3) $\underline{TR}_{\sum_{i=1}^m C_i}(\sim X) = \sim \overline{TR}_{\sum_{i=1}^m C_i}(X)$, $\overline{TR}_{\sum_{i=1}^m C_i}(\sim X) = \sim \underline{TR}_{\sum_{i=1}^m C_i}(X)$;
(4) $\underline{LR}_{\sum_{i=1}^m C_i}(\sim X) = \sim \overline{LR}_{\sum_{i=1}^m C_i}(X)$, $\overline{LR}_{\sum_{i=1}^m C_i}(\sim X) = \sim \underline{LR}_{\sum_{i=1}^m C_i}(X)$.

Proof. We only prove (1) here, the rest can be proved in a similar way.

(1) From Definition 7, we have that

$$\begin{aligned}\underline{FR}_{\sum_{i=1}^m C_i}(\sim X) &= \{x \in U \mid \cap md_{C_1}(x) \subseteq \sim X \text{ or } \cap md_{C_2}(x) \subseteq \sim X \text{ or } \ldots \text{ or } \cap md_{C_m}(x) \subseteq \sim X\} \\ &= \{x \in U \mid \cap md_{C_1}(x) \cap X = \emptyset \text{ or } \cap md_{C_2}(x) \cap X = \emptyset \text{ or } \ldots \text{ or } \cap md_{C_m}(x) \cap X = \emptyset\} \\ &= \sim \{x \in U \mid \cap md_{C_1}(x) \cap X \neq \emptyset \text{ and } \cap md_{C_2}(x) \cap X \neq \emptyset \text{ and } \ldots \text{ and } \cap md_{C_m}(x) \cap X \neq \emptyset\} \\ &= \sim \overline{FR}_{\sum_{i=1}^m C_i}(X).\end{aligned}$$

Analogously, we have that

$$\begin{aligned}\overline{FR}_{\sum_{i=1}^m C_i}(\sim X) &= \{x \in U \mid (\cap md_{C_1}(x)) \cap \sim X \neq \emptyset \text{ and } (\cap md_{C_2}(x)) \cap \sim X \neq \emptyset \text{ and } \ldots \text{ and } (\cap md_{C_m}(x)) \cap \sim X \neq \emptyset\} \\ &= \sim \{x \in U \mid (\cap md_{C_1}(x)) \cap \sim X = \emptyset \text{ or } (\cap md_{C_2}(x)) \cap \sim X = \emptyset \text{ or } \ldots \text{ or } (\cap md_{C_m}(x)) \cap \sim X = \emptyset\} \\ &= \sim \{x \in U \mid \cap md_{C_1}(x) \subseteq X \text{ or } \cap md_{C_2}(x) \subseteq X \text{ or } \ldots \text{ or } \cap md_{C_m}(x) \subseteq X\} \\ &= \sim \underline{FR}_{\sum_{i=1}^m C_i}(X). \qquad \square\end{aligned}$$

From Theorem 1, we can see that for any subset $X \subseteq U$, the lower and upper approximations of X in each of the above four MGCRS models satisfy the property of duality, i.e., the upper approximation of $\sim X$ can be defined by the complement of the lower approximation of X, and the lower approximation of $\sim X$ can also be defined by the complement of the upper approximation of X.

We respectively call the operators $\underline{FR}_{\sum_{i=1}^m C_i}$, $\underline{SR}_{\sum_{i=1}^m C_i}$, $\underline{TR}_{\sum_{i=1}^m C_i}$ and $\underline{LR}_{\sum_{i=1}^m C_i}$ the lower approximation operators on U, and $\overline{FR}_{\sum_{i=1}^m C_i}$, $\overline{SR}_{\sum_{i=1}^m C_i}$, $\overline{TR}_{\sum_{i=1}^m C_i}$ and $\overline{LR}_{\sum_{i=1}^m C_i}$ the upper approximation operators on U. Table 1 shows whether the eight approximation operators satisfy the properties $(L1)$–$(L9)$ and $(U1)$–$(U9)$ given in Section 2.1, where 'y' denotes "yes" and 'n' denotes "no".

From Table 1, we can see that the eight approximation operators defined in our MGCRS models satisfy all the properties listed above, with the exception of $(L4)$, $(U4)$ and $(U9)$.

Next, we only prove some of properties listed in Table 1. It is not difficult to prove the other properties of the eight approximation operators. Moreover, we also give some examples to show the properties listed in Table 1. In the following, we shall use symbol ∇ to denote FR, SR, TR or LR.

Proposition 1. *Let $\langle U, \mathbf{C}\rangle$ be a multi-granulation covering approximation space and $C_1, C_2, \ldots, C_m \in \mathbf{C}$, where m is a natural number. For any $X \subseteq U$, the lower approximation operators satisfy property $(L8)$ and the upper approximation operators satisfy property $(U8)$, that is,*

(1) $\underline{\nabla_{\sum_{i=1}^{m} C_i}}(\underline{\nabla_{\sum_{i=1}^{m} C_i}}(X)) = \underline{\nabla_{\sum_{i=1}^{m} C_i}}(X)$;
(2) $\overline{\nabla_{\sum_{i=1}^{m} C_i}}(\overline{\nabla_{\sum_{i=1}^{m} C_i}}(X)) = \overline{\nabla_{\sum_{i=1}^{m} C_i}}(X)$.

Proof. (1) From $(L1)$ in Table 1, we have that $\underline{\nabla_{\sum_{i=1}^{m} C_i}}(\underline{\nabla_{\sum_{i=1}^{m} C_i}}(X)) \subseteq \underline{\nabla_{\sum_{i=1}^{m} C_i}}(X)$. Moreover, according to (4) of Theorem 2 in Ref. [13], we can obtain that

$$\begin{aligned}
&\underline{\nabla_{\sum_{i=1}^{m} C_i}}\big(\underline{\nabla_{\sum_{i=1}^{m} C_i}}(X)\big)\\
&\quad= \underline{\nabla_{C_1}}\big(\underline{\nabla_{\sum_{i=1}^{m} C_i}}(X)\big) \cup \underline{\nabla_{C_2}}\big(\underline{\nabla_{\sum_{i=1}^{m} C_i}}(X)\big) \cup \cdots \cup \underline{\nabla_{C_m}}\big(\underline{\nabla_{\sum_{i=1}^{m} C_i}}(X)\big)\\
&\quad= \underline{\nabla_{C_1}}\big(\underline{\nabla_{C_1}}(X) \cup \underline{\nabla_{C_2}}(X) \cup \cdots \cup \underline{\nabla_{C_m}}(X)\big) \cup \underline{\nabla_{C_2}}\big(\underline{\nabla_{C_1}}(X) \cup \underline{\nabla_{C_2}}(X) \cup \cdots \cup \underline{\nabla_{C_m}}(X)\big)\\
&\qquad \cup \cdots \cup \underline{\nabla_{C_m}}\big(\underline{\nabla_{C_1}}(X) \cup \underline{\nabla_{C_2}}(X) \cup \cdots \cup \underline{\nabla_{C_m}}(X)\big)\\
&\quad\supseteq \underline{\nabla_{C_1}}\big(\underline{\nabla_{C_1}}(X)\big) \cup \underline{\nabla_{C_2}}\big(\underline{\nabla_{C_2}}(X)\big) \cup \cdots \cup \underline{\nabla_{C_m}}\big(\underline{\nabla_{C_m}}(X)\big)\\
&\quad= \underline{\nabla_{C_1}}(X) \cup \underline{\nabla_{C_2}}(X) \cup \cdots \cup \underline{\nabla_{C_m}}(X) = \underline{\nabla_{\sum_{i=1}^{m} C_i}}(X).
\end{aligned}$$

Hence, we can prove that $\underline{\nabla_{\sum_{i=1}^{m} C_i}}(\underline{\nabla_{\sum_{i=1}^{m} C_i}}(X)) = \underline{\nabla_{\sum_{i=1}^{m} C_i}}(X)$.

(2) From $(U1)$ in Table 1, we have that $\overline{\nabla_{\sum_{i=1}^{m} C_i}}(X) \subseteq \overline{\nabla_{\sum_{i=1}^{m} C_i}}(\overline{\nabla_{\sum_{i=1}^{m} C_i}}(X))$. Moreover, according to (5) of Theorem 2 in Ref. [13], we can obtain that

$$\begin{aligned}
&\overline{\nabla_{\sum_{i=1}^{m} C_i}}\big(\overline{\nabla_{\sum_{i=1}^{m} C_i}}(X)\big)\\
&\quad= \overline{\nabla_{C_1}}\big(\overline{\nabla_{\sum_{i=1}^{m} C_i}}(X)\big) \cap \overline{\nabla_{C_2}}\big(\overline{\nabla_{\sum_{i=1}^{m} C_i}}(X)\big) \cap \cdots \cap \overline{\nabla_{C_m}}\big(\overline{\nabla_{\sum_{i=1}^{m} C_i}}(X)\big)\\
&\quad= \overline{\nabla_{C_1}}\big(\overline{\nabla_{C_1}}(X) \cap \overline{\nabla_{C_2}}(X) \cap \cdots \cap \overline{\nabla_{C_m}}(X)\big) \cap \overline{\nabla_{C_2}}\big(\overline{\nabla_{C_1}}(X) \cap \overline{\nabla_{C_2}}(X) \cap \cdots \cap \overline{\nabla_{C_m}}(X)\big) \cap\\
&\qquad \cdots \cap \overline{\nabla_{C_m}}\big(\overline{\nabla_{C_1}}(X) \cap \overline{\nabla_{C_2}}(X) \cap \cdots \cap \overline{\nabla_{C_m}}(X)\big)\\
&\quad\subseteq \overline{\nabla_{C_1}}\big(\overline{\nabla_{C_1}}(X)\big) \cap \overline{\nabla_{C_2}}\big(\overline{\nabla_{C_2}}(X)\big) \cap \cdots \cap \overline{\nabla_{C_m}}\big(\overline{\nabla_{C_m}}(X)\big)\\
&\quad= \overline{\nabla_{C_1}}(X) \cap \overline{\nabla_{C_2}}(X) \cap \cdots \cap \overline{\nabla_{C_m}}(X) = \overline{\nabla_{\sum_{i=1}^{m} C_i}}(X).
\end{aligned}$$

Therefore, we can prove that $\overline{\nabla_{\sum_{i=1}^{m} C_i}}(\overline{\nabla_{\sum_{i=1}^{m} C_i}}(X)) = \overline{\nabla_{\sum_{i=1}^{m} C_i}}(X)$. □

Proposition 1 shows the idempotency of the eight approximation operators defined in our MGCRS models.

Remark 1. The following two properties do not hold generally.

(1) $\underline{\nabla_{\sum_{i=1}^{m} C_i}}(\underline{\nabla_{\sum_{i=1}^{m} C_i}}(\sim X)) = \sim \underline{\nabla_{\sum_{i=1}^{m} C_i}}(X)$;
(2) $\overline{\nabla_{\sum_{i=1}^{m} C_i}}(\overline{\nabla_{\sum_{i=1}^{m} C_i}}(\sim X)) = \sim \overline{\nabla_{\sum_{i=1}^{m} C_i}}(X)$.

Proof. According to Theorem 1 and Proposition 1, we have that

$$\begin{aligned}
\underline{\nabla_{\sum_{i=1}^{m} C_i}}\big(\underline{\nabla_{\sum_{i=1}^{m} C_i}}(\sim X)\big) &= \underline{\nabla_{\sum_{i=1}^{m} C_i}}\big(\sim \overline{\nabla_{\sum_{i=1}^{m} C_i}}(X)\big)\\
&= \sim \overline{\nabla_{\sum_{i=1}^{m} C_i}}\big(\overline{\nabla_{\sum_{i=1}^{m} C_i}}(X)\big)\\
&= \sim \overline{\nabla_{\sum_{i=1}^{m} C_i}}(X);\\
\overline{\nabla_{\sum_{i=1}^{m} C_i}}\big(\overline{\nabla_{\sum_{i=1}^{m} C_i}}(\sim X)\big) &= \overline{\nabla_{\sum_{i=1}^{m} C_i}}\big(\sim \underline{\nabla_{\sum_{i=1}^{m} C_i}}(X)\big)\\
&= \sim \underline{\nabla_{\sum_{i=1}^{m} C_i}}\big(\underline{\nabla_{\sum_{i=1}^{m} C_i}}(X)\big)\\
&= \sim \underline{\nabla_{\sum_{i=1}^{m} C_i}}(X).
\end{aligned}$$

Therefore, properties (1) and (2) are true only for all definable sets. If X is not a definable set, then (1) and (2) will not hold. □

Example 2 *(Continued from Example 1).* Since $X = \{a, d\}$, we have that $\sim X = \{b, c\}$. From Definitions 7–10, we have the following results.

(1) Since $\underline{FR_{C_1+C_2}}(\sim X) = \{b\}$, we have that $\underline{FR_{C_1+C_2}}(\underline{FR_{C_1+C_2}}(\sim X)) = \{b\}$. Moreover, since $\sim \underline{FR_{C_1+C_2}}(X) = \{b, c\}$, we can obtain that $\underline{FR_{C_1+C_2}}(\underline{FR_{C_1+C_2}}(\sim X)) \neq \sim \underline{FR_{C_1+C_2}}(X)$.

Since $\overline{FR_{C_1+C_2}}(\sim X) = \{b, c\}$, we have that $\overline{FR_{C_1+C_2}}(\overline{FR_{C_1+C_2}}(\sim X)) = \{b, c\}$. Moreover, since $\sim \overline{FR_{C_1+C_2}}(X) = \{b\}$, we have that $\overline{FR_{C_1+C_2}}(\overline{FR_{C_1+C_2}}(\sim X)) \neq \sim \overline{FR_{C_1+C_2}}(X)$.

(2) Since $\underline{SR_{C_1+C_2}}(\sim X) = \emptyset$, we have that $\underline{SR_{C_1+C_2}}(\underline{SR_{C_1+C_2}}(\sim X)) = \emptyset$. Moreover, since $\sim \underline{SR_{C_1+C_2}}(X) = \{a, b, c\}$, we have that $\underline{SR_{C_1+C_2}}(\underline{SR_{C_1+C_2}}(\sim X)) \neq \sim \underline{SR_{C_1+C_2}}(X)$.

Since $\overline{SR_{C_1+C_2}}(\sim X) = \{a, b, c\}$, we have that $\overline{SR_{C_1+C_2}}(\overline{SR_{C_1+C_2}}(\sim X)) = \{a, b, c\}$. Moreover, since $\sim \overline{SR_{C_1+C_2}}(X) = \emptyset$, we can obtain that $\overline{SR_{C_1+C_2}}(\overline{SR_{C_1+C_2}}(\sim X)) \neq \sim \overline{SR_{C_1+C_2}}(X)$.

(3) Since $\underline{TR_{C_1+C_2}}(\sim X) = \{b\}$, we have that $\underline{TR_{C_1+C_2}}(\underline{TR_{C_1+C_2}}(\sim X)) = \{b\}$. Moreover, since $\sim \underline{TR_{C_1+C_2}}(X) = \{b, c, d\}$, we have that $\underline{TR_{C_1+C_2}}(\underline{TR_{C_1+C_2}}(\sim X)) \neq \sim \underline{TR_{C_1+C_2}}(X)$.

Since $\overline{TR_{C_1+C_2}}(\sim X) = \{b, c, d\}$, we have that $\overline{TR_{C_1+C_2}}(\overline{TR_{C_1+C_2}}(\sim X)) = \{b, c, d\}$. Moreover, since $\sim \overline{TR_{C_1+C_2}}(X) = \{b\}$, we can obtain that $\overline{TR_{C_1+C_2}}(\overline{TR_{C_1+C_2}}(\sim X)) \neq \sim \overline{TR_{C_1+C_2}}(X)$.

(4) Since $\underline{LR_{C_1+C_2}}(\sim X) = \emptyset$, we have that $\underline{LR_{C_1+C_2}}(\underline{LR_{C_1+C_2}}(\sim X)) = \emptyset$. Moreover, since $\sim \underline{LR_{C_1+C_2}}(X) = \{a, b, c, d\}$, we have that $\underline{LR_{C_1+C_2}}(\underline{LR_{C_1+C_2}}(\sim X)) \neq \sim \underline{LR_{C_1+C_2}}(X)$.

Since $\overline{LR_{C_1+C_2}}(\sim X) = \{a, b, c, d\}$, we have that $\overline{LR_{C_1+C_2}}(\overline{LR_{C_1+C_2}}(\sim X)) = \{a, b, c, d\}$. Moreover, since $\sim \overline{LR_{C_1+C_2}}(X) = \emptyset$, we can obtain that $\overline{LR_{C_1+C_2}}(\overline{LR_{C_1+C_2}}(\sim X)) \neq \sim \overline{LR_{C_1+C_2}}(X)$. □

Proposition 2. *Let $\langle U, \mathbf{C}\rangle$ be a multi-granulation covering approximation space and $C_1, C_2, \ldots, C_m \in \mathbf{C}$. For any $X, Y \subseteq U$, the following properties are satisfied.*

(1) *If $X \subseteq Y$, then* $\underline{\nabla_{\sum_{i=1}^m C_i}}(X) \subseteq \underline{\nabla_{\sum_{i=1}^m C_i}}(Y)$;

(2) *If $X \subseteq Y$, then* $\overline{\nabla_{\sum_{i=1}^m C_i}}(X) \subseteq \overline{\nabla_{\sum_{i=1}^m C_i}}(Y)$;

(3) $\underline{\nabla_{\sum_{i=1}^m C_i}}(X \cap Y) \subseteq \underline{\nabla_{\sum_{i=1}^m C_i}}(X) \cap \underline{\nabla_{\sum_{i=1}^m C_i}}(Y)$;

(4) $\underline{\nabla_{\sum_{i=1}^m C_i}}(X \cup Y) \supseteq \underline{\nabla_{\sum_{i=1}^m C_i}}(X) \cup \underline{\nabla_{\sum_{i=1}^m C_i}}(Y)$;

(5) $\overline{\nabla_{\sum_{i=1}^m C_i}}(X \cup Y) \supseteq \overline{\nabla_{\sum_{i=1}^m C_i}}(X) \cup \overline{\nabla_{\sum_{i=1}^m C_i}}(Y)$;

(6) $\overline{\nabla_{\sum_{i=1}^m C_i}}(X \cap Y) \subseteq \overline{\nabla_{\sum_{i=1}^m C_i}}(X) \cap \overline{\nabla_{\sum_{i=1}^m C_i}}(Y)$.

Proof. Noting that properties (3), (4), (5) and (6) can be proved by employing (1) or (2), hence we only prove (1) and (2) here.

(1) From Definition 7, we have that $\underline{FR_{\sum_{i=1}^m C_i}}(X) = \{x \in U \mid \cap md_{C_1}(x) \subseteq X$ or $\cap md_{C_2}(x) \subseteq X$ or $\ldots$ or $\cap md_{C_m}(x) \subseteq X\}$. If $X \subseteq Y$, then $\cap md_{C_1}(x) \subseteq X \subseteq Y$, $\cap md_{C_2}(x) \subseteq X \subseteq Y, \ldots, \cap md_{C_m}(x) \subseteq X \subseteq Y$ hold. Therefore, $\underline{FR_{\sum_{i=1}^m C_i}}(X) \subseteq \underline{FR_{\sum_{i=1}^m C_i}}(Y)$.

One can prove the cases of type-2, type-3 and type-4 MGCRS in a similar manner, hence we omit them here.

(2) From Definition 7, we have that $\overline{FR_{\sum_{i=1}^m C_i}}(X) = \{x \in U \mid (\cap md_{C_1}(x)) \cap X \neq \emptyset$ and $(\cap md_{C_2}(x)) \cap X \neq \emptyset$ and $\ldots$ and $(\cap md_{C_m}(x)) \cap X \neq \emptyset\}$. If $X \subseteq Y$, then $(\cap md_{C_1}(x)) \cap Y \neq \emptyset$, $(\cap md_{C_2}(x)) \cap Y \neq \emptyset, \ldots, (\cap md_{C_m}(x)) \cap Y \neq \emptyset$. Therefore, $\overline{FR_{\sum_{i=1}^m C_i}}(X) \subseteq \overline{FR_{\sum_{i=1}^m C_i}}(Y)$.

One can prove the cases of type-2, type-3 and type-4 MGCRS in a similar manner, hence we also omit them here. This completes the proof of Proposition 2. □

In Proposition 2, (1) and (2) show the monotonicity of multi-granulation covering rough approximations, w.r.t. the variety of target or concept, and each of (3)–(6) expresses the relationship between the rough approximation of $X \cap Y$ (or $X \cup Y$) with the intersection (or union) of the two rough approximations of X and Y under the multi-covering environment.

Example 3 *(Continued from Example 1).* Let $Y = \{a, b\}$, we have that $X \cap Y = \{a\}$, $X \cup Y = \{a, b, d\}$. For simplicity, we only calculate the case of type-1 MGCRS, the results in other cases can be shown similarly. According to Definition 7, we have the following results.

From Definition 7, we can obtain that $\underline{FR_{C_1+C_2}}(X \cap Y) = \{a\}$, $\overline{FR_{C_1+C_2}}(X \cap Y) = \{a\}$, $\underline{FR_{C_1+C_2}}(X \cup Y) = \{a, b, d\}$, $\overline{FR_{C_1+C_2}}(X \cup Y) = \{a, b, c, d\}$, $\underline{FR_{C_1+C_2}}(Y) = \{a, b\}$, and $\overline{FR_{C_1+C_2}}(Y) = \{a, b\}$.

Therefore, we have that

$$\underline{FR_{C_1+C_2}}(Y) = \{a, b\} \subset \underline{FR_{C_1+C_2}}(X \cup Y) = \{a, b, d\};$$

$$\overline{FR_{C_1+C_2}}(Y) = \{a, b\} \subset \overline{FR_{C_1+C_2}}(X \cup Y) = \{a, b, c, d\};$$

$$\underline{FR_{C_1+C_2}}(X \cup Y) = \{a, b, d\} \supseteq \underline{FR_{C_1+C_2}}(X) \cup \underline{FR_{C_1+C_2}}(Y) = \{a, b, d\};$$

$$\overline{FR_{C_1+C_2}}(X \cap Y) = \{a\} \subseteq \overline{FR_{C_1+C_2}}(X) \cap \overline{FR_{C_1+C_2}}(Y) = \{a\};$$

$$\underline{FR_{C_1+C_2}}(X\cap Y)=\{a\}\subseteq \underline{FR_{C_1+C_2}}(X)\cap \underline{FR_{C_1+C_2}}(Y)=\{a\};$$
$$\overline{FR_{C_1+C_2}}(X\cup Y)=\{a,b,c,d\}\supseteq \overline{FR_{C_1+C_2}}(X)\cup \overline{FR_{C_1+C_2}}(Y)=\{a,b,c,d\}.$$

From Example 3, we can see that the results obtained from the above example are in accordance with those given in Proposition 2.

Proposition 3. *Let $\langle U,\mathbf{C}\rangle$ be a multi-granulation covering approximation space, and $C_1,C_2,\dots,C_m\in\mathbf{C}$, where $C_i=\{C_{i1},C_{i2},\dots,C_{in}\}$, $1\leqslant i\leqslant m$ and m,n are two natural numbers. For any $K_j\in C_i$, where $j\in\{i1,i2,\dots,in\}$, we have that $\underline{\nabla_{\sum_{i=1}^m C_i}}(K_j)=K_j$.*

Proof. It is straightforward according to the definitions of MGCRS. □

From Proposition 3, we can see that for any element in the coverings which construct a given MGCRS model, its lower approximation in that model is itself, but as shown in Remark 2, its upper approximation in that model may not be itself.

Remark 2. $\overline{\nabla_{\sum_{i=1}^m C_i}}(K_j)=K_j$ may not hold.

Remark 2 can be proved by the following example.

Example 4 *(Continued from Example 3).* As shown in Example 3, $X\cup Y=\{a,b,d\}$ is an element of C_2, but from Example 3, we have that

$$\overline{FR_{C_1+C_2}}(X\cup Y)=\{a,b,c,d\}\neq X\cup Y,\qquad \overline{SR_{C_1+C_2}}(X\cup Y)=\{a,b,c,d\}\neq X\cup Y,$$
$$\overline{TR_{C_1+C_2}}(X\cup Y)=\{a,b,c,d\}\neq X\cup Y,\qquad \overline{LR_{C_1+C_2}}(X\cup Y)=\{a,b,c,d\}\neq X\cup Y.$$

Definition 11. Let $\langle U,\mathbf{C}\rangle$ be a multi-granulation covering approximation space, $C_1,C_2\in\mathbf{C}$ and $reduct(C_1)=\{C_{11},C_{12},\dots,C_{1p}\}$, $reduct(C_2)=\{C_{21},C_{22},\dots,C_{2q}\}$. If for each $C_{1i}\in reduct(C_1)$, $1\leqslant i\leqslant p$, there exists $C_{2j}\in reduct(C_2)$, $1\leqslant j\leqslant q$, such that $C_{1i}\subseteq C_{2j}$, then we shall say that C_1 is finer than C_2, denoted by $C_1\preceq C_2$.

Obviously, if $C_1\preceq C_2$, then for any $x\in U$, we have that $md_{C_1}(x)\subseteq md_{C_2}(x)$ and $MD_{C_1}(x)\subseteq MD_{C_2}(x)$.

Theorem 2. *Let $\langle U,\mathbf{C}\rangle$ be a multi-granulation covering approximation space, and $C_1,C_2,\dots,C_m\in\mathbf{C}$. For any $X\subseteq U$, if $C_1\preceq C_2\preceq\cdots\preceq C_m$, then $\underline{\nabla_{\sum_{i=1}^m C_i}}(X)=\underline{\nabla_{C_1}}(X)$ and $\overline{\nabla_{\sum_{i=1}^m C_i}}(X)=\overline{\nabla_{C_1}}(X)$.*

Proof. For any $X\subseteq U$, if $C_1\preceq C_2\preceq\cdots\preceq C_m$, then from Definition 11, we have that $\underline{FR_{\sum_{i=1}^m C_i}}(X)=\{x\in U\mid \cap md_{C_1}(x)\subseteq X$ or $\cap md_{C_2}(x)\subseteq X$ or $\dots$ or $\cap md_{C_m}(x)\subseteq X\}=\{x\in U\mid \cap md_{C_1}(x)\subseteq X\}=\underline{FR_{C_1}}(X)$.

For any $X\subseteq U$, if $C_1\preceq C_2\preceq\cdots\preceq C_m$, then from Definition 11, we have that $\overline{FR_{\sum_{i=1}^m C_i}}(X)=\{x\in U\mid \cup md_{C_1}(x)\subseteq X$ and $\cup md_{C_2}(x)\subseteq X$ and $\dots$ and $\cup md_{C_m}(x)\subseteq X\}=\{x\in U\mid \cup md_{C_1}(x)\subseteq X\}=\overline{FR_{C_1}}(X)$.

One can prove the cases of type-2, type-3 and type-4 MGCRS in a similar manner, hence we omit them here. □

From Theorem 2, we can see that for any $X\subseteq U$, the approximations of X under multi-coverings equal to the approximations of X under the covering which is the finest among all the coverings.

Theorem 3. *Let $\langle U,\mathbf{C}\rangle$ be a multi-granulation covering approximation space, $C_1,C_2,\dots C_m\in\mathbf{C}$ and $reduct(C_1)$, $reduct(C_2),\dots,reduct(C_m)$ are the reducts of $C_1,C_2,\dots,C_m$ respectively. $FR_{\sum_{i=1}^m C_i}$ and $FR_{\sum_{i=1}^m reduct(C_i)}$, $SR_{\sum_{i=1}^m C_i}$ and $SR_{\sum_{i=1}^m reduct(C_i)}$ produce the identical multi-granulation covering rough sets.*

Proof. For any $x\in U$, C and $reduct(C)$ have the same $md(x)$, therefore, from Definitions 7 and 8, we can prove Theorem 3. □

Remark 3. $TR_{\sum_{i=1}^m C_i}$ and $TR_{\sum_{i=1}^m reduct(C_i)}$, $LR_{\sum_{i=1}^m C_i}$ and $LR_{\sum_{i=1}^m reduct(C_i)}$ may not produce the identical multi-granulation covering rough sets.

Proof. For any $x\in U$, C and $reduct(C)$ may produce different $MD(x)$, therefore, $TR_{\sum_{i=1}^m C_i}$ and $TR_{\sum_{i=1}^m reduct(C_i)}$, $LR_{\sum_{i=1}^m C_i}$ and $LR_{\sum_{i=1}^m reduct(C_i)}$ may not produce the identical multi-granulation covering rough sets. □

Theorem 3 and Remark 3 tell us that for the cases of type-1 and type-2 MGCRS, the coverings and their corresponding reducts produce identical multi-granulation covering rough sets, but it is not true for the cases of type-3 and type-4 MGCRS.

Definition 12. (See [8].) Let C be a covering of U, and K an element of C. If there exists another element K' of C such that $K \subset K'$, we shall say that K is an immured element of covering C.

Definition 13. (See [8].) Let C be a covering of U. If we remove all immured elements from C, and the new set is still a covering of U, then we call this new covering an **exclusion** of C, which is denoted by $exclusion(C)$.

Theorem 4. *Let $\langle U, \mathbf{C}\rangle$ be a multi-granulation covering approximation space, $C_1, C_2, C_3, C_4 \in \mathbf{C}$ and $reduct(C_1)$, $reduct(C_2)$, $reduct(C_3)$ and $reduct(C_4)$ be the reducts of C_1, C_2, C_3 and C_4 respectively. $\nabla_{C_1+C_2}$ and $\nabla_{C_3+C_4}$ produce the identical lower approximation operators of MGCRS, if at least one of the following conditions is satisfied.*

(1) $reduct(C_1) = reduct(C_3)$ *and* $exclusion(C_1) = exclusion(C_3)$;
(2) $reduct(C_1) = reduct(C_4)$ *and* $exclusion(C_1) = exclusion(C_4)$;
(3) $reduct(C_2) = reduct(C_3)$ *and* $exclusion(C_2) = exclusion(C_3)$;
(4) $reduct(C_2) = reduct(C_4)$ *and* $exclusion(C_2) = exclusion(C_4)$.

Proof. It is straightforward according to the definitions of MGCRS. □

Theorem 4 gives the sufficient conditions for two different MGCRS which produce the identical lower approximation operators.

Remark 4. The reverse of Theorem 4 is not true, i.e., $\nabla_{C_1+C_2}$ and $\nabla_{C_3+C_4}$ may produce the identical lower approximation operators of MGCRS, but their reducts or exclusions are not equal to each other.

Theorem 5. *Let $\langle U, \mathbf{C}\rangle$ be a multi-granulation covering approximation space, $C_1, C_2, C_3, C_4 \in \mathbf{C}$ and $reduct(C_1)$, $reduct(C_2)$, $reduct(C_3)$ and $reduct(C_4)$ be the reducts of C_1, C_2, C_3 and C_4 respectively. $FR_{C_1+C_2}$ and $FR_{C_3+C_4}$, $SR_{C_1+C_2}$ and $SR_{C_3+C_4}$ produce the identical multi-granulation covering rough sets, if one of the following conditions is satisfied.*

(1) $reduct(C_1) = reduct(C_3)$ *and* $reduct(C_2) = reduct(C_4)$;
(2) $reduct(C_1) = reduct(C_4)$ *and* $reduct(C_2) = reduct(C_3)$.

Proof. We only prove (1) here, since (2) can be analogously proved.

For any $x \in U$, C and $reduct(C)$ yield the identical $md(x)$. If $reduct(C_1) = reduct(C_3)$ and $reduct(C_2) = reduct(C_4)$, then C_1 and C_3 yield the same $md(x)$, and C_2 and C_4 share the another $md(x)$. From Definitions 7 and 8, $FR_{C_1+C_2}$ and $FR_{C_3+C_4}$, $SR_{C_1+C_2}$ and $SR_{C_3+C_4}$ produce the identical multi-granulation covering rough sets. □

Theorem 5 gives the sufficient conditions for two different type-1 or type-2 MGCRS which produce the identical lower and upper approximation operators.

Remark 5. The reverse of Theorem 5 does not hold, i.e., $FR_{C_1+C_2}$ and $FR_{C_3+C_4}$, $SR_{C_1+C_2}$ and $SR_{C_3+C_4}$ may produce the same multi-granulation covering rough sets, but their reducts may not be the same.

Remark 5 can be proved by the following example.

Example 5. Let $\langle U, \mathbf{C}\rangle$ be a multi-granulation covering approximation space, and $C_1, C_2, C_3, C_4 \in \mathbf{C}$, where $U = \{a, b, c\}$, $C_1 = \{\{a\}, \{b\}, \{c\}\}$, $C_2 = \{\{a, b\}, \{a, b, c\}, \{a, c\}\}$, $C_3 = \{\{a, b\}, \{b, c\}, \{a, c\}\}$, $C_4 = \{\{a, c\}, \{a, b, c\}\}$. From Definition 4, we can obtain that $reduct(C_1) = \{\{a\}, \{b\}, \{c\}\}$, $reduct(C_2) = \{\{a, b\}, \{a, c\}\}$, $reduct(C_3) = \{\{a, b\}, \{b, c\}, \{a, c\}\}$, and $reduct(C_4) = \{\{a, c\}, \{a, b, c\}\}$.

Hence, we have that

$$reduct(C_1) \neq reduct(C_3), \qquad reduct(C_1) \neq reduct(C_4);$$
$$reduct(C_2) \neq reduct(C_3), \qquad reduct(C_2) \neq reduct(C_4).$$

Suppose that $X = \{a, c\}$ and $Y = \{a, b, c\}$, from Definitions 7 and 8, we can obtain that

$$\underline{FR_{C_1+C_2}}(X) = \{a, c\} = \underline{FR_{C_3+C_4}}(X), \qquad \overline{FR_{C_1+C_2}}(X) = \{a, c\} = \overline{FR_{C_3+C_4}}(X);$$
$$\underline{SR_{C_1+C_2}}(Y) = \{a, b, c\} = \underline{SR_{C_3+C_4}}(Y), \qquad \overline{SR_{C_1+C_2}}(Y) = \{a, b, c\} = \overline{SR_{C_3+C_4}}(Y).$$

Theorem 6. *Let $\langle U, \mathbf{C}\rangle$ be a multi-granulation covering approximation space, $C_1, C_2, C_3, C_4 \in \mathbf{C}$ and $reduct(C_1)$, $reduct(C_2)$, $reduct(C_3)$ and $reduct(C_4)$ be the reducts of C_1, C_2, C_3 and C_4 respectively. $TR_{C_1+C_2}$ and $TR_{C_3+C_4}$, $LR_{C_1+C_2}$ and $LR_{C_3+C_4}$ produce the identical upper approximation operators of MGCRS, if one of the following conditions is satisfied.*

(1) $reduct(C_1) = reduct(C_3)$ *and* $reduct(C_2) = reduct(C_4)$;
(2) $reduct(C_1) = reduct(C_4)$ *and* $reduct(C_2) = reduct(C_3)$.

Proof. It can be analogously proved as Theorem 5. $\square$

Theorem 6 gives the sufficient conditions for two different type-3 or type-4 MGCRS which produce the identical upper approximation operators.

Remark 6. $TR_{C_1+C_2}$ and $TR_{C_3+C_4}$, $LR_{C_1+C_2}$ and $LR_{C_3+C_4}$ may produce different lower approximation operators of MGCRS, even if one of the following conditions is satisfied.

(1) $reduct(C_1) = reduct(C_3)$ and $reduct(C_2) = reduct(C_4)$;
(2) $reduct(C_1) = reduct(C_4)$ and $reduct(C_2) = reduct(C_3)$.

Remark 6 can be proved by the following example.

Example 6. Let $\langle U, \mathbf{C}\rangle$ be a multi-granulation covering approximation space, and $C_1, C_2, C_3, C_4 \in \mathbf{C}$, where $U = \{a, b, c\}$, $C_1 = \{\{a, c\}, \{b, c\}, \{a, b, c\}\}$, $C_2 = \{\{a, b\}, \{a, b, c\}, \{a, c\}\}$, $C_3 = \{\{a, c\}, \{b, c\}\}$, $C_4 = \{\{a, b\}, \{a, c\}\}$. From Definition 4, we can obtain that $reduct(C_1) = \{\{a, c\}, \{b, c\}\}$, $reduct(C_2) = \{\{a, b\}, \{a, c\}\}$, $reduct(C_3) = \{\{a, c\}, \{b, c\}\}$, $reduct(C_4) = \{\{a, b\}, \{a, c\}\}$.

Hence, we have that

$$reduct(C_1) = reduct(C_3), \qquad reduct(C_2) = reduct(C_4).$$

Suppose that $X = \{a, c\}$, from Definitions 8 and 9, we can obtain that

$$\emptyset = \underline{TR_{C_1+C_2}}(X) \neq \underline{TR_{C_3+C_4}}(X) = \{a, c\},$$
$$\emptyset = \underline{LR_{C_1+C_2}}(X) \neq \underline{LR_{C_3+C_4}}(X) = \{a, c\}.$$

4. Relationships among the four types of MGCRS models

In this section, we shall investigate the relationships among the four types of MGCRS models introduced in Section 3. Moreover, some interesting properties are also discussed here.

From the definitions of MGCRS, we can obtain the following theorem.

Theorem 7. *Let $\langle U, \mathbf{C}\rangle$ be a multi-granulation covering approximation space, and $C_1, C_2, \ldots, C_m \in \mathbf{C}$. For any $X \subseteq U$, we have that*

(1) $\underline{LR_{\sum_{i=1}^m C_i}}(X) \subseteq \underline{SR_{\sum_{i=1}^m C_i}}(X) \subseteq \underline{FR_{\sum_{i=1}^m C_i}}(X) \subseteq X$;
(2) $X \subseteq \overline{FR_{\sum_{i=1}^m C_i}}(X) \subseteq \overline{SR_{\sum_{i=1}^m C_i}}(X) \subseteq \overline{LR_{\sum_{i=1}^m C_i}}(X)$;
(3) $\underline{LR_{\sum_{i=1}^m C_i}}(X) \subseteq \underline{TR_{\sum_{i=1}^m C_i}}(X) \subseteq \underline{FR_{\sum_{i=1}^m C_i}}(X) \subseteq X$;
(4) $X \subseteq \overline{FR_{\sum_{i=1}^m C_i}}(X) \subseteq \overline{TR_{\sum_{i=1}^m C_i}}(X) \subseteq \overline{LR_{\sum_{i=1}^m C_i}}(X)$.

The proof of Theorem 7 is easy, so we omit it. What interests us is that for any $X \subseteq U$, the set of all the four types of lower and upper approximations in Theorem 7 equipped with the binary relation $\subseteq$ of inclusion, can construct a lattice. For the readers' convenience, the relationships among the four types of MGCRS models are shown in Fig. 1. In Fig. 1, each node denotes an approximation or a concept, and each line connects two approximations, where the lower node is a subset of the upper node.

From Fig. 1, we can obtain the following meaningful conclusions.

First, from Fig. 1, we can see that among the four pairs of lower and upper approximations, the pair $(\underline{FR_{\sum_{i=1}^m C_i}}(X), \overline{FR_{\sum_{i=1}^m C_i}}(X))$ is the best to describe X, since in Fig. 1 the two approximations are closer to X than other approximations. Moreover, the pair $(\underline{LR_{\sum_{i=1}^m C_i}}(X), \overline{LR_{\sum_{i=1}^m C_i}}(X))$ is the worst to describe X, since in Fig. 1 the two approximations have the furthest distance with X with respect to other approximations.

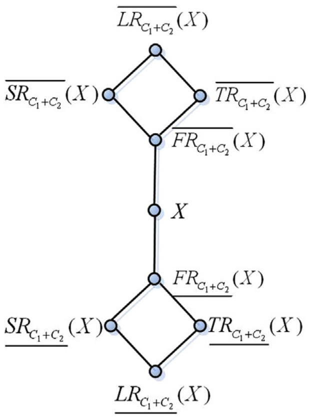

Fig. 1. Relationships among various MGCRS models.

Second, from Fig. 1, we can also see that $\underline{SR}_{\sum_{i=1}^m C_i}(X)$ and $\underline{TR}_{\sum_{i=1}^m C_i}(X)$ are generally not equal, and one does not contain another, that is, neither $\underline{SR}_{\sum_{i=1}^m C_i}(X) \subseteq \underline{TR}_{\sum_{i=1}^m C_i}(X)$ nor $\underline{TR}_{\sum_{i=1}^m C_i}(X) \subseteq \underline{SR}_{\sum_{i=1}^m C_i}(X)$, that is to say, $\underline{SR}_{\sum_{i=1}^m C_i}(X)$ and $\underline{TR}_{\sum_{i=1}^m C_i}(X)$ are independent. So do $\overline{SR}_{\sum_{i=1}^m C_i}(X)$ and $\overline{TR}_{\sum_{i=1}^m C_i}(X)$.

Finally, from Fig. 1, we can obtain the following theorem, which contains some important conclusions about the relationships among the four pairs of lower and upper approximations.

Theorem 8. *Let $\langle U, \mathbf{C}\rangle$ be a multi-granulation covering approximation space, and $C_1, C_2, \ldots, C_m \in \mathbf{C}$. For any $X \subseteq U$, we have that*

(1) $\underline{SR}_{\sum_{i=1}^m C_i}(X) \cup \underline{TR}_{\sum_{i=1}^m C_i}(X) = \underline{FR}_{\sum_{i=1}^m C_i}(X)$;
(2) $\underline{SR}_{\sum_{i=1}^m C_i}(X) \cap \underline{TR}_{\sum_{i=1}^m C_i}(X) = \underline{LR}_{\sum_{i=1}^m C_i}(X)$;
(3) $\overline{SR}_{\sum_{i=1}^m C_i}(X) \cup \overline{TR}_{\sum_{i=1}^m C_i}(X) = \overline{LR}_{\sum_{i=1}^m C_i}(X)$;
(4) $\overline{SR}_{\sum_{i=1}^m C_i}(X) \cap \overline{TR}_{\sum_{i=1}^m C_i}(X) = \overline{FR}_{\sum_{i=1}^m C_i}(X)$.

Proof. It can be obtained directly from Fig. 1. □

From Theorem 8, we can see that there exist some meaningful relationships among the four pairs of lower and upper approximations. For instance, from (1) of Theorem 8, we can see that for any $X \subseteq U$, the type-1 lower approximation of X is equal to the union of the type-2 and type-3 lower approximations of X. Moreover, from (2)–(4) of Theorem 8, we can obtain similar conclusions.

The above results about the four pairs of lower and upper approximations given in Fig. 1, can be shown by the following example.

Example 7 *(Continued from Example 1).* Let $U = \{a, b, c, d\}$, $C_1 = \{\{a, b\}, \{b, c, d\}, \{c, d\}\}$, $C_2 = \{\{a, c\}, \{b, d\}, \{a, b, d\}, \{d\}\}$, $X = \{a, d\}$.

From Definitions 7–10, we can obtain that $\underline{FR}_{C_1+C_2}(X) = \{a, d\}$, $\overline{FR}_{C_1+C_2}(X) = \{a, c, d\}$, $\underline{SR}_{C_1+C_2}(X) = \{d\}$, $\overline{SR}_{C_1+C_2}(X) = \{a, b, c, d\}$, $\underline{TR}_{C_1+C_2}(X) = \{a\}$, $\overline{TR}_{C_1+C_2}(X) = \{a, c, d\}$, $\underline{LR}_{C_1+C_2}(X) = \emptyset$, $\overline{LR}_{C_1+C_2}(X) = \{a, b, c, d\}$.

Hence, we have that

(1) $\underline{LR}_{C_1+C_2}(X) \subset \underline{SR}_{C_1+C_2}(X) \subset \underline{FR}_{C_1+C_2}(X) \subseteq X$;
(2) $X \subset \overline{FR}_{C_1+C_2}(X) \subset \overline{SR}_{C_1+C_2}(X) \subseteq \overline{LR}_{C_1+C_2}(X)$;
(3) $\underline{LR}_{C_1+C_2}(X) \subset \underline{TR}_{C_1+C_2}(X) \subset \underline{FR}_{C_1+C_2}(X) \subseteq X$;
(4) $X \subset \overline{FR}_{C_1+C_2}(X) \subseteq \overline{TR}_{C_1+C_2}(X) \subset \overline{LR}_{C_1+C_2}(X)$;
(5) $\underline{SR}_{C_1+C_2}(X) \cup \underline{TR}_{C_1+C_2}(X) = \underline{FR}_{C_1+C_2}(X)$;
(6) $\underline{SR}_{C_1+C_2}(X) \cap \underline{TR}_{C_1+C_2}(X) = \underline{LR}_{C_1+C_2}(X)$;
(7) $\overline{SR}_{C_1+C_2}(X) \cup \overline{TR}_{C_1+C_2}(X) = \overline{LR}_{C_1+C_2}(X)$;
(8) $\overline{SR}_{C_1+C_2}(X) \cap \overline{TR}_{C_1+C_2}(X) = \overline{FR}_{C_1+C_2}(X)$. □

Next, we review some basic definitions of lattice theory and discuss the properties of the lattice as shown in Fig. 1.

Definition 14. (See [38].) A lattice L is a distributive lattice if $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ or $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, for all $A, B, C \in L$.

Definition 15. (See [38].) Let $\langle L, \cup, \cap \rangle$ be a lattice. We say that L has a maximal element if there exists an element 1 such that $A \cap 1 = A$ for all $A \in L$. Dually, L is said to have a minimal element if there exists an element 0 such that $A \cup 0 = A$ for all $A \in L$. A lattice $\langle L, \cup, \cap \rangle$ possessing 0 and 1 is called a bounded lattice.

Definition 16. (See [38].) Let L be a bounded lattice with a maximal element 1 and a minimal element 0. For an element $A \in L$, we say that an element $B \in L$ is a complement of A if $A \cup B = 1$ and $A \cap B = 0$. A lattice L is a complemented lattice if each element has a complement.

Definition 17. (See [38].) A lattice L is called a boolean lattice if it is a complemented and distributive lattice.

Theorem 9. *The lattice as shown in Fig. 1 is a distributive lattice.*

Proof. The proof of Theorem 9 can be given according to Theorems 7 and 8. For any A, B, C belong to the lattice as shown in Fig. 1, one can prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. We may take $\overline{FR_{\sum_{i=1}^m C_i}}(X)$, $\underline{SR_{\sum_{i=1}^m C_i}}(X)$ and $\overline{LR_{\sum_{i=1}^m C_i}}(X)$ as an example, other cases can be proved in a similar manner and hence omitted.

On the one hand, we have that

$$\overline{FR_{\sum_{i=1}^m C_i}}(X) \cup \left(\underline{SR_{\sum_{i=1}^m C_i}}(X) \cap \overline{LR_{\sum_{i=1}^m C_i}}(X)\right) = \overline{FR_{\sum_{i=1}^m C_i}}(X) \cup \underline{SR_{\sum_{i=1}^m C_i}}(X) = \overline{FR_{\sum_{i=1}^m C_i}}(X).$$

On the other hand, we have that

$$\left(\overline{FR_{\sum_{i=1}^m C_i}}(X) \cup \underline{SR_{\sum_{i=1}^m C_i}}(X)\right) \cap \left(\overline{FR_{\sum_{i=1}^m C_i}}(X) \cup \overline{LR_{\sum_{i=1}^m C_i}}(X)\right) = \overline{FR_{\sum_{i=1}^m C_i}}(X) \cap \overline{LR_{\sum_{i=1}^m C_i}}(X) = \overline{FR_{\sum_{i=1}^m C_i}}(X).$$

Therefore,

$$\begin{aligned}&\overline{FR_{\sum_{i=1}^m C_i}}(X) \cup \left(\underline{SR_{\sum_{i=1}^m C_i}}(X) \cap \overline{LR_{\sum_{i=1}^m C_i}}(X)\right)\\ &\quad = \left(\overline{FR_{\sum_{i=1}^m C_i}}(X) \cup \underline{SR_{\sum_{i=1}^m C_i}}(X)\right) \cap \left(\overline{FR_{\sum_{i=1}^m C_i}}(X) \cup \overline{LR_{\sum_{i=1}^m C_i}}(X)\right). \qquad \square\end{aligned}$$

Theorem 10. *The lattice as shown in Fig. 1 is a bounded lattice.*

Proof. Obviously, the maximal element of the lattice is $\overline{LR_{\sum_{i=1}^m C_i}}(X)$ and the minimal element is $\underline{LR_{\sum_{i=1}^m C_i}}(X)$. $\square$

Theorem 11. *The lattice as shown in Fig. 1 is not a complemented lattice.*

Proof. Noting that elements $\underline{SR_{\sum_{i=1}^m C_i}}(X)$, $\overline{SR_{\sum_{i=1}^m C_i}}(X)$, $\underline{TR_{\sum_{i=1}^m C_i}}(X)$, $\overline{TR_{\sum_{i=1}^m C_i}}(X)$ of the lattice have no complements. Hence, from Definition 16, we can obtain that the lattice is not a complemented lattice. $\square$

Theorem 12. *The lattice as shown in Fig. 1 is not a boolean lattice.*

Proof. Noting that the lattice is not a complemented lattice, thus it is also not a boolean lattice. $\square$

Theorems 9–12 describe some important properties about the lattice given in Fig. 1.

Definition 18. (See [9].) Let C be a covering of a set U. C is said to be unary if for any $x \in U$, $|md(x)| = 1$, where $|\cdot|$ denotes the cardinality of a set.

C is also said to be unary if for any $x \in U$, $|MD(x)| = 1$.

Theorem 13. *Let $\langle U, \mathbf{C} \rangle$ be a multi-granulation covering approximation space, and $C_1, C_2, \ldots, C_m \in \mathbf{C}$. For any $X \subseteq U$, if both $C_1, C_2, \ldots, C_m$ are unary, then we have that*

(1) $\underline{FR_{\sum_{i=1}^m C_i}}(X) = \underline{SR_{\sum_{i=1}^m C_i}}(X)$,
(2) $\overline{FR_{\sum_{i=1}^m C_i}}(X) = \overline{SR_{\sum_{i=1}^m C_i}}(X)$,

(3) $\underline{TR_{\sum_{i=1}^{m} C_i}}(X) = \underline{LR_{\sum_{i=1}^{m} C_i}}(X)$,
(4) $\overline{TR_{\sum_{i=1}^{m} C_i}}(X) = \overline{LR_{\sum_{i=1}^{m} C_i}}(X)$.

Proof. We only prove (1) and (2) here, (3) and (4) can be analogously proved.

If $C_1, C_2, \dots, C_m$ are unary, then for any $x \in U$, we have that $|md_{C_1}(x)| = 1$, $|md_{C_2}(x)| = 1, \dots, |md_{C_m}(x)| = 1$. Therefore, for any $x \in U$, $\cap md_{C_1}(x) = \cup md_{C_1}(x), \cap md_{C_2}(x) = \cup md_{C_2}(x), \dots, \cap md_{C_2}(x) = \cup md_{C_m}(x)$. According to Definitions 7 and 8, we have that $\underline{FR_{\sum_{i=1}^{m} C_i}}(X) = \underline{SR_{\sum_{i=1}^{m} C_i}}(X)$ and $\overline{FR_{\sum_{i=1}^{m} C_i}}(X) = \overline{SR_{\sum_{i=1}^{m} C_i}}(X)$. □

Theorem 13 gives the sufficient condition for type-1 and type-2 MGCRS which produce the identical approximations. Moreover, Theorem 13 also gives the sufficient condition for type-3 and type-4 MGCRS which produce the identical approximations.

Remark 7. The reverse of Theorem 13 is not true, i.e., the lower (upper) approximation of type-1 MGCRS and the lower (upper) approximation of type-2 MGCRS are identical, and the lower (upper) approximation of type-3 MGCRS and the lower (upper) approximation of type-4 MGCRS are identical, but none of $C_1, C_2, \dots, C_m$ is unary.

Remark 7 can be proved by the following example.

Example 8. Let $\langle U, \mathbf{C} \rangle$ be a multi-granulation covering approximation space, and $C_1, C_2 \in \mathbf{C}$, where $U = \{a, b, c\}$, $C_1 = \{\{a, b\}, \{a, c\}\}$, $C_2 = \{\{a, c\}, \{b, c\}\}$.

Suppose that $X = \{a, c\}$, we can obtain that

$$md_{C_1}(a) = \{\{a, b\}, \{a, c\}\}, \quad md_{C_1}(b) = \{\{a, b\}\}, \quad md_{C_1}(c) = \{\{a, c\}\};$$
$$md_{C_2}(a) = \{\{a, c\}\}, \quad md_{C_2}(b) = \{\{b, c\}\}, \quad md_{C_2}(c) = \{\{a, c\}, \{b, c\}\};$$
$$MD_{C_1}(a) = \{\{a, b\}, \{a, c\}\}, \quad MD_{C_1}(b) = \{\{a, b\}\}, \quad MD_{C_1}(c) = \{\{a, c\}\};$$
$$MD_{C_2}(a) = \{\{a, c\}\}, \quad MD_{C_2}(b) = \{\{b, c\}\}, \quad MD_{C_2}(c) = \{\{a, c\}, \{b, c\}\}.$$

Hence, we can further obtain that

$$\cap md_{C_1}(a) = \{a\}, \quad \cap md_{C_1}(b) = \{a, b\}, \quad \cap md_{C_1}(c) = \{a, c\};$$
$$\cup md_{C_1}(a) = \{a, b, c\}, \quad \cup md_{C_1}(b) = \{a, b\}, \quad \cup md_{C_1}(c) = \{a, c\};$$
$$\cap md_{C_2}(a) = \{a, c\}, \quad \cap md_{C_2}(b) = \{b, c\}, \quad \cap md_{C_2}(c) = \{c\};$$
$$\cup md_{C_2}(a) = \{a, c\}, \quad \cup md_{C_2}(b) = \{b, c\}, \quad \cup md_{C_2}(c) = \{a, b, c\};$$
$$\cap MD_{C_1}(a) = \{a\}, \quad \cap MD_{C_1}(b) = \{a, b\}, \quad \cap MD_{C_1}(c) = \{a, c\};$$
$$\cup MD_{C_1}(a) = \{a, b, c\}, \quad \cup MD_{C_1}(b) = \{a, b\}, \quad \cup MD_{C_1}(c) = \{a, c\};$$
$$\cap MD_{C_2}(a) = \{a, c\}, \quad \cap MD_{C_2}(b) = \{b, c\}, \quad \cap MD_{C_2}(c) = \{c\};$$
$$\cup MD_{C_2}(a) = \{a, c\}, \quad \cup MD_{C_2}(b) = \{b, c\}, \quad \cup MD_{C_2}(c) = \{a, b, c\}.$$

Obviously, both C_1 and C_2 are not unary. But from the definitions of MGCRS, we have that

$$\underline{FR_{C_1+C_2}}(X) = \{a, c\} = \underline{SR_{C_1+C_2}}(X) \quad \text{and} \quad \overline{FR_{C_1+C_2}}(X) = \{a, b, c\} = \overline{SR_{C_1+C_2}}(X);$$
$$\underline{TR_{C_1+C_2}}(X) = \{a, c\} = \underline{LR_{C_1+C_2}}(X) \quad \text{and} \quad \overline{TR_{C_1+C_2}}(X) = \{a, b, c\} = \overline{LR_{C_1+C_2}}(X).$$

Theorem 14. *Let $\langle U, \mathbf{C} \rangle$ be a multi-granulation covering approximation space, and $C_1, C_2, \dots, C_m \in \mathbf{C}$. If both $C_1, C_2, \dots, C_m$ are partitions of U, then for any $X \subseteq U$, we have that*

(1) $\underline{FR_{\sum_{i=1}^{m} C_i}}(X) = \underline{SR_{\sum_{i=1}^{m} C_i}}(X) = \underline{TR_{\sum_{i=1}^{m} C_i}}(X) = \underline{LR_{\sum_{i=1}^{m} C_i}}(X)$;
(2) $\overline{FR_{\sum_{i=1}^{m} C_i}}(X) = \overline{SR_{\sum_{i=1}^{m} C_i}}(X) = \overline{TR_{\sum_{i=1}^{m} C_i}}(X) = \overline{LR_{\sum_{i=1}^{m} C_i}}(X)$.

The above lower and upper approximations are the same as the lower and upper approximations of Qian et al.'s MGRS, respectively.

Theorem 14 gives the sufficient and necessary conditions for the four types of MGCRS defined in this paper and Qian et al.'s MGRS which produce the identical approximations.

5. Conclusions

To extend the application domain of MGRS, this paper has introduced four types of multi-granulation covering rough set models and studied the basic properties of these models. We have shown that our MGCRS models are generalized versions of MGRS models. Moreover, we have given the sufficient conditions for two distinct MGCRS models which produce identical lower and upper approximations of a given concept in a covering approximation space. Finally, the relationships among various MGCRS models were discussed. We have shown that for any $X \subseteq U$, the set of all those lower and upper approximations equipped with the binary relation $\subseteq$ of inclusion, can construct a lattice.

In the future work, we shall further discuss other aspects of MGCRS, for example, the topological properties of MGCRS, or MGCRS in fuzzy settings.

Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments and helpful suggestions. The work is supported by the National Natural Science Foundation of China under Grant No. 61273304, China National Natural Science Foundation of Youth Science Foundation under Grant Nos. 61103067, 61203247, 61203285, 61305052.

References

[1] Z. Pawlak, Rough sets, Int. J. Comput. Inf. Sci. 11 (1982) 341–356.
[2] Y.Y. Yao, T.Y. Lin, Graded rough set approximations based on nested neighborhood systems, in: Proceedings of 5th European Congress on Intelligent Techniques and Soft Computing, vol. 1, Aachen, Verlag Mainz, 1997, pp. 196–200.
[3] Y.Y. Yao, Perspectives of granular computing, in: Proceedings of 2005 IEEE International Conference on Granular Computing, vol. 1, 2005, pp. 85–90.
[4] Y.Y. Yao, B. Yao, Covering based rough set approximations, Inf. Sci. 200 (2012) 91–107.
[5] W. Zakowski, Approximations in the space (U, Π), Demonstr. Math. 16 (1983) 761–769.
[6] Z. Bonikowski, E. Bryniarski, U.W. Skardowska, Extension and intentions in the rough set theory, Inf. Sci. 107 (1998) 149–167.
[7] E. Tsang, D. Cheng, J. Lee, D. Yeung, On the upper approximations of covering generalized rough sets, in: Proceedings of the 3rd International Conference on Machine Learning and Cybernetics, 2004, pp. 4200–4203.
[8] W. Zhu, F.Y. Wang, Reduction and axiomatization of covering generalized rough sets, Inf. Sci. 152 (2003) 217–230.
[9] W. Zhu, Generalized rough sets based on relations, Inf. Sci. 177 (22) (2007) 4997–5011.
[10] L.A. Zadeh, Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic, Fuzzy Sets Syst. 90 (1997) 111–127.
[11] Y.H. Qian, J.Y. Liang, Rough set method based on multi-granulations, in: Proc. 5th IEEE International Conference on Cognitive Informatics, Beijing, China, July 17–19, 2006, pp. 297–304.
[12] Y.H. Qian, J.Y. Liang, Y.Y. Yao, C.Y. Dang, MGRS: A multi-granulation rough set, Inf. Sci. 180 (2010) 949–970.
[13] Y.H. Qian, J.Y. Liang, C.Y. Dang, Incomplete multigranulation rough set, IEEE Trans. Syst. Man Cybern., Part A 20 (2010) 420–430.
[14] Y.H. Qian, J.Y. Liang, W. Pedrycz, C.Y. Dang, Positive approximation: an accelerator for attribute reduction in rough set theory, Artif. Intell. 174 (2010) 597–618.
[15] X.B. Yang, X.N. Song, Z.H. Chen, J.Y. Yang, On multigranulation rough sets in incomplete information system, Int. J. Mach. Learn. Cybern. 3 (2011) 223–232.
[16] X.B. Yang, X.N. Song, H.L. Dou, J.Y. Yang, Multi-granulation rough set: from crisp to fuzzy case, Ann. Fuzzy Math. Inform. 1 (1) (2011) 55–70.
[17] X.B. Yang, Y.H. Qian, J.Y. Yang, Hierarchical structures on multigranulation spaces, J. Comput. Sci. Technol. 27 (6) (2012) 1169–1183.
[18] C.H. Liu, D.Q. Miao, Covering rough set model based on multi-granulations, in: Proceedings of the Thirteenth International Conference on Rough Sets, Fuzzy Set, Data Mining and Granular Computing, in: LNCS (LNAI), vol. 6743, 2011, pp. 87–90.
[19] C.H. Liu, M.Z. Wang, Covering fuzzy rough set based on multi-granulations, in: Proceedings of International Conference on Uncertainty Reasoning and Knowledge Engineering, vol. 2, 2011, pp. 146–149.
[20] G.P. Lin, Y.H. Qian, J.J. Li, NMGRS: Neighborhood-based multigranulation rough sets, Int. J. Approx. Reason. 53 (7) (2012) 1080–1093.
[21] W.H. Xu, Q.R. Wang, X.T. Zhang, Multi-granulation fuzzy rough set model on tolerance relations, in: Proceedings of the Fourth International Workshop on Advanced Computational Intelligence, Wuhan, Hubei, China, October 19–21, 2006, pp. 357–364.
[22] W.H. Xu, W.X. Sun, X.Y. Zhang, W.X. Zhang, Multiple granulation rough set approach to ordered information systems, Int. J. Gen. Syst. 41 (5) (2012) 475–501.
[23] W.H. Xu, W.X. Zhang, Measuring roughness of generalized rough sets induced by a covering, Fuzzy Sets Syst. 158 (2007) 2443–2455.
[24] W.H. Xu, Q.R. Wang, Multi-granulation fuzzy rough sets, J. Intell. Fuzzy Syst., http://dx.doi.org/10.3233/IFS-130818.
[25] W.H. Xu, Q.R. Wang, X.T. Zhang, Multi-granulation rough sets based on tolerance relations, Soft Comput. 17 (7) (2013) 1241–1252.
[26] W.H. Xu, Q.R. Wang, X.T. Zhang, Multi-granulation fuzzy rough sets in a fuzzy tolerance approximation space, Int. J. Fuzzy Syst. 13 (4) (2011) 246–259.
[27] W.H. Xu, X.T. Zhang, Q.R. Wang, A Generalized Multi-Granulation Rough Set Approach, in: Lecture Notes in Bioinformatics, vol. 6840, 2012, pp. 681–689.
[28] Y.H. She, X.L. He, On the structure of the multigranulation rough set model, Knowl.-Based Syst. 36 (2012) 81–92.
[29] D.W. Pei, Z.B. Xu, Transformation of rough set models, Knowl.-Based Syst. 20 (2007) 745–751.
[30] A. Skowron, J. Stepaniuk, Tolerance approximation spaces, Fundam. Inform. 27 (1996) 245–253.
[31] R. Slowinski, J. Stefanowski, Rough-set reasoning about uncertain data, Fundam. Inform. 27 (1996) 229–243.
[32] M. Kryszkiewicz, Rough set approach to incomplete information systems, Inf. Sci. 112 (1998) 39–49.
[33] D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, Int. J. Gen. Syst. 17 (1990) 191–209.
[34] W. Ziarko, Variable precision rough set model, J. Comput. Syst. Sci. 46 (1) (1993) 39–59.
[35] W. Ziarko, Analysis of uncertain information in the framework of variable precision rough sets, Found. Comput. Decision Sci. 18 (1993) 381–396.
[36] J.A. Pomykala, Approximation operations in approximation space, Bull. Polish Acad. Sci. 35 (9–10) (1987) 653–662.
[37] J. Mordeson, Rough set theory applied to (fuzzy) ideal theory, Fuzzy Sets Syst. 121 (2001) 315–324.
[38] G. Birkhoff, Lattice Theory, American Mathematics Society, 1995.
[39] J.Y. Liang, F. Wang, C.Y. Dang, Y.H. Qian, An efficient rough feature selection algorithm with a multi-granulation view, Int. J. Approx. Reason. 53 (6) (2012) 912–926.
[40] L.W. Ma, On some types of neighborhood-related covering rough sets, Int. J. Approx. Reason. 53 (6) (2012) 901–911.
[41] P. Zhu, Q.Y. Wen, Entropy and co-entropy of a covering approximation space, Int. J. Approx. Reason. 53 (4) (2012) 528–540.
[42] P. Zhu, Covering rough sets based on neighborhoods: an approach without using neighborhoods, Int. J. Approx. Reason. 52 (3) (2011) 461–472.
[43] G.P. Lin, J.Y. Liang, Y.H. Qian, Multigranulation rough sets: from partition to covering, Inf. Sci. 241 (2013) 101–118.
[44] Y.H. Qian, H. Zhang, Y.L. Sang, J.L. Liang, Multigranulation decision-theoretic rough sets, Int. J. Approx. Reason. 55 (1) (2014) 225–237.