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Region-based quantitative and hierarchical attribute reduction in the two-category decision theoretic rough set model

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ABSTRACT

Quantitative attribute reduction exhibits applicability but complexity when compared to qualitative reduction. According to the two-category decision theoretic rough set model, this paper mainly investigates quantitative reducts and their hierarchies (with qualitative reducts) from a regional perspective. (1) An improved type of classification regions is proposed, and its preservation reduct (CRP-Reduct) is studied. (2) Reduction targets and preservation properties of set regions are analyzed, and the set-region preservation reduct (SRP-Reduct) is studied. (3) Separability of set regions and rule consistency is verified, and the quantitative and qualitative double-preservation reduct (DP-Reduct) is established. (4) Hierarchies of CRP-Reduct, SRP-Reduct, and DP-Reduct are explored with two qualitative reducts: the Pawlak-Reduct and knowledge-preservation reduct (KP-Reduct). (5) Finally, verification experiments are provided. CRP-Reduct, SRP-Reduct, and DP-Reduct expand layer by layer Pawlak-Reduct and exhibit quantitative applicability, and the experimental results indicate their effectiveness and hierarchies regarding Pawlak-Reduct and KP-Reduct.

1. Introduction

Rough set theory (RS-Theory) [1,2] is a novel mathematical theory for uncertainty descriptions and an important applicable methodology for knowledge discovery. In particular, it can effectively process uncertain, imprecise, and incomplete information. Thus, the model-based uncertainty description and reduction-based knowledge discovery become its two main issues, where the qualitative mechanism-based quantitative extension plays an increasingly important role.

The classical Pawlak-Model [1,2] is qualitative and thus has accuracy. However, the qualitative absoluteness can also cause some limitations and problems, such as over-fitting. In fact, Pawlak-Model cannot fully capture latent useful knowledge in the uncertainty boundary. In contrast, quantitative models resort to some measures and thresholds to express quantization approximation and fault tolerance, so they can tackle data sets with noises, thus holding important application significance; moreover, they usually conduct theoretical expansion for qualitative Pawlak-Model. Thus, the probabilistic rough set (PRS) [3–6] utilizes the probability uncertainty measure to exhibit application merits regarding measurability, generality, and robustness, and it includes several concrete models, such as the decision-theoretic rough set (DTRS) [7] and variable precision rough set (VPRS) [8].

Attribute reduction is a fundamental subject in RS-Theory due to its optimization and generalization for data mining. The classical reduction is related to Pawlak-Model and thus reflects a qualitative approach, and different reduction algorithms were extensively explored in [9–14]. In contrast, quantitative reduction mainly utilizes the quantitative mechanisms and advantages to achieve deep development and extensive applications; for example, Refs. [15–26] studied DTRS-Reduction and VPRS-Reduction, respectively. For the decision table, the classical reduction theory mainly depends on the classification-positive region (C-POS). Thus, Pawlak-Reduction directly preserves C-POS due to the change monotonicity of qualitative C-POS. However, quantitative region exhibits the change non-monotonicity, and quantitative reduction usually accompanies some anomalies [15,24,25]. In fact, Ref. [27] verified that quantitative regions have the essential change uncertainty, which determines the change non-monotonicity. Thus, quantitative reduction has already transcended qualitative Pawlak-Reduction and thus becomes a complex problem. For this

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difficulty, we aim to conduct some systematical studies by virtue of a concrete quantitative model.

PRS usually needs thresholds for quantitative applications, so threshold determination becomes a critical task. In particular, DTRS achieves thresholds' semantics and calculation by using the Bayesian risk decision and three-way decision semantics [7]; moreover, DTRS also establishes a basic platform for quantitative explorations via its expansion and representativeness. For DTRS, Refs. [28,29] analyzed three-way decisions and their superiority, Refs. [30–33] discussed model development and threshold calculation, Refs. [34–38] researched model applications (regarding regression, clustering, and semi-supervised leaning), Refs. [38–41] exploited multi-category construction. For DTRS-Reduction, Ref. [15] proposed general reducts by mining transcendental measures for the dependency degree; moreover, Refs. [16–20] summarized the existing methods, including the positive-based reduct, nonnegative-based reduct, cost-based reduct, and distribution-based reduct.

Against the above backgrounds, quantitative reduction exhibits applicability but complexity, and DTRS is a fundamental PRS and its attribute reduction can reflect some essence of quantitative reduction. Thus, this paper concentrates on DTRS-Reduction in the decision table. Note that the two-category case corresponds to the fundamental issue for DTRS, and it is also linked to a usual classification task in the decision table. In fact, it causes relatively clear regional structure for RS-Theory by complementary simplification, thus underling multiple-category generalization; moreover, it can also provide some verification analyzes by degeneration. Therefore, our discussion is mainly within the two-category framework, and this restriction becomes a rational strategy in view of the complexity of quantitative reduction. In particular, granular computing (GrC) [42,43] emphasizes multiple levels and provides a structural approach for hierarchical information processing, and Refs. [44–48] conducted GrC studies for RS-Theory. Based on the GrC technology, we will construct hierarchical regional targets to systematically investigate hierarchical DTRS-Reduction on a basic premise of reduction expansion.

According to the two-category DTRS-Model, this paper mainly investigates quantitative reducts and their hierarchies (with qualitative reducts) from a regional perspective. It involves the following five parts. (1) An improved type of classification regions is proposed, and its preservation reduct (CRP-Reduct) is studied. (2) Reduction targets and preservation properties of set regions are analyzed, and the set-region preservation reduct (SRP-Reduct) is studied. (3) Separability of set regions and rule consistency is verified, and the quantitative and qualitative double-preservation reduct (DP-Reduct) is established. (4) Hierarchies of CRP-Reduct, SRP-Reduct, and DP-Reduct are explored with two qualitative reducts: the Pawlak-Reduct and knowledge-preservation reduct (KP-Reduct). (5) Finally, verification experiments are provided. In summary, the main contribution of our works is to construct three types of quantitative reducts and to further investigate their hierarchies with two types of qualitative reducts, and structural regions act as a main perspective in view of the two-category feature. As a result, CRP-Reduct, SRP-Reduct, and DP-Reduct expand layer by layer Pawlak-Reduct and exhibit quantitative applicability, and the experimental results indicate their effectiveness and hierarchies regarding Pawlak-Reduct and KP-Reduct.

The rest of this paper is organized as follows. Section 2 reviews basic models and reducts. Section 3 constructs an improved type of classification regions, and CRP-Reduct. Section 4 studies set-region preservation and SRP-Reduct. Section 5 discusses double-preservation and DP-Reduct. Section 6 investigates hierarchies of five reduction types. Section 7 conducts experimental analyzes. Finally, Section 8 concludes this paper.

2. Preliminaries

For simplification, abbreviations are first provided for several repeated terms. First alphabet-based replacement includes: Set → S, Classification → C, Region → R, Preservation → P, Double → D, and Knowledge → K.

(1) S-Region and C-Region denote the set region and classification region, respectively. Concretely, POS, BND, and NEG denote the set positive, boundary, negative regions, respectively, while C-POS, C-BND, and C-NEG denote the classification positive, boundary, negative regions, respectively.
(2) CR-Preservation, SR-Preservation, D-Preservation, and K-Preservation denote C-Region preservation, S-Region preservation, double preservation (of set regions and rule consistency), and knowledge preservation, respectively. Furthermore, CRP-Reduct, SRP-Reduct, DP-Reduct, and KP-Reduct denote corresponding preservation reducts.

Next, this section reviews Pawlak-Model, DTRS-Model, and their reducts.

2.1. Pawlak-Model and Pawlak-Reduct

Pawlak-Model and Pawlak-Reduct [1,2] are first reviewed.

U is a finite universe, $\mathscr{R}$ is a family of equivalence relations, and $(U,\mathscr{R})$ constitutes a knowledge base. Let $\emptyset \neq R \subseteq \mathscr{R}, \cap R$ determines an equivalence relation – $IND(R)$. Knowledge R refers to classified structure $U/IND(R)$ with granule $[x]_R$. Thus, (U,R) constitutes an approximate space, where set $X \subseteq U$ is also called a concept. In Pawlak-Model, the lower and upper approximations of X are defined by

$$\underline{apr}_R X = \{x|[x]_R \subseteq X\}, \overline{apr}_R X = \{x|[x]_R \cap X \neq \emptyset\}.$$

$$\begin{cases} \mathrm{POS}_R(X) = \underline{apr}_R X, \\ \mathrm{NEG}_R(X) = U - \overline{apr}_R X, \\ \mathrm{BND}_R(X) = \overline{apr}_R X - \underline{apr}_R X \end{cases} \tag{1}$$

further denotes POS, NEG, and BND.

The decision table (D-Table) is an important information table with classification tasks. In D-Table $(U, C \cup D), C$ and D include condition and decision attributes, respectively, and the decision rule is related to the function $d_x(a) = a(x)$, where $x \in U, a \in C \cup D$. D-Table is consistent, if all its decision rules are consistent, i.e., arbitrary decision rule d_x satisfies $d_x|C = d_y|C \Rightarrow d_x|D = d_y|D, \forall x \neq y$; otherwise, it is inconsistent. Moreover, condition attribute subset A determines an equivalence relation and knowledge, where $\emptyset \neq A \subseteq C$.

Definition 2.1 (*Qualitative Type*). In Pawlak-Model, C-Regions are qualitative and are composed of following C-POS and C-BND:

$$\begin{cases} \mathrm{POS}_A(D) = \bigcup\limits_{X \in U/IND(D)} \underline{apr}_{IND(A)} X, \\ \mathrm{BND}_A(D) = U - \mathrm{POS}_A(D), \end{cases} \tag{2}$$

$\mathrm{POS}_A(D)$ describes certain granules for classification. $\mathrm{POS}_{B'}(D) \subseteq \mathrm{POS}_B(D)$ if $B' \subseteq B \subseteq C$, so C-POS change has monotonicity. Thus, Pawlak-Reduct is naturally established by preserving C-POS; moreover, dependency degree $\gamma_A(D) = \frac{|\mathrm{POS}_A(D)|}{|U|}$ is important for evaluating classification quality.

Definition 2.2 (*Pawlak-Reduct*). B is Pawlak-Reduct of C, if it satisfies C-POS preservation and set independence, i.e.,

(1) $\mathrm{POS}_B(D) = \mathrm{POS}_C(D)$;
(2) $\mathrm{POS}_{B-\{b\}}(D) \neq \mathrm{POS}_B(D), \forall b \in B$.

$Core(C) = \{c \in C | \mathrm{POS}_{C-\{c\}}(D) \neq \mathrm{POS}_C(D)\}$ becomes the reduction core, and $RED(C)$ denotes the reduction set.

Proposition 2.3. *If $U/IND(D) = \{X, \neg X\}$, then*

$$\begin{cases} \mathrm{POS}_A(D) = \mathrm{POS}_A(X) \cup \mathrm{POS}_A(\neg X) = \mathrm{POS}_A(X) \cup \mathrm{NEG}_A(X), \\ \mathrm{BND}_A(D) = \mathrm{BND}_A(X), \end{cases}$$
$$\gamma_A(D) = \frac{|\mathrm{POS}_A(X)|}{|U|} + \frac{|\mathrm{NEG}_A(X)|}{|U|}.$$

Proposition 2.3 provides the two-category result, where $|U/IND(D)| = 2$. Thus, C-Regions and S-Regions exhibit clear relationships, e.g., C-POS is composed of two S-Regions – POS and NEG; moreover, the dependency degree is integrated by two measures of POS and NEG.

2.2. DTRS-Model and DTRS-Reduct

Herein, DTRS-Model [7,28] is reviewed by the two-category problem.

Two states $X, \neg X$ indicate that an element is in X and not in X, respectively, while three actions a_P, a_B, a_N decide that an object is in sets $\mathrm{POS}(X), \mathrm{BND}(X), \mathrm{NEG}(X)$, respectively. When an object belongs to X, let $\lambda_{PP}, \lambda_{BP}, \lambda_{NP}$ denote costs of taking actions a_P, a_B, a_N, respectively; in contrast, $\lambda_{PN}, \lambda_{BN}, \lambda_{NN}$ denote costs of taking the same three actions in the contrary condition. Thus, the loss functions are expressed as the following matrix:

	a_P	a_B	a_N
X	λ_{PP}	λ_{BP}	λ_{NP}
$\neg X$	λ_{PN}	λ_{BN}	λ_{NN}

By the Bayesian minimum-risk decision, the three-way decisions (regarding acceptance, rejection, and deferment) are obtained as follows:

$$\begin{cases} \text{If } P(X|[x]_R) \geqslant \alpha, \text{ then decide } [x]_R \subseteq \mathrm{POS}(X); \\ \text{If } P(X|[x]_R) \leqslant \beta, \text{ then decide } [x]_R \subseteq \mathrm{NEG}(X); \\ \text{If } \beta < P(X|[x]_R) < \alpha, \text{ then decide } [x]_R \subseteq \mathrm{BND}(X). \end{cases}$$

Herein, $P(X|[x]_R) = \frac{|X \cap [x]_R|}{|[x]_R|}$ denotes the conditional probability for state X, and thresholds follow calculation formulas:

$$\begin{cases} \alpha = \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})}, \\ \beta = \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})}, \\ \gamma = \frac{\lambda_{PN} - \lambda_{NN}}{(\lambda_{PN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{PP})}. \end{cases}$$

Usually, $0 \leqslant \beta < 0.5 < \alpha \leq 1$. Furthermore, three S-Regions are established as follows:

$$\begin{cases} \mathrm{POS}_R^{\alpha,\beta}(X) = \{x | P(X|[x]_R) \geqslant \alpha\}, \\ \mathrm{NEG}_R^{\alpha,\beta}(X) = \{x | P(X|[x]_R) \leqslant \beta\}, \\ \mathrm{BND}_R^{\alpha,\beta}(X) = \{x | \beta < P(X|[x]_R) < \alpha\}. \end{cases} \quad (3)$$

The two-category case is fundamental for DTRS, where regional structure is relatively clear and can be fully described by only three S-Regions. For multi-category extension, there are two main approaches of DTRS development. One utilizes multiple two-category bases to make composite construction, while the other resorts to performing a Bayesian decision in a high-dimensional space. Accordingly, based on the S-Region construction and optimal decision, two types of DTRS C-Regions are used currently and widely.

Definition 2.4 (*Type I*). [15–17]

$$\begin{cases} {}^{\mathrm{I}}\mathrm{POS}_A^{\alpha,\beta}(D) = \bigcup_{X \in U/IND(D)} \mathrm{POS}_A^{\alpha,\beta}X, \\ {}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(D) = \bigcup_{X \in U/IND(D)} \mathrm{BND}_A^{\alpha,\beta}X, \\ {}^{\mathrm{I}}\mathrm{NEG}_A^{\alpha,\beta}(D) = U - (\mathrm{POS}_A^{\alpha,\beta}(D) \cup \mathrm{BND}_A^{\alpha,\beta}(D)). \end{cases} \quad (4)$$

Definition 2.5 (*Type II*). [18–20]

$$\begin{cases} {}^{\mathrm{II}}\mathrm{POS}_A^{\alpha,\beta}(D) = \{x | P(D_{max}([x]_A)|[x]_A) \geqslant \alpha\}, \\ {}^{\mathrm{II}}\mathrm{BND}_A^{\alpha,\beta}(D) = \{x | \beta < P(D_{max}([x]_A)|[x]_A) < \alpha\}, \\ {}^{\mathrm{II}}\mathrm{NEG}_A^{\alpha,\beta}(D) = \{x | P(D_{max}([x]_A)|[x]_A) \leqslant \beta\}, \end{cases} \quad (5)$$

$$D_{max}([x]_A) = arg \max_{X \in U/IND(D)} \left\{ \frac{|[x]_A \cap X|}{|[x]_A|} \right\}.$$

Furthermore, there are two corresponding types of DTRS-Reduct by adopting C-POS preservation. Next, a uniform form is provided but has two different C-POS connotations.

Definition 2.6 (*DTRS-Reduct*). [17,20] B is DTRS-Reduct of C, if it satisfies two conditions:

(1) $\mathrm{POS}_B^{\alpha,\beta}(D) = \mathrm{POS}_C^{\alpha,\beta}(D)$;
(2) $\mathrm{POS}_{B-\{b\}}^{\alpha,\beta}(D) \neq \mathrm{POS}_B^{\alpha,\beta}(D), \forall b \in B$.

Note that we mainly investigate two-category DTRS-Reduction and $U/IND(D) = \{X, \neg X\}$ becomes the general but omitted premise in the whole paper. Thus, we will make simple but complete descriptions by only X, because it can express $\neg X$ by complementation; in other words, the two-category feature essentially determines efficiencies of the region-based approach. For the multi-category issue, our works can make degeneration inspection and also underlie generalization explorations.

3. An improved type of classification regions and its preservation reduct (CRP-Reduct)

By virtue of S-Regions, this section proposes an improved type of C-Regions by degenerate analyzes for the existing two types. Furthermore, the relevant preservation reduct (CRP-Reduct) is investigated.

3.1. An improved type of classification regions

This subsection first makes degenerate analyzes of Types I and II, and it further proposes a new type and analyzes its relevant improvement.

Theorem 3.1 (Type I's Formula).

$$\begin{cases} {}^{\mathrm{I}}\mathrm{POS}_A^{\alpha,\beta}(D) = \{x | P(X|[x]_A) \geqslant \alpha\} \cup \{x | P(X|[x]_A) \leq 1-\alpha\}, \\ {}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(D) = \{x | \beta < P(X|[x]_A) < \alpha\} \cup \{x | 1-\alpha < P(X|[x]_A) < 1-\beta\}, \\ {}^{\mathrm{I}}\mathrm{NEG}_A^{\alpha,\beta}(D) = \emptyset, \end{cases}$$

$${}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(D) = \begin{cases} \{x | 1-\alpha < P(X|[x]_A) < \alpha\}, & \text{if } \alpha+\beta \geqslant 1, \\ \{x | \beta < P(X|[x]_A) < 1-\beta\}, & \text{if } \alpha+\beta < 1. \end{cases}$$

Proof. $P(\neg X|[x]_A) = 1 - P(X|[x]_A)$. Thus, ${}^{\mathrm{I}}\mathrm{POS}_A^{\alpha,\beta}(X) = \{x|P(X|[x]_A) \geqslant \alpha\}$, ${}^{\mathrm{I}}\mathrm{POS}_A^{\alpha,\beta}(\neg X) = \{x|P(\neg X|[x]_A) \geqslant \alpha\} = \{x|P(X|[x]_A) \leqslant 1-\alpha\}$; ${}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(X) = \{x|\beta < P(X|[x]_A) < \alpha\}$, ${}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(\neg X) = \{x|1-\alpha < P(X|[x]_A) < 1-\beta\}$. Furthermore, (1) if $\alpha+\beta \geqslant 1$, then $0 \leq 1-\alpha \leqslant \beta < 0.5 < 1-\beta \leqslant \alpha \leq 1$. Thus, ${}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(D) = \{x|1-\alpha < P(X|[x]_A) < \alpha\}$, ${}^{\mathrm{I}}\mathrm{POS}_A^{\alpha,\beta}(D) \cap {}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(D) = \emptyset$. (2) If $\alpha+\beta < 1$, then $0 \leqslant \beta < 1-\alpha < 0.5 < \alpha < 1-\beta \leq 1$. Thus, ${}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(D) = \{x|\beta < P(X|[x]_A) < 1-\beta\}$, ${}^{\mathrm{I}}\mathrm{POS}_A^{\alpha,\beta}(D) \cap {}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(D) \neq \emptyset$. Hence, the C-Region results are obtained. $\square$

Corollary 3.2 (Type I's Structure).

$$
\begin{aligned}
&{}^{\mathrm{I}}\mathrm{POS}_A^{\alpha,\beta}(D) \cup {}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(D) = U,\\
&{}^{\mathrm{I}}\mathrm{POS}_A^{\alpha,\beta}(D) \cap {}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(D) = \begin{cases} \emptyset, & \text{if } \alpha+\beta \geqslant 1,\\ \{x|\beta < P(X|[x]_A) \leq 1-\alpha\} \cup \{x|\alpha \\ \leqslant P(X|[x]_A) < 1-\beta\} \neq \emptyset, & \text{if } \alpha+\beta < 1. \end{cases}
\end{aligned}
$$

Corollary 3.3 (Type I's Expansion).

(1) ${}^{\mathrm{I}}\mathrm{POS}_A^{\alpha,\beta}(D) \supseteq \mathrm{POS}_A(D)$, ${}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(D) \subseteq \mathrm{BND}_A(D)$.
(2) *When* $\alpha = 1$, ${}^{\mathrm{I}}\mathrm{NEG}_A^{\alpha,\beta}(D) = \phi$, ${}^{\mathrm{I}}\mathrm{POS}_A^{\alpha,\beta}(D) = \mathrm{POS}_A(D)$, ${}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(D) = \mathrm{BND}_A(D)$.

For Type I, Theorem 3.1 provides a basic formula, Corollary 3.2 reflects systemic structure, while Corollary 3.3 exhibits expansion for Qualitative Type, where $\mathrm{POS}_A(D) = \{x|P(X|[x]_A) = 1\} \cup \{x|P(X|[x]_A) = 0\}$ and $\mathrm{BND}_A(D) = \{x|0 < P(X|[x]_A) < 1\}$. Next, its improvement spaces are analyzed.

(1) Type I is directly constructed by its basic element – S-Regions, thus exhibiting a structural characteristic. However, it relates NEG to C-NEG rather than C-POS, but NEG more yields classification certainty. Thus, C-NEG has vague classification semantics, while C-POS completeness can be improved.
(2) Both C-POS and C-BND are related to only an independent threshold – α or β. In fact, DTRS is a double-threshold system and the double-threshold representation can exhibit systematicness and stability. Thus, we need a more systematic and rational description based on α and β.
(3) Type I has an improvement space for systemic structure because ${}^{\mathrm{I}}\mathrm{POS}_A^{\alpha,\beta}(D) \cap {}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(D) \neq \emptyset$ when $\alpha+\beta < 1$.
(4) Based on S-Regions, DTRS-Model exhibits point expansion regarding $(\alpha,\beta) = (1,0)$ for Pawlak-Model. However, Type I corresponds to wider plane expansion regarding $\alpha = 1$. Thus, the essential point expansion is worth inheriting for C-Regions.

Next, we provide Type II's degenerate results and improvement spaces.

Theorem 3.4 (Type II's Formula).

$$
\begin{cases}
{}^{\mathrm{II}}\mathrm{POS}_A^{\alpha,\beta}(D) = \{x|P(X|[x]_A) \geqslant \alpha\} \cup \{x|P(X|[x]_A) \leq 1-\alpha\},\\
{}^{\mathrm{II}}\mathrm{BND}_A^{\alpha,\beta}(D) = \{x|1-\alpha < P(X|[x]_A) < \alpha\},\\
{}^{\mathrm{II}}\mathrm{NEG}_A^{\alpha,\beta}(D) = \emptyset.
\end{cases}
$$

Proof. Type II is mainly described by $MAX = max(P(X|[x]_A), P(\neg X|[x]_A)) = max(P(X|[x]_A), 1-P(X|[x]_A))$.

$$
MAX = \begin{cases} P(X|[x]_A), & \text{if } P(X|[x]_A) \geqslant 0.5,\\ 1-P(X|[x]_A), & \text{if } P(X|[x]_A) \leq 0.5. \end{cases}
$$

Thus, $MAX \geqslant 0.5 > \beta$, so ${}^{\mathrm{II}}\mathrm{NEG}_A^{\alpha,\beta}(D) = \{x|MAX \leqslant \beta\} = \emptyset$. Moreover, ${}^{\mathrm{II}}\mathrm{POS}_A^{\alpha,\beta}(D) = \{x|MAX \geqslant \alpha\} = \{x|P(X|[x]_A) \geqslant \alpha\} \cup \{x|P(X|[x]_A) \leq 1-\alpha\}$, ${}^{\mathrm{II}}\mathrm{BND}_A^{\alpha,\beta}(D) = \{x|\beta < MAX < \alpha\} = \{x|0.5 \leqslant P(X|[x]_A) < \alpha\} \cup \{x|1-\alpha < P(X|[x]_A) \leq 0.5\} = \{x|1-\alpha < P(X|[x]_A) < \alpha\}$. $\square$

Corollary 3.5 (Type II's Structure).

$$
\begin{cases}
{}^{\mathrm{II}}\mathrm{POS}_A^{\alpha,\beta}(D) \cup {}^{\mathrm{II}}\mathrm{BND}_A^{\alpha,\beta}(D) = U,\\
{}^{\mathrm{II}}\mathrm{POS}_A^{\alpha,\beta}(D) \cap {}^{\mathrm{II}}\mathrm{BND}_A^{\alpha,\beta}(D) = \emptyset.
\end{cases}
$$

Corollary 3.6 (Type II's Expansion).

(1) ${}^{\mathrm{II}}\mathrm{POS}_A^{\alpha,\beta}(D) \supseteq \mathrm{POS}_A(D)$, ${}^{\mathrm{II}}\mathrm{BND}_A^{\alpha,\beta}(D) \subseteq \mathrm{BND}_A(D)$.
(2) *When* $\alpha = 1$, ${}^{\mathrm{II}}\mathrm{NEG}_A^{\alpha,\beta}(D) = \emptyset$, ${}^{\mathrm{II}}\mathrm{POS}_A^{\alpha,\beta}(D) = \mathrm{POS}_A(D)$, ${}^{\mathrm{II}}\mathrm{BND}_A^{\alpha,\beta}(D) = \mathrm{BND}_A(D)$.

For Type II, Theorem 3.4 provides a basic formula, Corollary 3.5 reflects systemic structure (where C-POS and C-BND construct a universe division), while Corollary 3.6 exhibits expansion for Qualitative Type. Next, the improvement spaces are simply provided by referring to Type I's relevant analyzes.

(1) Type II adopts an optimal strategy on decisions and thus has application optimization. However, it lacks a necessary link with essential S-Regions; thus, its classification semantics (especially C-NEG's) becomes weak and ambiguous.
(2) Both C-POS and C-BND are related to only one independent threshold – α, and this result implies a flaw due to the neglect of β. Thus, a systemic double-threshold description is needed.
(3) The essential point expansion is also expected in view of the same plane expansion.

Types I and II are fundamental for relevant DTRS-Reduction, and the above studies provide their degenerate results at the two-category level. First, C-NEG is always empty (though it may be non-empty in the multi-category case), so there are only two C-Regions. Second, Type I concerns only α if $\alpha+\beta \geqslant 1$, and ${}^{\mathrm{I}}\mathrm{POS}_A^{\alpha,\beta}(D) \cap {}^{\mathrm{I}}\mathrm{BND}_A^{\alpha,\beta}(D) \neq \emptyset$ if $\alpha+\beta < 1$; in contrast, Type II concerns only α. Moreover, they exhibit plane expansion for Qualitative Type by virtue of only α. In view of quantitative essence of DTRS, a double-threshold description becomes more reasonable; moreover, a structural division is also needed for C-Regions. According to these improvement spaces, we next propose an improvement type at the two-category level by completely considering the S-Region system.

Definition 3.7 (*Type III*). Type III of C-Regions is defined by following C-POS, C-BND, and C-NEG:

$$
\begin{cases}
\mathrm{POS}_A^{\alpha,\beta}(D) = \mathrm{POS}_A^{\alpha,\beta}(X) \cup \mathrm{NEG}_A^{\alpha,\beta}(X),\\
\mathrm{BND}_A^{\alpha,\beta}(D) = \mathrm{BND}_A^{\alpha,\beta}(X),\\
\mathrm{NEG}_A^{\alpha,\beta}(D) = U - (\mathrm{POS}_A^{\alpha,\beta}(D) \cup \mathrm{BND}_A^{\alpha,\beta}(D)).
\end{cases} \tag{6}
$$

Proposition 2.3 provides S-Region-based C-Regions for Pawlak-Model, where C-NEG does not exist. In contrast, Definition 3.7 proposes S-Region-based C-Regions for DTRS-Model, so Type III naturally simulates Qualitative Type by promoting S-Regions. Herein, three S-Regions are completely used and NEG information is related to C-POS; moreover, C-NEG is always empty but is also set up for multi-category generalization.

Theorem 3.8 (Type III's Formula).

$$\begin{cases} \mathrm{POS}_A^{\alpha,\beta}(D) = \{x|P(X|[x]_A) \geqslant \alpha\} \cup \{x|P(X|[x]_A) \leqslant \beta\}, \\ \mathrm{BND}_A^{\alpha,\beta}(D) = \{x|\beta < P(X|[x]_A) < \alpha\}, \\ \mathrm{NEG}_A^{\alpha,\beta}(D) = \emptyset. \end{cases}$$

Corollary 3.9 (Type III's Structure).

$$\begin{cases} \mathrm{POS}_A^{\alpha,\beta}(D) \cup \mathrm{BND}_A^{\alpha,\beta}(D) = U, \\ \mathrm{POS}_A^{\alpha,\beta}(D) \cap \mathrm{BND}_A^{\alpha,\beta}(D) = \emptyset. \end{cases}$$

Corollary 3.10 (Type III's Expansion).

(1) $\mathrm{POS}_A^{\alpha,\beta}(D) \supseteq \mathrm{POS}_A(D)$, $\mathrm{BND}_A^{\alpha,\beta}(D) \subseteq \mathrm{BND}_A(D)$.
(2) *If* $(\alpha,\beta) = (1,0)$, *then* $\mathrm{NEG}_A^{\alpha,\beta}(D) = \emptyset$, $\mathrm{POS}_A^{\alpha,\beta}(D) = \mathrm{POS}_A(D)$, $\mathrm{BND}_A^{\alpha,\beta}(D) = \mathrm{BND}_A(D)$.

For Type III, Theorem 3.8 provides a basic formula, Corollary 3.9 reflects divided systemic structure, and Corollary 3.10 exhibits point expansion for Qualitative Type. These performances are compared to those of Types I and II.

Finally, Type III's superiority and improvement are concluded by some comparison. Type II has weak structural relationships with S-Regions in spite of its application optimization. In contrast, Types I and III mainly utilize S-Regions to construct C-Regions, and this constructive strategy more adheres to the hierarchical development of RS-theory. In particular, there are two equivalent forms of qualitative C-POS, i.e., $\mathrm{POS}_A(D) = \mathrm{POS}_A(X) \cup \mathrm{POS}_A(\neg X)$ and $\mathrm{POS}_A(D) = \mathrm{POS}_A(X) \cup \mathrm{NEG}_A(X)$. In the quantitative expansion process, Type I uses the former with two POS regions – $\mathrm{POS}_A^{\alpha,\beta}(X)$ and $\mathrm{POS}_A^{\alpha,\beta}(\neg X)$, while Type III uses the latter with two usual certainty S-Regions – $\mathrm{POS}_A^{\alpha,\beta}(X)$ and $\mathrm{NEG}_A^{\alpha,\beta}(X)$. Accordingly, Type III more adheres to the double-approximation thought and the two-category feature, thus exhibiting completeness and stationarity from the S-Region viewpoint.

(1) Type III originates from S-Regions' integration and promotion, so it has the thorough construction mechanism and clear classification semantics. In particular, $\mathrm{POS}_A^{\alpha,\beta}(D)$ describes certain and approximate granules for classification in a condition of tolerant threshold (α,β).
(2) Type III concerns both thresholds α,β, thus exhibiting a systematic description. In fact, Type III's C-POS mainly introduces threshold β to add initial NEG information. Thus, Type III becomes reasonable for it closely adheres to the DTRS feature.
(3) Type III has a perfect division feature, which improves upon Type I.
(4) Type III inherits the essential point expansion to extend Qualitative Type. Thus, Type III has a thorough expansion mechanism and more quantitative application spaces.

In summary, we analyze Types I and II by degeneration and discover their improvement spaces, so we propose novel Type III. In fact, Type III rationally considers not only POS but also NEG, so it more adheres to the RS-Theory thought and DTRS-Model feature. Therefore, Type III becomes rational and effective in view of its thorough mechanism, clear semantics, perfect completeness, scientific system, benign expansion, and natural formulas; moreover, it particularly improves upon usual Types I and II at the two-category level.

3.2. Classification-region preservation reduct (CRP-Reduct)

Based on improved Type III, this subsection investigates its preservation reduct – CRP-Reduct. Note that only Type III is concerned in the surplus research.

Proposition 3.11. *DTRS C-POS and C-BND exhibit change non-monotonicity in attribute deletion.*

Example 1. In D-Table $(U, C \cup D)$ from Table 1, $U = \{x_1, x_2, \ldots, x_{12}\}$, $C = \{a,b\}$, $D = \{d\}$. $U/\{a,b\} = U/\{a\} = \{Y_1, Y_2, Y_3, Y_4\}$, $Y_1 = \{x_1,x_2,x_3\}$, $Y_2 = \{x_4,x_5,x_6\}$, $Y_3 = \{x_7,x_8,x_9\}$, $Y_4 = \{x_{10},x_{11},x_{12}\}$; $U/\{b\} = \{Z_1, Z_2\}$, $Z_1 = \{x_1,\ldots,x_6\}$, $Z_2 = \{x_7,\ldots,x_{12}\}$; $U/IND(D) = \{X, \neg X\}$, $X = \{x_1,x_2,x_3,x_4,x_7,x_8,x_9,x_{10},x_{11}\}$.

$P(X|Y_1) = 1, P(X|Y_2) = 1/3, P(X|Y_3) = 1, P(X|Y_4) = 2/3; P(X|Z_1) = 2/3, P(X|Z_2) = 5/6$. Let $0 < \beta < 1/3 < 2/3 < \alpha < 5/6 < 1$. Thus, $\mathrm{POS}_{\{a,b\}}^{\alpha,\beta}(X) = Y_1 \cup Y_3, \mathrm{BND}_{\{a,b\}}^{\alpha,\beta}(X) = Y_2 \cup Y_4, \mathrm{NEG}_{\{a,b\}}^{\alpha,\beta}(X) = \emptyset, \mathrm{POS}_{\{a,b\}}^{\alpha,\beta}(D) = Y_1 \cup Y_3$. When deleting $\{a\}, \mathrm{POS}_{\{b\}}^{\alpha,\beta}(X) = Z_2 = Y_3 \cup Y_4, \mathrm{BND}_{\{b\}}^{\alpha,\beta}(X) = Z_1 = Y_1 \cup Y_2, \mathrm{NEG}_{\{b\}}^{\alpha,\beta}(X) = \emptyset, \mathrm{POS}_{\{b\}}^{\alpha,\beta}(D) = Z_2 = Y_3 \cup Y_4$. Fig. 1 exhibits the old and new S-Regions and their following change. (1) Neither $\mathrm{POS}_{\{b\}}^{\alpha,\beta}(X) \subseteq \mathrm{POS}_{\{a,b\}}^{\alpha,\beta}(X)$ nor $\mathrm{POS}_{\{b\}}^{\alpha,\beta}(X) \supseteq \mathrm{POS}_{\{a,b\}}^{\alpha,\beta}(X)$ holds; (2) neither $\mathrm{BND}_{\{b\}}^{\alpha,\beta}(X) \subseteq \mathrm{BND}_{\{a,b\}}^{\alpha,\beta}(X)$ nor $\mathrm{BND}_{\{b\}}^{\alpha,\beta}(X) \supseteq \mathrm{BND}_{\{a,b\}}^{\alpha,\beta}(X)$ holds; (3) C-POS exhibits a similar result (for it is equal to POS), i.e., $\mathrm{POS}_{\{b\}}^{\alpha,\beta}(D) \nsubseteq \mathrm{POS}_{\{a,b\}}^{\alpha,\beta}(D)$ and $\mathrm{POS}_{\{b\}}^{\alpha,\beta}(D) \nsupseteq \mathrm{POS}_{\{a,b\}}^{\alpha,\beta}(D)$. Thus, the change non-monotonicity is verified for POS/BND and C-POS. Moreover, $|\mathrm{POS}_{\{b\}}^{\alpha,\beta}(X)| = |\mathrm{POS}_{\{a,b\}}^{\alpha,\beta}(X)|, |\mathrm{BND}_{\{b\}}^{\alpha,\beta}(X)| = |\mathrm{BND}_{\{a,b\}}^{\alpha,\beta}(X)|$, $|\mathrm{POS}_{\{b\}}^{\alpha,\beta}(D)| = |\mathrm{POS}_{\{a,b\}}^{\alpha,\beta}(D)|$, because $|Y_1| = |Y_4|$. □

Qualitative C-POS has the change monotonicity, so C-POS preservation becomes a natural target for Pawlak-Reduct to achieve the initial classification ability. However, DTRS C-POS produces the change non-monotonicity, which is verified by Proposition 3.11 and Example 1. According to the reduction thought and initial D-Table, DTRS C-POS preservation becomes a reasonable and reliable reduction strategy in the new situation with the C-Region change non-monotonicity. Moreover, the requirement of C-POS

Table 1
D-Table of Example 1.

U	a	b	d	U	a	b	d
x_1	0	1	1	x_7	2	2	1
x_2	0	1	1	x_8	2	2	1
x_3	0	1	1	x_9	2	2	1
x_4	1	1	1	x_{10}	3	2	1
x_5	1	1	0	x_{11}	3	2	1
x_6	1	1	0	x_{12}	3	2	0

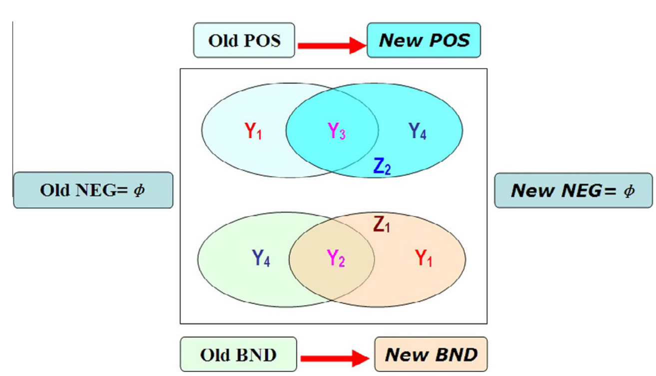

Fig. 1. Regions and their non-monotonicity change.

preservation also becomes natural for expanding Pawlak-Reduct. Based on this reduction target, we next discuss the corresponding DTRS-Reduct. In particular, C-POS preservation is equivalent to C-Region preservation (CR-Preservation), so we usually use the CR-Preservation form but the C-POS essence.

Definition 3.12 (*CRP-Reduct*). c is called an indispensable attribute in C, if $POS^{\alpha,\beta}_{C-\{c\}}(D) \neq POS^{\alpha,\beta}_{C}(D)$; otherwise, c is dispensable. B is independent, if b is indispensable in $B, \forall b \in B$. B is a C-Region preservation reduct (CRP-Reduct) of C, if B is independent and $POS^{\alpha,\beta}_{B}(D) = POS^{\alpha,\beta}_{C}(D)$. Herein, $RED^{\alpha,\beta}_{CRP}(C)$ denotes the set of all CRP-Reducts, and $CORE^{\alpha,\beta}_{CRP}(C)$ denotes the core with all indispensable attributes in C.

Based on CR-Preservation, Definition 3.12 defines CRP-Reduct by using the set independence. Next, the set independence will be equivalently described with the set maximality. For this purpose, a basic principle is first established.

Theorem 3.13 (Squeeze Principle). *Let* $\emptyset \neq B_1 \subseteq B \subseteq B_2 \subseteq C$. *If* $POS^{\alpha,\beta}_{B_1}(D) = POS^{\alpha,\beta}_{B_2}(D)$, *then* $POS^{\alpha,\beta}_{B_1}(D) = POS^{\alpha,\beta}_{B}(D) = POS^{\alpha,\beta}_{B_2}(D)$.

The regional change monotonicity is usually utilized in qualitative reduction but no longer holds in quantitative reduction. To study DTRS-Reduction, its function should be replaced by a more general property. Thus, Squeeze Principle is constructed by bidirectional clamp approximations. Concretely, if C-POS has the same classification ability at a coarser and a finer knowledge levels, then it necessarily has the same classification ability at a middle level. Clearly, Squeeze Principle originates from the essential reduction feature – knowledge monotonicity, thus holding general significance for DTRS-Reduction.

Proposition 3.14. *The following two items are equivalent:*

(1) $POS^{\alpha,\beta}_{B-\{b\}}(D) \neq POS^{\alpha,\beta}_{B}(D), \forall b \in B$;
(2) $POS^{\alpha,\beta}_{B-\{B'\}}(D) \neq POS^{\alpha,\beta}_{B}(D), \forall \emptyset \neq B' \subset B$.

Proof. $\forall \emptyset \neq B' \subset B, \exists b \in B - B'$, s.t., $B' \subseteq B - \{b\} \subset B$. Based on Squeeze Principle, if $POS^{\alpha,\beta}_{B'}(D) = POS^{\alpha,\beta}_{B}(D)$, then $POS^{\alpha,\beta}_{B'}(D) = POS^{\alpha,\beta}_{B-\{b\}}(D) = POS^{\alpha,\beta}_{B}(D)$, but this result contradicts $POS^{\alpha,\beta}_{B-\{b\}}(D) \neq POS^{\alpha,\beta}_{B}(D)$. Hence, (1) $\Rightarrow$ (2) holds. In contrast, (2) $\Rightarrow$ (1) is easy by setting up $B' = \{b\}$. $\square$

For CR-Preservation, Proposition 3.14 reflects equivalence of the set independence (Item (1)) and set maximality (Item (2)). Thus, an equivalent CRP-Reduct is defined by the latter.

Definition 3.15 (*CRP-Reduct*). B is CRP-Reduct of C, if

(1) $POS^{\alpha,\beta}_{B}(D) = POS^{\alpha,\beta}_{C}(D)$;
(2) $POS^{\alpha,\beta}_{B'}(D) \neq POS^{\alpha,\beta}_{B}(D), \forall \emptyset \neq B' \subset B$.

Theorem 3.16 (CRP-Reduct's Expansion). *CRP-Reduct expands Pawlak-Reduct to DTRS-Reduct, and it degenerates into Pawlak-Reduct when* $(\alpha, \beta) = (1, 0)$.

Herein, CRP-Reduct is established by two approaches. CR-Preservation acts as the natural reduction target, and the set independence and maximality are equivalently required. Thus, CRP-Reduct is a minimal subset to provide the initial classification ability. By virtue of rational CR-Preservation, CRP-Reduct naturally expands qualitative reduction to quantitative reduction, thus becoming scientific and valuable. Moreover, in view of Type III's improvement, CRP-Reduct also improves upon relevant two-category DTRS-Reduct based on Types I and II [17,20], thus holding more powerful efficiencies.

Theorem 3.17. $CORE^{\alpha,\beta}_{CRP}(C) = \bigcap RED^{\alpha,\beta}_{CRP}(C)$.

Proof. (1) If $c \notin \bigcap RED^{\alpha,\beta}_{CRP}(C)$, then $\exists B \in RED^{\alpha,\beta}_{CRP}(C)$ but $c \notin B$. Thus, $B \subseteq C - \{c\} \subset C, POS^{\alpha,\beta}_{B}(D) = POS^{\alpha,\beta}_{C}(D)$. Based on Squeeze Principle, $POS^{\alpha,\beta}_{C-\{c\}}(D) = POS^{\alpha,\beta}_{C}(D)$. Hence, $c \notin CORE^{\alpha,\beta}_{CRP}(C), CORE^{\alpha,\beta}_{CRP}(C) \subseteq \bigcap RED^{\alpha,\beta}_{CRP}(C)$. (2) If $c \notin CORE^{\alpha,\beta}_{CRP}(C)$, then $POS^{\alpha,\beta}_{C-\{c\}}(D) = POS^{\alpha,\beta}_{C}(D)$. $\exists B \in RED^{\alpha,\beta}_{CRP}(C - \{c\})$, so $POS^{\alpha,\beta}_{C-\{c\}}(D) = POS^{\alpha,\beta}_{B}(D)$ and B is independent. Hence, $POS^{\alpha,\beta}_{C}(D) = POS^{\alpha,\beta}_{B}(D), B \in RED^{\alpha,\beta}_{CRP}(C)$, but $c \notin B$. Hence, $c \notin \bigcap RED^{\alpha,\beta}_{CRP}(C), \bigcap RED^{\alpha,\beta}_{CRP}(C) \subseteq CORE^{\alpha,\beta}_{CRP}(C)$. $\square$

Theorem 3.17 provides a fundamental relationship between CRP-Reducts and the core, which is similar to the classical conclusion of qualitative reduction. Example 1 can provide some illustration by $CORE^{\alpha,\beta}_{CRP}(\{a, b\}) = \{a\}$ and $RED^{\alpha,\beta}_{CRP}(\{a, b\}) = \{\{a\}\}$. Thus, the core holds significance for all CRP-Reducts computation, because it is included in every CRP-Reduct and its computation is straightforward. Next, a core-based algorithm for CRP-Reduct is constructed by adopting an attribute addition strategy.

Algorithm 1. A core-based attribute addition algorithm for CRP-Reduct

Input:
D-Table $(U, C \cup D)$ and threshold (α, β);
Output:
CRP-Reduct $B \in RED^{\alpha,\beta}_{CRP}(C)$;
1: Compute $CORE^{\alpha,\beta}_{CRP}(C)$.
2: $B = CORE^{\alpha,\beta}_{CRP}(C)$.
3: **while** $POS^{\alpha,\beta}_{B}(D) \neq POS^{\alpha,\beta}_{C}(D)$ **do**
4: c is randomly chosen in $C - B$ and let $B = B \cup \{c\}$;
5: **end while**
6: **return** B.

In Algorithm 1, Step 1 yields the core, and Steps 3, 4 further seek an attribute subset including the core on the CR-Preservation premise, where the added attribute is random. Thus, this algorithm is convergent and effective, and it can usually obtain one CRP-Reduct regardless of the set independence/maximality. As is known, attribute heuristic information can accelerate relevant algorithms, so the basic dependency degree is inspected.

Definition 3.18 (*Dependency Degree*). Dependency degree of D on A is defined by ${}^{CRP}\gamma^{\alpha,\beta}_{A}(D) = \frac{|POS^{\alpha,\beta}_{A}(D)|}{|U|}$.

Proposition 3.19. *If* $A_1 \subseteq A_2 \subseteq C$, *then* $POS^{\alpha,\beta}_{A_1}(D) = POS^{\alpha,\beta}_{A_2}(D) \Rightarrow {}^{CRP}\gamma^{\alpha,\beta}_{A_1}(D) = {}^{CRP}\gamma^{\alpha,\beta}_{A_2}(D)$, *and the opposite does not hold.*

Herein, qualitative ${}^{CRP}\gamma^{\alpha,\beta}_{A}(D)$ is naturally proposed by referring to qualitative $\gamma_A(D)$. Clearly, ${}^{CRP}\gamma^{\alpha,\beta}_{A}(D) \in [0, 1], {}^{CRP}\gamma^{\alpha,\beta}_{A}(D) \geqslant \gamma_A(D)$, and ${}^{CRP}\gamma^{\alpha,\beta}_{A}(D) = \gamma_A(D)$ when $(\alpha, \beta) = (1, 0)$. In qualitative reduction, $\gamma_A(D)$ inherits the change monotonicity and can make a sufficient and necessary measurement. However, ${}^{CRP}\gamma^{\alpha,\beta}_{A}(D)$ inherits the change non-monotonicity. Furthermore, Proposition 3.19 shows that ${}^{CRP}\gamma^{\alpha,\beta}_{A}(D)$ equality acts as only a necessary condition for

CR-Preservation. In particular, the insufficiency can be verified by Example 1, where ${}^{CRP}\gamma^{\alpha,\beta}_{\{a,b\}}(D) = 0.5 = {}^{CRP}\gamma^{\alpha,\beta}_{\{b\}}(D)$ but $POS^{\alpha,\beta}_{\{a,b\}}(D) \neq POS^{\alpha,\beta}_{\{b\}}(D)$. Thus, the dependency degree acts as only a necessary measure for CR-Preservation, so it can provide only some heuristic information for CRP-Reduct; furthermore, benign heuristic measures are worth deep mining.

4. Set-region preservation and set-region preservation reduct (SRP-Reduct)

At the C-Region level, Section 3 studies CRP-Reduct by constructing an improved type of C-Regions, where S-Regions play a core role in integrated C-Regions construction. In fact, original S-Regions can fully realize regional descriptions in view of the two-category characteristic. Thus, at the S-Region level, this section mainly discusses reduction targets, preservation properties, and SRP-Reduct.

4.1. Set-region preservation target

This subsection first analyzes reduction targets according to S-Regions' monotonic expansion, and it finally concludes the rational criterion of S-Region preservation (SR-Preservation).

Proposition 4.1. *In Pawlak-Reduction, the following six targets are equivalent: (1) CR-Preservation; (2) C-POS preservation; (3) C-BND preservation; (4) SR-Preservation; (5) POS and NEG preservation; (6) BND preservation.*

Proof. Herein, only (2) ⇒ (5) needs proving. When obtaining B from C, $POS_B(X) \subseteq POS_C(X), NEG_B(X) \subseteq NEG_C(X)$, so $POS_B(D) = POS_B(X) \cup NEG_B(X) \subseteq POS_C(X) \cup NEG_C(X) = POS_C(D)$. If $(POS_B(X), NEG_B(X)) \neq (POS_C(X), NEG_C(X))$, then suppose $POS_B(X) \subset POS_C(X)$. Thus, $POS_B(X) \cap NEG_B(X) = \emptyset, POS_C(X) \cap NEG_C(X) = \emptyset$, so $POS_B(D) = POS_B(X) \cup NEG_B(X) \subset POS_C(X) \cup NEG_C(X) = POS_C(D)$. However, $POS_B(D) \neq POS_C(D)$ contradicts $POS_B(D) = POS_C(D)$. Hence, $(POS_B(X), NEG_B(X)) = (POS_C(X), NEG_C(X))$ if $POS_B(D) = POS_C(D)$. □

Proposition 4.1 provides several equivalent targets for qualitative reduction. Thus, SR-Preservation is equivalent to CR-Preservation (where regional change monotonicity plays a key role in granularity transformation), so SR-Preservation becomes a new reduction viewpoint within the two-category framework. In particular, SR-Preservation implies that only three = symbols emerge for qualitative S-Region change, and this conclusion should be highly consulted in quantitative reduction construction.

In fact, DTRS-Reduct should rationally expand Pawlak-Reduct because DTRS-Model expands Pawlak-Model. Thus, this necessary expansion becomes an important requirement to choose potential targets and to further construct DTRS-Reducts. Next, only reduction targets are analyzed due to their fundamental role for reduct construction, and we begin to seek potential DTRS-Reduction targets.

Proposition 4.2. *DTRS S-Regions exhibit change non-monotonicity in attribute deletion.*

Example 1 shows the change non-monotonicity of DTRS S-Region, which underlies the change non-monotonicity of DTRS C-POS. Thus, three kinds of DTRS S-Regions have much change possibility, and we mainly analyze the monotonic expansion change from the usual set theory. In other words, = can be reasonably extended to only three cases regarding $=, \subseteq, \supseteq$, which express new S-Region's equality change, non-enlargement change, and non-lessening change for corresponding old S-Region, respectively.

Proposition 4.3. *Within a framework of monotonic expansion, there are and only are 13 types of DTRS S-Region change, i.e., Cases (1) (2) (3a) (3b) (3c) (4a) (4b) (4c) (5–9) in Table 2.*

Proof. The systemic change symbols are easily obtained by first choosing $=, \subseteq, \supseteq$ for POS and NEG, because independent POS and NEG determine BND. □

The 13 cases originate from both the Pawlak-Reduction target and DTRS extension feature, and based on the set inclusion relation, each one has the expansion property for SR-Preservation. Moreover, each case can exactly emerge, and following Example 2 provides the example illustration. In Table 2, they can be further concluded to only nine cases, i.e., Cases (1–9), where $*$ means arbitrary symbols of $=, \subseteq, \supseteq$. In particular, Cases (1–4, 7) exhibit transcendence, because there are only four types (Cases (5) (6) (8) (9)) for Pawlak-Model. Moreover, many criteria exhibit symmetry regarding three S-Regions.

Example 2. In D-Table $(U, C \cup D)$ from Table 3, $U = \{x_1, \ldots, x_6\}$, $C = \{a, b\}, D = \{d\}, U/\{d\} = \{\{x_1, x_2, x_4, x_5, x_6\}, \{x_3\}\}, X = \{x_1, x_2, x_4, x_5, x_6\}$. Thus, $U/\{a, b\} = \{\{x_1\}, \{x_2, x_3\}, \{x_4, x_5, x_6\}\}, U/\{a\} = \{\{x_1\}, \{x_2, x_3, x_4, x_5, x_6\}\}$. $P(X|\{x_1\}) = 1, P(X|\{x_2, x_3\}) = 0.5, P(X|\{x_4, x_5, x_6\}) = 1; P(X|\{x_2, x_3, x_4, x_5, x_6\}) = 0.8$. Let $0 < \beta < 0.5 < \alpha < 0.8$. Old S-Regions are $POS^{\alpha,\beta}_{\{a,b\}}(X) = \{x_1, x_4, x_5, x_6\}, BND^{\alpha,\beta}_{\{a,b\}}(X) = \{x_2, x_3\}, NEG^{\alpha,\beta}_{\{a,b\}}(X) = \emptyset$; when deleting b, new S-Regions become $POS^{\alpha,\beta}_{\{a\}}(X) = U, BND^{\alpha,\beta}_{\{a\}}(X) = \emptyset, NEG^{\alpha,\beta}_{\{a\}}(X) = \emptyset$. This regional change reflects practical existence of Table 2's Case (2), which never happens in Pawlak-Model. □

The 13 cases fully provide a rational range for DTRS-Reduction targets from a usual viewpoint of set theory. In other words, we yield complete targets of DTRS-Reduction. Next, our task is to select a rational target by some analyzes. Cases (5) (6) (9) which lessen C-POS are suspectable, because Pawlak-Reduct never less-

Table 2
DTRS-Reduction targets based on S-Regions' monotonic expansion.

Case	POS change	NEG change	BND change
(1)	$\supseteq$	$\supseteq$	$\subseteq$
(2)	$\supseteq$	$=$	$\subseteq$
(3)	$\supseteq$	$\subseteq$	$*$
(3a)	$\supseteq$	$\subseteq$	$\supseteq$
(3b)	$\supseteq$	$\subseteq$	$\subseteq$
(3c)	$\supseteq$	$\subseteq$	$=$
(4)	$\subseteq$	$\supseteq$	$*$
(4a)	$\subseteq$	$\supseteq$	$\supseteq$
(4b)	$\subseteq$	$\supseteq$	$\subseteq$
(4c)	$\subseteq$	$\supseteq$	$=$
(5)	$\subseteq$	$=$	$\supseteq$
(6)	$\subseteq$	$\subseteq$	$\supseteq$
(7)	$=$	$\supseteq$	$\subseteq$
(8)	$=$	$=$	$=$
(9)	$=$	$\subseteq$	$\supseteq$

Table 3
D-Table of Example 2.

U	a	b	d
x_1	1	1	1
x_2	0	1	1
x_3	0	1	0
x_4	0	0	1
x_5	0	0	1
x_6	0	0	1

ens C-POS even when it has operational possibility. In contrast, enlarging POS, enlarging NEG, or lessening BND seem preferable because POS, NEG, and BND correspond to positive, negative, and deferred decisions, respectively. For example, Cases (1) (2) seem to be perfect directions. Next, an imaginary reduct is provided as an example.

Definition 4.4. B is an imaginary reduct of C, if it satisfies two conditions:

(1) $\mathrm{NEG}_B^{\alpha,\beta}(X) = \mathrm{NEG}_C^{\alpha,\beta}(X), \mathrm{POS}_B^{\alpha,\beta}(X) \supseteq \mathrm{POS}_C^{\alpha,\beta}(X)$;
(2) If $\mathrm{NEG}_{B'}^{\alpha,\beta}(X) = \mathrm{NEG}_C^{\alpha,\beta}(X)$, then $\mathrm{POS}_{B'}^{\alpha,\beta}(X) \not\supseteq \mathrm{POS}_C^{\alpha,\beta}(X), \forall \emptyset \neq B' \subset B$.

The imaginary reduct is mainly according to Case (2) – POS non-lessening, NEG preservation (and BND non-enlargement). In Example 2, $\{a\}$ acts as the sole imaginary reduct of $\{a,b\}$, and $\mathrm{POS}_{\{a\}}^{\alpha,\beta}(X) = U$. Thus, x_2 and x_3 originally adopt the deferred decision for initial $\{a,b\}$, but by attribute reduction, they finally correspond to the positive decision for reduct $\{a\}$. In a vivid treatment explanation, two deferred patients are transferred to be treated only by removing a medical examination of b. This contradiction originates from inconformity between theoretical change and practical processing for approximation. In fact, deleting attribute b necessarily leads to uncertainty reinforcement, and DTRS-POS can be enlarged because it has already concerned uncertainty; however, DTRS-POS is directly related to the positive decision in the practical approximation processing. Thus, the imaginary reduct is worth deeply discussing. In DRTS-Model, three DTRS S-Regions actually exhibit equality in some degree within the framework of quantitative uncertainty, and S-Region change implies decision change, so other regional change reducts are not sure as well.

In fact, only SR-Preservation of Case (8) does not change all S-Regions and decisions, thus becoming the surest target in view of the S-Region change non-monotonicity. Based on DTRS modeling, SR-Preservation can necessarily keep optimal structure which is related to original application decisions. Moreover, SR-Preservation is the most natural for DTRS-Reduction because it is exactly Pawlak-Reduction target. In a word, for DTRS-Reduction targets based on quantitative expansion, more choices do not necessarily bring better reducts, but the initial one must be sure and feasible. Therefore, SR-Preservation becomes the most appropriate DTRS-Reduction target at the S-Region level.

Theorem 4.5. *In DTRS-Model, SR-Preservation and CR-Preservation are separate. Furthermore, SR-Preservation induces CR-Preservation, but the opposite does not hold.*

In Pawlak-Model, SR-Preservation and CR-Preservation are equivalent (Proposition 4.1), which is mainly attributed to the regional change monotonicity. However, in DTRS-Model, SR-Preservation is not only different from but also stronger than CR-Preservation; herein, C-POS integration leads to the natural induction and its opposite for both targets. Thus, this target separability provides us new development spaces beyond CR-Preservation and CRP-Reduct.

4.2. Set-region preservation properties

SubSection 4.1 analyzes the SR-Preservation rationality. Thus, this subsection further provides several SR-Preservation properties, which underlie later SRP-Reduct development. Herein, in D-Table $(U,C,D), C = B \cup C', C' = \{a_1, a_2, \ldots, a_t\}$ $(t \geqslant 2)$; $C \xrightarrow{-C'} C - C'$ or $C \xrightarrow{-} C - C'$ describe the process of deleting C', and $C \xrightarrow{-a} C - \{a\}$ is also used. Moreover, SRP and $SRNP$ are used to denote S-Region preservation and S-Region none-preservation, respectively; thus, for attribute deletion $\varphi : . \xrightarrow{-} .$, "$. \xrightarrow{-} . \models SRP$" means that φ preserves S-Regions, while "$. \xrightarrow{-} . \models SRNP$" denotes the opposite.

Lemma 4.6. $\forall a_1, a_2, \ldots, a_t$,

$$C \xrightarrow{-a_1} B - \{a_1\} \xrightarrow{-a_2} \cdots \xrightarrow{-a_t} B - \{a_1, a_2, \ldots, a_t\} \iff C \xrightarrow{-C'} B.$$

According to element deletion, Lemma 4.6 describes interchangeability and composition for a deletion sequence. Furthermore, this result is also applicable to subset deletion. Herein, let $C_0 \subseteq C', . \xrightarrow{-C_0} .$ is in sequence T for $C \xrightarrow{-C'} B$.

Proposition 4.7 (Infection Principle).

$$\exists T, \exists . \xrightarrow{-C_0} . \models SRNP \iff C \xrightarrow{-C'} B \models SRNP,$$
$$\exists C_0^* \subseteq C', C \xrightarrow{-C_0^*} C - C_0^* \models SRNP \iff C \xrightarrow{-C'} C - C' \models SRNP.$$

Proof. (1) For $. \xrightarrow{-C_0} . \models SRNP$, two granules $[x]_1$ and $[x]_2$ from two different old S-Regions will be merged into a new granule $[x]_3$, i.e., $[x]_1 \cup [x]_2 \subseteq [x]_3$. Based on Lemma 4.6, $. \xrightarrow{-C_0} .$ is supposed to be the first SRNP process in the sequence. In the later deletion, $[x]_3$ would not be decomposed due to knowledge roughening. Hence, $C \xrightarrow{-C'} B \models SRNP$, because both $[x]_1$ and $[x]_2$ are finally in a new S-Region of B. The opposite is clear by setting up $C_0 = C'$ and $T = \{C \xrightarrow{-C'} B\}$. (2) $C \xrightarrow{-C'} B \iff C \xrightarrow{-C_0^*} C - C_0^* \xrightarrow{-} B$. $C \xrightarrow{-C_0^*} C - C_0^* \models SRNP$, so $C \xrightarrow{-C'} B \models SRNP$ according to (1). The opposite can be obtained by setting up $C_0^* = C'$. □

Corollary 4.8 (Purity Principle).

$$C \xrightarrow{-C'} B \models SRP \iff \forall T, \forall . \xrightarrow{-C_0} . \models SRP,$$
$$C \xrightarrow{-C'} C - C' \models SRP \iff \forall C_0^* \subseteq C', C \xrightarrow{-C_0^*} C - C_0^* \models SRP.$$

Based on Infection Principle, arbitrary SRNP sub-deletion at arbitrary a stage necessarily leads to SRNP for the whole deletion. In other words, SRNP has infection from interior deletion. Thus, Infection Principle mainly cancels our worry whether global SRP can be implemented by local SRNP. In contrast, dual Purity Principle reflects SRP harmony between global and local deletion.

To extract SRP and SRNP elements, let $SRP(A) = \{a \in A | A \xrightarrow{-a} A - \{a\} \models SRP\}$ and $SRNP(A) = \{a \in A | A \xrightarrow{-a} A - \{a\} \models SRNP\}$. Thus, $SRP(A)$ defines the elements in A, whose deletion on A preserves S-Regions, while $SRNP(A)$ describes the opposites.

Proposition 4.9 (Core Principle). $SRNP(C) \cap C' \neq \emptyset \Rightarrow C \xrightarrow{-C'} B \models SRNP$.

Proof. $SRNP(C) \cap C' \neq \emptyset$ implies that $\exists a_i \in C', C \xrightarrow{-a_i} C - \{a_i\} \models SRNP$. Based on Infection Principle, $C \xrightarrow{-C'} B \models SRNP$. □

Corollary 4.10. $C \xrightarrow{-C'} B \models SRP \Rightarrow C' \subseteq SRP(C)$.

According to SRP in single attribute deletion, condition attributes are divided into two classes – $SRP(C)$ and $SRNP(C)$. For SRP, we cannot delete attributes in $SRNP(C)$ but may delete some in $SRP(C)$, so $SRNP(C)$ acts as the attribute core. However, neither opposite of Core Principle and its dual Corollary 4.10 holds.

The change monotonicity of qualitative regions is a specific characteristic for Pawlak-Model. It is so perfect that it determines many important processing, such as the SR-Preservation requirement and Pawlak-Reduction construction. However, for DTRS S-Region, the change monotonicity has been collapsed while the change non-monotonicity emerges. In this new situation, Infection and Purity Principles transcend the regional change monotonicity by utilizing initial knowledge monotonicity, so they play a fundamental role for quantitative DTRS-Reduction. In fact, they ensure the SR-Preservation stability and provide the attribute core, thus underling the next SRP-Reduct construction.

4.3. Set-region preservation reduct (SRP-Reduct)

Based on SR-Preservation and its properties, this subsection naturally constructs SRP-Reduct.

Definition 4.11 (*SRP-Reduct*). c is an indispensable attribute in C, if $C \xrightarrow{-c} C - \{c\} \models SRNP$; otherwise, c is dispensable. B is independent, if b is indispensable in $B, \forall b \in B$. B is SRP-Reduct of C, if B is independent and $C \xrightarrow{-} B \models SRP$. Moreover, $CORE^{\alpha,\beta}_{SRP}(C)$ and $RED^{\alpha,\beta}_{SRP}(C)$ denote the core and reduct set, respectively.

Proposition 4.12. *The following two items are equivalent:*

(1) $B \xrightarrow{-b} B - \{b\} \models SRNP, \forall b \in B$;
(2) $B \xrightarrow{-} B' \models SRNP, \forall \emptyset \neq B' \subset B$.

Proof. $\forall \emptyset \neq B' \subset B, \exists b \in B - B'$, s.t., $\emptyset \neq B' \subseteq B - \{b\}$. Based on Lemma 4.6, $B \xrightarrow{-} B' \Longleftrightarrow B \xrightarrow{-b} B - \{b\} \xrightarrow{-} B'$. $B \xrightarrow{-b} B - \{b\} \models SRNP$, so $B \xrightarrow{-} B' \models SRNP$ according to Infection Principle; thus, (1) $\Rightarrow$ (2) holds. In contrast, (2) $\Rightarrow$ (1) can be proved by setting up $B' = B - \{b\}$. □

According to SR-Preservation, Definition 4.11 defines SRP-Reduct by using the set independence. Furthermore, Proposition 4.12 reflects equivalence of the set independence and maximality for SR-Preservation. Thus, SRP-Reduct can also be given by the set maximality.

Definition 4.13 (*SRP-Reduct*). B is SRP-Reduct of C, if it satisfies two conditions:

(1) $C \xrightarrow{-} B \models SRP$;
(2) $B \xrightarrow{-} B' \models SRNP, \forall \emptyset \neq B' \subset B$.

Theorem 4.14 (SRP-Reduct's Expansion). *SRP-Reduct expands Pawlak-Reduct to DTRS-Reduct, and it degenerates into Pawlak-Reduct when* $(\alpha, \beta) = (1, 0)$.

SRP-Reduct is equivalently defined by the set independence and maximality to achieve SR-Preservation, so it is a minimal subset to provide the optimal structure and decision regarding C. By virtue of rational SR-Preservation, SRP-Reduct naturally expands qualitative reduction to quantitative reduction in the environment of the change non-monotonicity, thus becoming scientific and valuable. In particular, SRP-Reduct becomes a distinctive reduct for the two-category case.

Theorem 4.15. $CORE^{\alpha,\beta}_{SRP}(C) = \bigcap RED^{\alpha,\beta}_{SRP}(C)$.

Proof. (1) If $c \notin \bigcap RED^{\alpha,\beta}_{SRP}(C)$, then $\exists B \in RED^{\alpha,\beta}_{SRP}(C)$ but $c \notin B$. Thus, $C \xrightarrow{-(C-B)} B \models SRP, c \in C - B \subseteq C$; based on Purity Principle, $C \xrightarrow{-c} C - \{c\} \models SRP$, i.e., $c \in SRP(C), c \notin SRNP(C)$. Hence, $c \notin CORE^{\alpha,\beta}_{SRP}(C)$, $CORE^{\alpha,\beta}_{SRP}(C) \subseteq \bigcap RED^{\alpha,\beta}_{SRP}(C)$. (2) If $c \notin CORE^{\alpha,\beta}_{SRP}(C)$, then $C \xrightarrow{-c} C - \{c\} \models SRP$. $\exists B \in RED^{\alpha,\beta}_{SRP}(C - \{c\})$, so $C - \{c\} \longrightarrow B \models SRP$ and B is independent. Based on Lemma 4.6, $C \xrightarrow{-} B \models SRP$. Thus, $B \in RED^{\alpha,\beta}_{SRP}(C)$, but $c \notin B$. Hence, $c \notin \bigcap RED^{\alpha,\beta}_{SRP}(C), \bigcap RED^{\alpha,\beta}_{SRP}(C) \subseteq CORE^{\alpha,\beta}_{SRP}(C)$. □

To construct SRP-Reduction, Infection and Purity Principles are utilized to prove the set maximality/independence and attribute core, so they fulfil basic functions of the qualitative change monotonicity. Herein, core $CORE^{\alpha,\beta}_{SRP}(C)$ exhibits the similar relationship and significance for SRP-Reduct, and the related conclusion can be verified by the previous two examples. In Example 1, $CORE^{\alpha,\beta}_{SRP}(\{a,b\}) = \{a\}$ and $RED^{\alpha,\beta}_{SRP}(\{a,b\}) = \{\{a\}\}$. In Example 2, $\{a,b\} \xrightarrow{-b} \{a\} \models SRNP$, so $b \in CORE^{\alpha,\beta}_{SRP}(\{a,b\})$. When deleting a, there is only a granular merging process – $\{x_1\} \cup \{x_2, x_3\} = \{x_1, x_2, x_3\}$; $\{x_1\} \subseteq POS^{\alpha,\beta}_{\{a,b\}}(X), \{x_2, x_3\} \subseteq BND^{\alpha,\beta}_{\{a,b\}}(X)$, so $\{a,b\} \xrightarrow{-a} \{b\} \models SRNP$. Thus, $CORE^{\alpha,\beta}_{SRP}(\{a,b\}) = \{a,b\}, RED^{\alpha,\beta}_{SRP}(\{a,b\}) = \{\{a,b\}\}$.

Definition 4.16 (*Dependency Degree*). Positive, boundary, negative dependency degree of D on A are defined by:

$$\begin{cases} \gamma^{POS}_A(D) = \frac{|POS^{\alpha,\beta}_A(X)|}{|U|}, \\ \gamma^{BND}_A(D) = \frac{|BND^{\alpha,\beta}_A(X)|}{|U|}, \\ \gamma^{NEG}_A(D) = \frac{|NEG^{\alpha,\beta}_A(X)|}{|U|}, \end{cases}$$
$${}^{SRP}\gamma_A(D) = (\gamma^{POS}_A(D), \gamma^{NEG}_A(D))$$

is further called the dependency degree array of D on A.

In Pawlak-Model, $\gamma_A(D)$ describes only a part – C-POS. In DTRS-Model, three S-Regions are utilized to construct three measures, i.e., $\gamma^{POS}_A(D), \gamma^{BND}_A(D), \gamma^{NEG}_A(D)$, and they measure classification proportions for three parts – POS, NEG, BND, respectively. Moreover, $\gamma_A(D)$ is addition fusion of two related regional measures (Proposition 2.3); in contrast, array ${}^{SRP}\gamma_A(D)$ mainly makes a systemic description by independent POS and NEG. Clearly, $\gamma^{POS}_A(D), \gamma^{NEG}_A(D), \gamma^{BND}_A(D) \in [0,1], \gamma^{POS}_A(D) + \gamma^{NEG}_A(D) + \gamma^{BND}_A(D) = 1$.

Proposition 4.17.

(1) ${}^{CRP}\gamma_A(D) = \gamma^{POS}_A(D) + \gamma^{NEG}_A(D)$. ${}^{CRP}\gamma_A(D) \geqslant |{}^{SRP}\gamma_A(D)|; {}^{CRP}\gamma_A(D) = |{}^{SRP}\gamma_A(D)|$, if $\gamma^{POS}_A(D) = 0$ or $\gamma^{NEG}_A(D) = 0$.
(2) $\gamma^{POS}_A(D) + \gamma^{NEG}_A(D) \geqslant \gamma_A(D); \gamma^{POS}_A(D) + \gamma^{NEG}_A(D) = \gamma_A(D)$, *if* $(\alpha, \beta) = (1, 0)$.

Proposition 4.18. *Let* $A_1 \subseteq A_2 \subseteq C$. *If* $A_2 \xrightarrow{-} A_1 \models SRP$,
$\gamma^{POS}_{A_1}(D) + \gamma^{NEG}_{A_1}(D) = \gamma^{POS}_{A_2}(D) + \gamma^{NEG}_{A_2}(D)$,
${}^{SRP}\gamma_{A_1}(D) = {}^{SRP}\gamma_{A_2}(D)$.

Based on Proposition 4.17, addition fusion $\gamma^{POS}_A(D) + \gamma^{NEG}_A(D)$ of SRP-Reduct is equal to ${}^{CRP}\gamma_A(D)$ of CRP-Reduct because of C-POS's combination, and it also exhibits monotonicity for $\gamma_A(D)$ of Pawlak-Reduct because of S-Regions' expansion. Based on Proposition 4.18, ${}^{SRP}\gamma_A(D)$ equality acts as only a necessary condition for SR-Preservation, which is attributed to the change non-monotonicity of $\gamma^{POS}_A(D), \gamma^{BND}_A(D), \gamma^{NEG}_A(D)$. In particular, the insufficiency can be verified by Example 1, where ${}^{SRP}\gamma_{\{a,b\}}(D) = (0.5, 0) = {}^{SRP}\gamma_{\{b\}}(D)$ but

$\{a, b\} \xrightarrow{-a} \{b\} \models SRNP$. Thus, the dependency degree array, as only a necessary measure, can partly reflect attribute significance and heuristic information for SRP-Reduct.

5. Double-preservation and double-preservation reduct (DP-Reduct)

By analyzing rule consistency and its regional essence, this section first establishes consistency preservation, which is essentially qualitative preservation. Furthermore, DP-Reduct is proposed to dually preserve S-Region and consistency, which is essentially a quantitative and qualitative dual reduct.

Except the usual regional way, there is another fundamental approach for Pawlak-Reduct, i.e., the consistency method. In fact, qualitative reduction mainly preserves rule consistency of both consistent and inconsistent D-Tables in the decision logic and algorithm [2]. Thus, the consistency requirement further provides us a novel viewpoint and development power for quantitative reduction construction. Since the two-category feature more adheres to the region-based approach, we first explore regional essence of rule consistency and consistency preservation.

Definition 5.1. $\forall E \in 2^U$, mappings $cst_A, icst_A : 2^U \longrightarrow 2^U$ are defined as follows:

$$\begin{cases} cst_A(E) = \{x \in E | \forall [x]_A, ([x]_A \subseteq X) \vee ([x]_A \subseteq \neg X)\}, \\ icst_A(E) = \{x \in E | \forall [x]_A, ([x]_A \cap X \neq \emptyset) \wedge ([x]_A \cap \neg X \neq \emptyset)\}. \end{cases}$$

$cst_A(E)$ and $icst_A(E)$ are called consistent and inconsistent regions of E on A, respectively.

Proposition 5.2

$$\begin{cases} cst_C(E) = E - \mathrm{BND}_C(X), \\ icst_C(E) = E \cap \mathrm{BND}_C(X). \end{cases}$$

By using mappings, consistent and inconsistent regions are mined (in Definition 5.1) to extract objects connected with consistent and inconsistent rules, respectively. Thus, they equivalently describe rule consistency and inconsistency from the regional perspective, and they are mainly related to $\mathrm{BND}_C(X)$ according to their macroscopical formulas (Proposition 5.2).

Corollary 5.3

$$\begin{cases} cst_C(U) = U - \mathrm{BND}_C(X), \\ icst_C(U) = \mathrm{BND}_C(X). \end{cases}$$

Regional essence of consistency and inconsistency are reflected (in Corollary 5.3) by assessing consistent and inconsistent regions of the universe. Thus, consistency and inconsistency are actually related to only qualitative Pawlak-BND, so they are completely determined by D-Table's formal structure on C and D (rather than by concrete models or quantitative regions). In other words, qualitative S-Regions (especially Pawlak-BND) have been promoted to be related to not only Pawlak-Regions but also consistency/inconsistency (for all models).

Corollary 5.4 (Consistency Change Monotonicity). *In attribute deletion, consistency has change monotonicity, i.e., the consistent and inconsistent regions of the universe are not enlarged and lessened, respectively.*

For quantitative reduction, consistency preservation is also a considerable target, because qualitative reduction acts as the sole reference. It becomes a rational criterion based on the consistency change monotonicity, which is related to the consistency's regional essence. In particular, consistency preservation is equivalent to qualitative CR-Preservation, i.e., it maintains qualitative Pawlak-Regions. Thus, let us further analyze its scientific nature in the quantitative environment. In fact, preserving rule consistency aims to maintain absolute and complete certainty due to qualitative factors. In other words, the certainty position cannot change even in slight degree, and real certainty cannot be fused with other approximate certainty because both are completely two different notions from the qualitative viewpoint; in contrast, inconsistency and uncertainty also need qualitative preservation. In summary, based on certainty/uncertainty, qualitative regions are no longer limited to only Pawlak-Reduction but further determine all models reduction; thus, based on consistency/inconsistency, Pawlak-Reduct has been promoted to extensively describe quantitative reduction.

Theorem 5.5. *Consistency preservation is equivalent to qualitative CR-Preservation. CR-Preservation, SR-Preservation, and consistency preservation are equivalent in Pawlak-Model but are separate in DTRS-Model.*

Theorem 5.5 reflects uniformity and separability of usual three targets of Pawlak-Reduction and DTRS-Reduction, respectively. In Pawlak-Model, SR-Preservation and consistency preservation are uniformly implemented by CR-Preservation, i.e., qualitative C-POS plays a dual role for both region and consistency parts. However, in DTRS-Model, both CR-Preservation and SR-Preservation correspond to only quantitative regions rather than qualitative consistency/inconsistency blocks. Thus, consistency preservation further provides a new development space for hierarchical construction of DTRS-Reduction, and the integration of SR-Preservation and consistency preservation becomes a rational and novel strategy due to their separability.

Definition 5.6 (*D-Preservation*). In DTRS-Model, D-Preservation means both SR-Preservation and consistency preservation, i.e., $C \xrightarrow{-} B \models SRP$ and $\mathrm{BND}_B(X) = \mathrm{BND}_C(X)$ for $C \xrightarrow{-} B$. Herein, $C \xrightarrow{-} B \models DP$ and $C \xrightarrow{-} B \models DNP$ denote D-Preservation and its opposite.

Theorem 5.7. *D-Preservation induces SR-Preservation and consistency preservation, but neither opposite holds.*

Clearly, integrated D-Preservation holds a dual preservation function. In fact, it perfectly fulfils all Pawlak-reduction targets, and it essentially preserves five parts by introducing Pawlak-BND into three DTRS S-Regions. By adding consistency factors and qualitative ideas, D-Preservation becomes a novel and valuable criterion for quantitative reduction.

Theorem 5.8 (DP-Reduct). *B is DP-Reduct of C, if it satisfies two conditions:*

(1) $C \xrightarrow{-} B \models DP$;
(2) $B \xrightarrow{-} B' \models DNP, \forall \emptyset \neq B' \subset B$.

Herein, $RED_{DP}^{\alpha,\beta}(C)$ denotes the reduct set, and the intersection of all reducts constitutes core $CORE_{DP}^{\alpha,\beta}(C)$.

Theorem 5.9 (DP-Reduct's Expansion). *DP-Reduct expands Pawlak-Reduct to DTRS-Reduct, and it degenerates into Pawlak-Reduct when $(\alpha, \beta) = (1, 0)$.*

By virtue of D-Preservation, DP-Reduct naturally expands qualitative reduction to quantitative reduction. In particular, D-Preservation and DP-Reduct adopt a quantitative and qualitative dual strategy to implement expansion completeness, and they go beyond SR-Preservation and SRP-Reduct. Therefore, they provide a novel thought and a nice prospect for quantitative reduction.

Based on complete D-Preservation, we can further make in-depth structural construction. For example, accurate reducts can be considered by feasible internal structure. For explanation, we finally provide an optimal reduct with measures. Herein, a set of measures $M = (m_1, m_2, \ldots)$ is given for evaluating C regarding D, and Ref. [15] provided multiple measures.

Definition 5.10 (*Optimal Reduct*). *B* is an optimal reduct of *C* on measure set *M*, if it satisfies two conditions:

(1) $C \xrightarrow{-} B \models DP$, and $m(B) \succeq m(C), \forall m \in M$;
(2) $\forall \emptyset \neq B' \subset B$, if $B \xrightarrow{-} B' \models DP$, then $m(B') \prec m(B), \forall m \in M$.

Proposition 5.11. *On the D-Preservation premise, optimal reducts exhibit nondecreasing optimization for given measures.*

Based on D-Preservation, the optimal reduct mainly makes an optimal adjustment of internal structure by considering multiple measures. Thus, it can seek optimization for given measures within the structural framework of D-Preservation.

6. Reduction hierarchies

Thus far, we have established three systems of quantitative reduction, i.e., (1) CR-Preservation and CRP-Reduct, (2) SR-Preservation and SRP-Reduct, (3) D-Preservation and DP-Reduct. In this section, we first introduce the fourth reduction system – (4) knowledge preservation (K-Preservation) and K-Preservation reduct (KP-Reduct); furthermore, we emphatically analyze reduction hierarchies of the four systems and the Pawlak-Reduction system (which is labeled by symbol (0)). Finally, a D-Table example is provided for illustration.

Definition 6.1 (*K-Preservation and KP-Reduct*). $C \xrightarrow{-} A$ satisfies K-Preservation, if $IND(A) = IND(C)$. *B* is KP-Reduct of *C*, if it satisfies two conditions:

(1) $IND(B) = IND(C)$;
(2) $IND(B') \neq IND(B), \forall \emptyset \neq B' \subset B$.

Herein, $CORE_{KP}(C)$ and $RED_{KP}(C)$ denote the natural core and reduct set, respectively.

For D-Table reduction, K-Preservation is the highest requirement and the surest strategy. In particular, KP-Reduct is applicable to D-Table $(U, C \cup D)$, but it is actually the usual qualitative reduct for information table (U, C). Moreover, KP-Reduct also falls into the category of quantitative reduction though it essentially corresponds to qualitative reduction.

Theorem 6.2 (Target Hierarchies).

(1) *If* $IND(A) = IND(C)$, *then* $C \xrightarrow{-} A \models DP$.
(2) *If* $C \xrightarrow{-} A \models DP$, *then* $C \xrightarrow{-} A \models SRP$ *and* $\mathrm{POS}_A(D) = \mathrm{POS}_C(D)$.
(3) *If* $C \xrightarrow{-} A \models SRP$, *then* $\mathrm{POS}_A^{\alpha,\beta}(D) = \mathrm{POS}_C^{\alpha,\beta}(D)$.

Moreover, all opposites do not hold.

For four DTRS-Reduction systems, Theorem 6.2 provides their target hierarchies, i.e., K-Preservation induces D-Preservation, D-Preservation induces SR-Preservation, while SR-Preservation induces CR-Preservation. Note that all opposites cannot hold, and some verification will be provided in next Example 3. For the four targets, their hierarchies reflect their strong and weak features. In fact, CR-Preservation and K-Preservation act as the lowest and highest targets, respectively, so they determine a complete range which includes SR-Preservation and D-Preservation. From a regional perspective, CR-Preservation, SR-Preservation, and D-Preservation actually correspond to macroscopical two-way, three-way, and five-way region preservation, respectively;

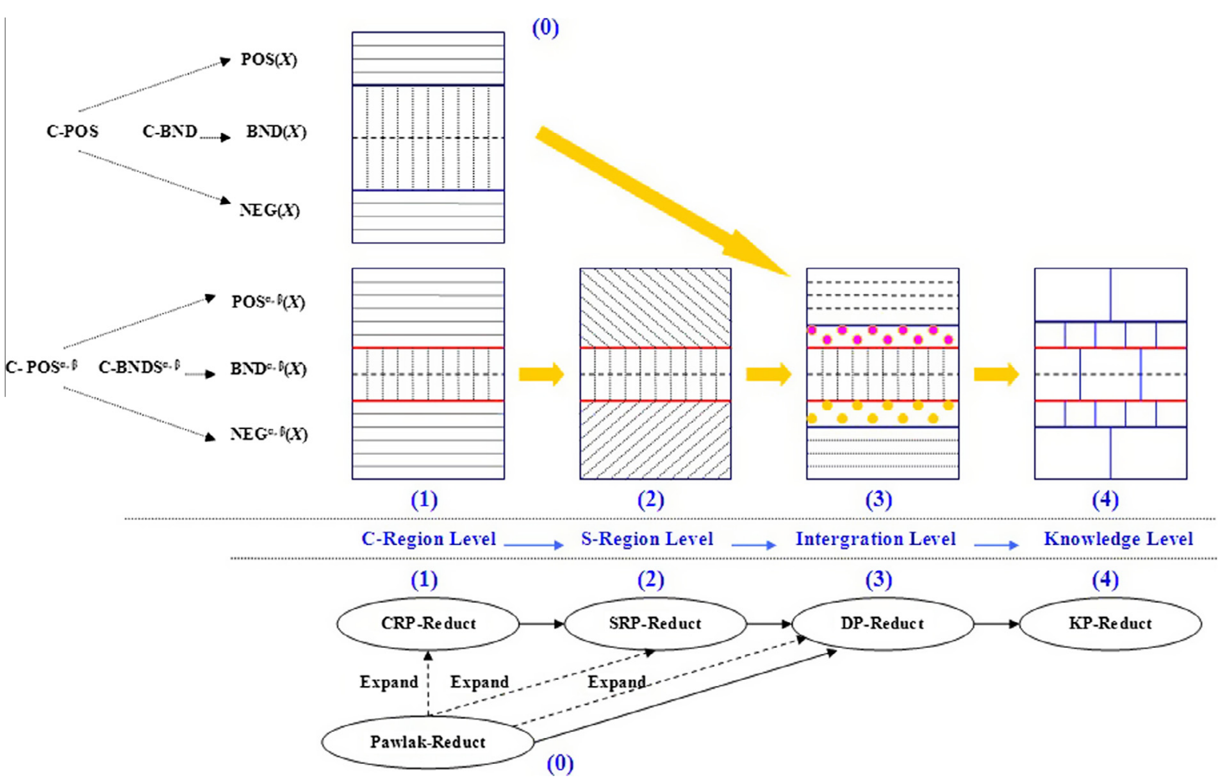

Fig. 2. Hierarchies of five types of reduction.

moreover, K-Preservation aims to preserve microscopical granular structure with $|U/IND(C)|$ regions. In particular, these targets' regional structure and their hierarchies are exhibited by the upper half of Fig. 2, where Pawlak-Reduction target is also added by using its two-way region preservation at the C-Region level. In this sub-figure, the five targets are marked by symbols (0–4), and their development and induction are reflected by the arrow directions and opposite directions, respectively.

In view of targets' pivotal position, the target hierarchies essentially determine the following core hierarchies and reduct hierarchies. For convenience, the core and reduct with a weak target is called the weak core and reduct, respectively, and so are the strong core and reduct. Thus, CRP-Reduct and KP-Reduct become the weakest and strongest DTRS-Reducts, respectively.

Theorem 6.3 (Core Hierarchies). *$CORE^{\alpha,\beta}_{CRP}(C) \subseteq CORE^{\alpha,\beta}_{SRP}(C) \subseteq CORE^{\alpha,\beta}_{DP}(C) \subseteq CORE_{KP}(C)$. Moreover, $CORE(C) \subseteq CORE^{\alpha,\beta}_{DP}(C)$.*

Theorem 6.4 (Reduct Hierarchies).

(1) $\forall B_{KP} \in RED_{KP}(C)$, $\exists B_{DP} \in RED^{\alpha,\beta}_{DP}(C)$, s.t., $B_{DP} \subseteq B_{KP}$.
(2) $\forall B_{DP} \in RED^{\alpha,\beta}_{DP}(C)$, $\exists B_{SRP} \in RED^{\alpha,\beta}_{SRP}(C)$, s.t., $B_{SRP} \subseteq B_{DP}$; $\exists B \in RED(C)$, s.t., $B \subseteq B_{DP}$.
(3) $\forall B_{SRP} \in RED^{\alpha,\beta}_{SRP}(C)$, $\exists B_{CRP} \in RED^{\alpha,\beta}_{CRP}(C)$, s.t., $B_{CRP} \subseteq B_{SRP}$.

Theorem 6.3 reflects the core hierarchies – a weak core is necessarily included by a strong one. Theorem 6.4 reflects the reduct hierarchies – each strong reduct necessarily includes at least a weak reduct; as a result, strong reducts can provide some guidance for weak reducts. However, a weak reduct could still exist beyond the embedded relationship from its strong reduct. In fact, the above hierarchies have provided the imperfect but complete conclusion for reducts, because the reduct requirement of the set maximality/independency hinders more necessary hierarchical results.

By following targets' strong evolutionary, *CRP-Reduct⟶SRP-Reduct⟶DP-Reduct⟶KP-Reduct* reflects the corresponding reduct development from the weak to the strong. In particular, the lower half of Fig. 2 exhibits hierarchies for the four reducts and Pawlak-Reduct. There, development and expansion of reducts are reflected by the solid and virtual arrows, respectively, and marked symbols (0–4) are also used. In fact, CRP-Reduct, SRP-Reduct, and DP-Reduct exhibit expansion for Pawlak-Reduct, and DP-Reduct is also stronger than Pawlak-Reduct. Moreover, Pawlak-Reduct and KP-Reduct are qualitative, while the surplus four are quantitative. Next, generalization and accuracy are analyzed for four DTRS-Reducts. In fact, the weak/strong reducts imply their strong/weak generalization and weak/strong accuracy. Thus, the weakest CRP-Reduct has the strongest generalization but the weakest accuracy, while the strongest KP-Reduct exhibits the opposites; moreover, SRP-Reduct and DP-Reduct hold a middle level.

In Fig. 2, the target and reduct hierarchies are exhibited in the upper and lower halves, respectively. Moreover, from the regional perspective, the granular development is also marked in the middle location. In fact, this evolutionary of *C-Region Level⟶S-Region Level⟶Integration Level⟶Knowledge Level* corresponds to granular decomposition and fine granulation. In particular, reduction hierarchies have natural transitivity, and the inclusion relation of subsets plays an important role, especially for both the core and reduct hierarchies. Thus, by adopting a dendrogram approach, we will particularly organize a reduction hierarchy figure (e.g., Figs. 3–5) to vividly exhibit reduction results, where arrow $\longrightarrow$ and its opposite represent relations $\subseteq$ and $\supseteq$, respectively.

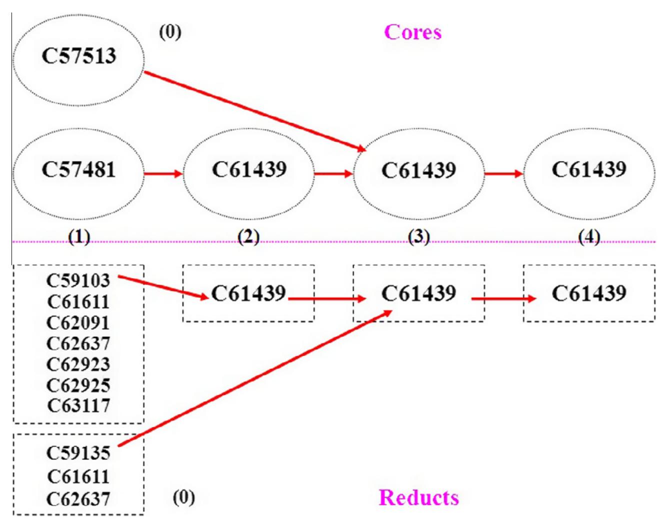

Fig. 4. Reduction hierarchy figure of (3) Voting database.

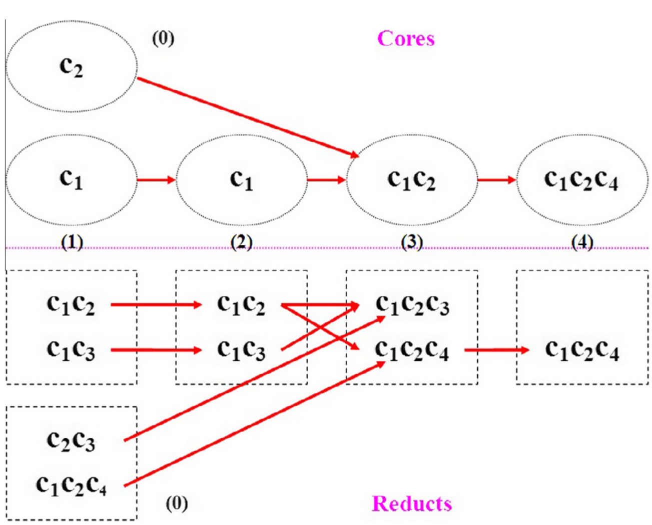

Fig. 3. Reduction hierarchy figure of Example 3.

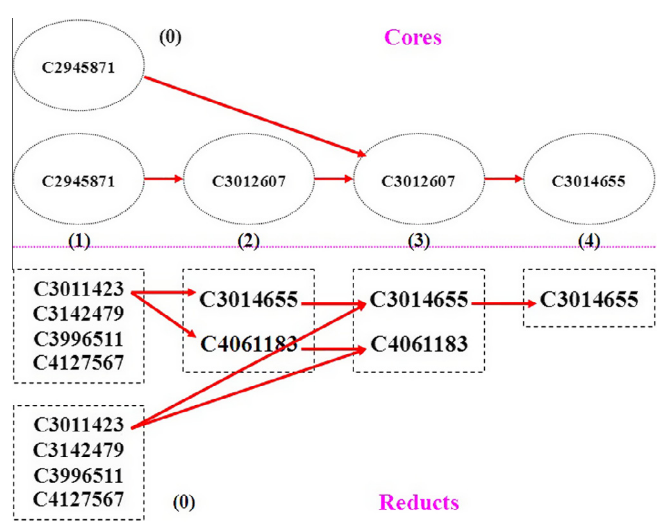

Fig. 5. Reduction hierarchy figure of (4) SPECT Heart database.

Based on the reduct hierarchies, a strong reduct can provide some guidance for seeking weak reducts (especially only one). Thus, we propose a strong reduct-based hierarchial algorithm for weak reducts. In Algorithm 2, Step 1 establishes the guidance of a strong reduct, Step 2 yields the weak core, and Step 3 searches the weak reducts between the weak core and strong reduct. Clearly, this algorithm is convergent and effective and can necessarily achieve all weak reducts included in the given strong reduct. In particular, this hierarchial algorithm's result can be vividly reflected (by the opposite of arrow $\longrightarrow$) in the reduction hierarchy figure (e.g., Figs. 3–5). Moreover, KP-Reduct can be particularly utilized due to its strength and simplicity.

Algorithm 2. A strong reduct-based hierarchial algorithm for weak reducts

Input:
D-Table $(U, C \cup D)$;
Output:
Weak reduct;
1: Give a strong reduct B_{strong};
2: Yield the weak core – $CORE_{weak}(C)$;
3: Search B_{weak} in range $CORE_{weak}(C) \subseteq B_{weak} \subseteq B_{strong}$ to satisfy the weak reduct requirement;
4: **return** B_{weak}.

Finally, for illustrating the reduction hierarchies, a D-Table example is provided and analyzed.

Example 3. In D-Table $(U, C \cup D), U = \{x_1, x_2, \ldots, x_{36}\}, C = \{c_1, c_2, c_3, c_4\}, U/IND(C) = \{[x]_1, \ldots, [x]_8\}, U/IND(D) = \{X, \neg X\}$, and C class-based statistical information is provided by Table 4. Herein, let $(\alpha, \beta) = (0.8, 0.2)$.

Qualitative S-Regions are $\mathrm{POS}_C(X) = [x]_7 \cup [x]_8, \mathrm{NEG}_C(X) = [x]_1 \cup [x]_2, \mathrm{BND}_C(X) = [x]_3 \cup \ldots \cup [x]_6$. In contrast, DTRS S-Regions become $\mathrm{POS}_C^{\alpha,\beta}(X) = [x]_6 \cup [x]_7 \cup [x]_8, \mathrm{NEG}_C^{\alpha,\beta}(X) = [x]_1 \cup [x]_2 \cup [x]_3, \mathrm{BND}_C^{\alpha,\beta}(X) = [x]_4 \cup [x]_5$. Thus, Table 5 provides reduction results of five types, and Fig. 3 further constructs the reduction hierarchy figure.

Based on the reduction results, we provide the following illustration for the reduction hierarchies.

(1) By virtue of $\{c_1, c_2\}, C \xrightarrow{-\{c_3,c_4\}} \{c_1, c_2\} \models SRP$ but $C \xrightarrow{-\{c_3,c_4\}} \{c_1, c_2\} \models DNP$, so SR-Preservation cannot induce D-Preservation; by virtue of $\{c_1, c_2, c_3\}, C \xrightarrow{-c_4} \{c_1, c_2, c_3\} \models DP$ but $IND(\{c_1, c_2, c_3\}) \neq IND(C)$, so D-Preservation cannot induce K-Preservation; moreover, by virtue of $\{c_2, c_3\}, \mathrm{POS}_{\{c_2,c_3\}}(D) = \mathrm{POS}_C(D)$ but $C \xrightarrow{-\{c_1,c_4\}} \{c_2, c_3\} \models DNP$, so qualitative CR-Preservation cannot induce D-Preservation. Thus, these results act as counterexamples to illustrate several opposites of the target hierarchies (Theorem 6.2), where positive induction is clear.

(2) $CORE_{CRP}^{\alpha,\beta}(C) = CORE_{SRP}^{\alpha,\beta}(C) = \{c_1\} \subseteq \{c_1, c_2\} = CORE_{DP}^{\alpha,\beta}(C) \subseteq \{c_1, c_2, c_4\} = CORE_{KP}(C)$; moreover, $CORE(C) = \{c_2\} \subseteq \{c_1, c_2\} = CORE_{DP}^{\alpha,\beta}(C)$. The results reflect the core hierarchies (Theorem 6.3).

(3) Next, we verify the reduct hierarchies (Theorem 6.4). Sole KP-Reduct $\{c_1, c_2, c_4\}$ is also DP-Reduct, and this DP-Reduct includes SRP-Reduct $\{c_1, c_2\}$; the other DP-Reduct $\{c_1, c_2, c_3\}$ includes all SRP-Reducts $\{c_1, c_2\}$ and $\{c_1, c_3\}$, and both SRP-Reducts are also CRP-Reducts. Moreover, DP-Reduct $\{c_1, c_2, c_4\}$ is also Pawlak-Reduct, and the other DP-Reduct $\{c_1, c_2, c_3\}$ includes Pawlak-Reduct $\{c_2, c_3\}$. In particular, DP-Reduct $\{c_1, c_2, c_3\} \not\subseteq \{c_1, c_2, c_4\}$ though $\{\{c_1, c_2, c_4\}\} = RED_{KP}(C)$, so this result acts as a counterexample to verify a specific conclusion that *a weak reduct could exist by transcending the embedded relation regarding its strong reducts*.

For the hierarchial algorithm (Algorithm 2), the reduction hierarchy figure (Fig. 3) exhibits the relevant results. For example, if strong KP-Reduct/DP-Reduct $\{c_1, c_2, c_4\}$ is provided, then weak SRP-Reduct/CRP-Reduct $\{c_1, c_2\}$ will be achieved; if strong DP-Reducts $\{c_1, c_2, c_3\}$ is given, then weak SRP-Reducts/CRP-Reducts $\{c_1, c_2\}$ and $\{c_1, c_3\}$ as well as weak Pawlak-Reduct $\{c_2, c_3\}$ will be obtained. □

Table 4
C class-based statistical information of Example 3.

$[x]_i$	$\lvert [x]_i \rvert$	c_1	c_2	c_3	c_4	$\lvert [x]_i \cap X \rvert$
$[x]_1$	5	2	1	3	1	0
$[x]_2$	3	1	1	1	2	0
$[x]_3$	6	1	2	1	2	1
$[x]_4$	6	3	1	2	1	3
$[x]_5$	4	3	1	2	2	3
$[x]_6$	6	2	2	1	2	5
$[x]_7$	4	2	2	2	1	4
$[x]_8$	2	1	3	2	2	2

Table 5
Five types of reducts and their cores of Example 3.

Type	Core	Reduct
(0) Pawlak-Reduct	$\{c_2\}$	$\{c_2, c_3\}, \{c_1, c_2, c_4\}$
(1) CRP-Reduct	$\{c_1\}$	$\{c_1, c_2\}, \{c_1, c_3\}$
(2) SRP-Reduct	$\{c_1\}$	$\{c_1, c_2\}, \{c_1, c_3\}$
(3) DP-Reduct	$\{c_1, c_2\}$	$\{c_1, c_2, c_3\}, \{c_1, c_2, c_4\}$
(4) KP-Reduct	$\{c_1, c_2, c_4\}$	$\{c_1, c_2, c_4\}$

7. Experiments

This section mainly performs experiments to illustrate effectiveness of relevant reduction systems and their hierarchies, and they are three quantitative reduction systems (regarding CRP-Reduct, SRP-Reduct, and DP-Reduct) and two qualitative reduction systems (regarding Pawlak-Reduct and KP-Reduct).

Four data sets from the UCI Machine Learning Repository [49] were used in our empirical study, and their information regarding D-Table $(U, C \cup D)$ are summarized in Table 6. Herein, each data set has only two decision classes, and decision concept X selectively corresponds to the first element.

For convenience of calculations, symbols $1, 0$ are used to represent selectivity of a condition attribute, and a binary code and its decimal result are further used to represent and label a condition attribute subset. For example, $\{c_1, c_2, c_3, c_9, c_{11}, c_{13}, c_{16}\}$ with $|C| = 16$ corresponds to binary code 1110000010101001 and decimal Result 57513, so it is labeled by $C57513 =$

Table 6
Description of four UCI data sets.

ID	Data sets	$\lvert U \rvert$	$\lvert C \rvert$	$\lvert U/IND(D) \rvert$
(1)	Monks-3	432	6	2
(2)	Tic-Tac-Toe	958	9	2
(3)	Voting	435	16	2
(4)	SPECT Heart	267	22	2

1110000010101001. Moreover, six groups of threshold (α, β) are mainly considered in experiments, i.e., $(0.9, 0.1), (0.9, 0.2), (0.8, 0.1), (0.8, 0.2), (0.7, 0.3), (0.6, 0.4)$. Next, relevant experimental results are exhibited and analyzed one by one, and some comprehensive conclusions are provided finally.

(I) Table 7 provides the reduction results of (1) Monk-3 database for the first four threshold groups. Thus, the cores and reducts are correspondingly equal, and their hierarchies are clear by virtue of relation $\subseteq$. Moreover, when $(\alpha, \beta) = (0.7, 0.3)$, only CRP-Core and CRP-Reduct become $\{c_2, c_4\}$; when $(\alpha, \beta) = (0.6, 0.4)$, only CRP-Core and CRP-Reduct become $\emptyset$ and $\{c_2\}, \{c_5\}$, respectively.

(II) For (2) Tic-Tac-Toe database, all reduction systems yield the same results – the empty core and nine reducts $C - \{c_i\}$ $(i = 1, 2, \ldots, 9)$ – for all the six threshold groups. Thus, the attribute system has relative redundancy, and the symmetry of the nine reducts highly accords with the geometrical feature of this database. In particular, this database also illustrates effectiveness of qualitative reduction, but it cannot clearly reflect reduction hierarchies because all reduction results become the same. In other words, five types of reduction systems exhibit indifference and thus cannot be vividly distinguished for Tic-Tac-Toe database. As is shown by this example, the reduction hierarchies may be masked in practice, though they objectively exist in theory. Therefore, the reduction hierarchies also depend on practical data structure.

Table 7
Five types of reducts and their cores of (1) Monks-3 database.

Type	Core	Reduct
(0) Pawlak-Reduct	$\{c_2, c_4, c_5\}$	$\{c_2, c_4, c_5\}$
(1) CRP-Reduct	$\{c_2, c_4, c_5\}$	$\{c_2, c_4, c_5\}$
(2) SRP-Reduct	$\{c_1, c_2, c_3, c_4, c_5\}$	$\{c_1, c_2, c_3, c_4, c_5\}$
(3) DP-Reduct	$\{c_1, c_2, c_3, c_4, c_5\}$	$\{c_1, c_2, c_3, c_4, c_5\}$
(4) KP-Reduct	$\{c_1, c_2, c_3, c_4, c_5, c_6\}$	$\{c_1, c_2, c_3, c_4, c_5, c_6\}$

Herein, (α, β) is $(0.9, 0.1)$ or $(0.9, 0.2)$ or $(0.8, 0.1)$ or $(0.8, 0.2)$.

(III) For (3) Voting database, only CRP-Core and CRP-Reduct exhibit difference for the six threshold groups, and Table 8 provides relevant results, where the inclusion/extension relation does not necessarily exist among different types of reducts. According to threshold $(0.8, 0.1)$ or $(0.7, 0.3)$, Table 9 provides all reduction results, where the reducts include (or equal) the corresponding cores, and Fig. 4 further exhibits the core hierarchies and reduct hierarchies.

(IV) Table 10 provides the reduction results of (4) SPECT Heart database for the first four threshold groups, where the reducts include (or equal) the corresponding cores, and Fig. 5 further exhibits the core hierarchies and reduct hierarchies. Moreover, when $(\alpha, \beta) = (0.7, 0.3)$, only CRP-Reduction results change as follows: $C2683725$ becomes the core and there are six reducts – C2685791, C2816847, C2880335, C3011423, C3865421, C3928909; when $(\alpha, \beta) = (0.6, 0.4)$, only CRP-Core and CRP-Reduct variedly correspond to $C2679369$ and 40 reducts, respectively.

Based on these reduction results and experiment analyzes, the three quantitative reduction systems (regarding CRP-Reduct, SRP-Reduct, and DP-Reduct) exist with structural difference to pursue different hierarchical reduction targets, so they and their hierarchies with the two qualitative reduction systems (regarding Pawlak-Reduct and KP-Reduct) are usually effective for practical applications. For complexities, the two qualitative reduction approaches do well in experiments because they can be equivalently described by some basic measures (such as the granular number and dependency degree); moreover, because of the target integration, DP-Reduction needs more computing time when compared to the other two quantitative reduction methods. In particular, all reduction ways can achieve better performances if their non-empty cores are fully utilized, and practical data structure also underlies application efficiencies. Moreover, the hierarchical algorithm (Algorithm 2) also has good efficiencies (to seek part reducts) by virtue of a strong reduct's guidance, and relevant processes can be demonstrated by the reduction hierarchy figures of Voting database and SPECT Heart database (i.e., Figs. 4 and 5).

Table 8
CRP-Core and CRP-Reduct of (3) Voting database.

Threshold (α, β)	CRP-Core	CRP-Reduct	Reduct number
$(0.9, 0.1), (0.9, 0.2)$	$C57513$	$C59135, C61611, C62637$	3
$(0.8, 0.1), (0.7, 0.3)$	$C57481$	$C59103, C61611, C62091, C62637, C62923, C62925, C63117$	7
$(0.8, 0.2)$	$C57513$	$C59103, C61611, C62091, C62637, C62923, C62925, C63117$	7
$(0.6, 0.4)$	$C57481$	$C57999, C61583, C61609, C62089, C62619, C62621, C62923, C62925$	8

Table 9
Five types of reducts and their cores of (3) Voting database.

Type	Core	Reduct
(0) Pawlak-Reduct	$C57513 = 1110000010101001$	$C59135 = 1110011011111111$
		$C61611 = 1111000010101011$
		$C62637 = 1111010010101101$
(1) CRP-Reduct	$C57481 = 1110000010001001$	$C59103 = 1110011011011111$
		$C61611 = 1111000010101011$
		$C62091 = 1111001010001011$
		$C62637 = 1111010010101101$
		$C62923 = 1111010111001011$
		$C62925 = 1111010111001101$
		$C63117 = 1111011010001101$
(2) SRP-Reduct	$C61439 = 1110111111111111$	$C61439 = 1110111111111111$
(3) DP-Reduct	$C61439 = 1110111111111111$	$C61439 = 1110111111111111$
(4) KP-Reduct	$C61439 = 1110111111111111$	$C61439 = 1110111111111111$

Herein, (α, β) is $(0.8, 0.1)$ or $(0.7, 0.3)$.

Table 10
Five types of reducts and their cores of (4) SPECT Heart database.

Type	Core	Reduct
(0) Pawlak-Reduct	$C2945871 = 1011001111001101001111$	$C3011423 = 1011011111001101011111$
		$C3142479 = 1011111111001101001111$
		$C3996511 = 1111001111101101011111$
		$C4127567 = 1111101111101101001111$
(1) CRP-Reduct	$C2945871 = 1011001111001101001111$	$C3011423 = 1011011111001101011111$
		$C3142479 = 1011111111001101001111$
		$C3996511 = 1111001111101101011111$
		$C4127567 = 1111101111101101001111$
(2) SRP-Reduct	$C3012607 = 1011011111011111111111$	$C3014655 = 1011011111111111111111$
		$C4061183 = 1111011111011111111111$
(3) DP-Reduct	$C3012607 = 1011011111011111111111$	$C3014655 = 1011011111111111111111$
		$C4061183 = 1111011111011111111111$
(4) KP-Reduct	$C3014655 = 1011011111111111111111$	$C3014655 = 1011011111111111111111$

Herein, (α, β) is $(0.9, 0.1)$ or $(0.9, 0.2)$ or $(0.8, 0.1)$ or $(0.8, 0.2)$.

Herein, we summarize the effect of threshold change for the reduction systems. By adjusting approximation degrees or controlling tolerance measures, (α, β) can impact not qualitative reduction but quantitative reduction, and it determines only region-structural measures regarding reduction targets rather than necessary reduction hierarchies regarding parameter variation. However, appropriate thresholds can better exhibit reduction hierarchies among the five reduction systems. In the experiments, threshold change mainly affects CRP-Reduction results and reduction hierarchies related to them.

Furthermore, based on these data sets and their quantitative reducts, decision rules can be extracted to establish rule bases. Finally, we provide an example for illustration. For (3) Voting database, $C62925$ is CRP-Reduct for threshold $(0.8, 0.1)$ or $(0.7, 0.3)$ or $(0.8, 0.2)$ or $(0.6, 0.4)$; thus, within the threshold-approximate framework, $C62925$ can be used to establish a rule base with the following 259 decision rules (where d_{rep} and d_{dem} simply denote $d_{republican}$ and $d_{democrat}$, respectively):

$$(c_1)_n(c_2)_y(c_3)_n(c_4)_y(c_6)_y(c_8)_n(c_9)_n(c_{10})_y(c_{13})_y(c_{14})_y(c_{16})_y \rightarrow d_{rep},$$
$$(c_1)_n(c_2)_y(c_3)_n(c_4)_y(c_6)_y(c_8)_n(c_9)_n(c_{10})_n(c_{13})_y(c_{14})_y(c_{16})_? \rightarrow d_{rep},$$
$$(c_1)_?(c_2)_y(c_3)_y(c_4)_?(c_6)_y(c_8)_n(c_9)_n(c_{10})_n(c_{13})_y(c_{14})_y(c_{16})_n \rightarrow d_{dem},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$(c_1)_y(c_2)_n(c_3)_y(c_4)_n(c_6)_n(c_8)_y(c_9)_y(c_{10})_y(c_{13})_n(c_{14})_?(c_{16})_y \rightarrow d_{dem},$$
$$(c_1)_n(c_2)_n(c_3)_y(c_4)_y(c_6)_y(c_8)_n(c_9)_y(c_{10})_y(c_{13})_y(c_{14})_y(c_{16})_y \rightarrow d_{rep},$$
$$(c_1)_n(c_2)_n(c_3)_n(c_4)_y(c_6)_y(c_8)_?(c_9)_?(c_{10})_?(c_{13})_y(c_{14})_y(c_{16})_y \rightarrow d_{rep}.$$

8. Conclusions

Quantitative reduction exhibits applicability but complexity when compared to qualitative reduction. According to two-category DTRS-Model, this paper mainly investigates quantitative reducts and their hierarchies (with qualitative reducts) from a regional perspective. First, we summarize equivalent Pawlak-Reduction targets at different levels, i.e., CR-Preservation, SR-Preservation, and D-Preservation at the C-Regional, S-Regional, and integrated levels, respectively. Thus, quantitative CRP-Reduct, SRP-Reduct, and DP-Reduct are naturally constructed to expand Pawlak-Reduct. Furthermore, hierarchies among these three types of quantitative reducts and two types of qualitative reducts (Pawlak-Reduct and KP-Reduct) are emphatically investigated. Finally, the experiments verify effectiveness of relevant reducts and their hierarchies.

Qualitative reducts pursue a perfect absoluteness. In contrast, quantitative reducts mainly consider the quantitative approximation and tolerance. In fact, they exhibit measurability, generality, and robustness so can avoid over-fitting. Thus, CRP-Reduct, SRP-Reduct, and DP-Reduct are particularly useful for processing databases with noises due to their approximation accuracy and fault tolerance. Furthermore, the three quantitative reducts and their hierarchies (with Pawlak-Reduct and KP-Reduct) express the granularity pursuit for different measures of generalization and accuracy. In fact, CRP-Reduct more adheres to Pawlak-Reduct and holds extensive generalization; in contrast, DP-Reduct fully integrates qualitative and quantitative reduction essence to achieve powerful accuracy; moreover, SRP-Reduct – a distinctive reduct of two categories – seeks a balance at a middle level. Therefore, the three quantitative reducts can be chosen according to different requirements of generalization and accuracy. Moreover, these reducts and their hierarchies can make good performances if the databases have inherent structural diversity, such as Voting database and SPECT Heart database used in our experiments.

The proposed DTRS-Reduction systems and their hierarchies have basic expansion, quantitative applicability, and structural systematicness; in particular, we have already utilized the reduction target structures to conduct some subsequent and in-depth works in Ref. [50]. The relevant thoughts and results provide some new insight into quantitative reduction and thus are worth generalizing for the multi-category problem and other quantitative models (especially PRS-Models).

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