A Further Investigation to Relative Reducts of Decision Information Systems

Duoqian Miao[1](✉) and Guangming Lang[1,2]

[1] Department of Computer Science and Technology, Tongji University, Shanghai 201804, People's Republic of China
{dqmiao,langguangming1984}@tongji.edu.cn

[2] School of Mathematics and Computer Science, Changsha University of Science and Technology, Changsha 410114, Hunan, People's Republic of China

Abstract. In practical situations, there are many definitions of relative reducts with respect to different criterions, but researchers don't notice their application backgrounds, and less efforts have been done on investigating the relationship between them. In this paper, we first discuss the relationship between these relative reducts and present the generalized relative reduct. Then we investigate the relationship between several discernibility matrixes and present the generalized discernibility matrix and discernibility function. Finally, we employ several examples to illustrate the related results accordingly.

Keywords: Rough sets · Discernibility matrix · Discernibility function · Relative reduct

1 Introduction

Rough set theory [20], the theoretical aspects and applications of which have been intensively investigated, is a useful mathematical tool for knowledge acquisition of information systems. Originally, Pawlak's rough set theory was constructed on the basis of an equivalence relation, and it has been developed by extending the equivalence relation to the tolerance relation, similarity relation and so on. Until now, rough set theory has successfully been applied in many fields such as granular computing, pattern recognition, data mining, and knowledge discovery, with applications increasingly being adopted in the development of rough set theory.

In rough set theory, attribute reduction [1–6,8–17,19,21,22,24,27–37,39–41,43] plays an important role in classification and feature selection, and a reduct is interpreted as a minimal set of attributes that can provide the same descriptive or classification ability as the entire set of attributes. Many attempts have been done on attribute reduction of consistent and inconsistent information systems. On one hand, many concepts of reducts have been proposed for attribute reduction with respect to different criteria. For example, Chen et al. [1] presented the concept of reduct for consistent and inconsistent covering decision

D. Ciucci et al. (Eds.): RSKT 2015, LNAI 9436, pp. 26–38, 2015.
DOI: 10.1007/978-3-319-25754-9_3

information systems with covering rough sets. Jia et al. [6] introduced the generalized attribute reduct of information systems in rough set theory. Kryszkiewicz et al. [7] focused on boundary region reduct of decision information systems. Mi et al. [17] proposed the concepts of β lower and β upper distribution reducts of information systems based on variable precision rough set model. Miao et al. [18] presented relative reducts in consistent and inconsistent decision information systems. Pawlak et al. [20] proposed the concept of positive region reduct of decision information systems. Qian et al. [22] provided the concepts of the lower and upper approximation reducts of decision information systems. Slezak et al. [23] studied decision reduct of decision information systems. Zhao et al. [42] presented a general definition of an attribute reduct. Zhang et al. [40] proposed the concept of assignment reducts and maximum assignment reducts for decision information systems. In practical situations, different reducts are equivalent to each other in consistent decision information systems, but they are not equivalent in inconsistent decision information systems. For example, the boundary region of a consistent decision information system is empty, and the positive region reducts are equivalent to boundary region reducts in consistent decision information systems, but they are not equivalent to each other in inconsistent decision information systems since the boundary regions of inconsistent decision information systems are not empty. Therefore, we only need to consider one of the positive region reducts and boundary region reducts in consistent information systems, and we need to consider the positive region reducts and boundary region reducts simultaneously in inconsistent information systems. But researchers usually utilize the concepts of relative reducts without considering their application backgrounds and many efforts have been paid to useless work. Therefore, it is urgent to discuss the relationship between them.

On the other hand, many methods [1,7,9,11,14,17–19,23,25,26,38,39] have been proposed for constructing relative reducts of decision information systems. For example, Kryszkiewicz et al. [7] provided the notion of discernibility matrix and discernibility function for computing decision reducts. Leung et al. [11] investigated dependence space-based attribute reduction in inconsistent decision information systems. Li et al. [14] studied quick attribute reduction in inconsistent decision information systems. Miao et al. [18] presented the generalized discernibility matrix and discernibility function of three types of relative reducts. Meng et al. [19] studied extended rough set-based attribute reduction in inconsistent incomplete decision systems. Slezak et al. [23] introduced the notion of discernibility matrix and discernibility function for constructing distribution reducts. Skowron et al. [25] proposed the concept of the classical discernibility matrix and discernibility function for constructing relative reducts of decision information systems. Yao et al. [37] investigated discernibility matrix simplification for constructing attribute reducts. Ye et al. [38] also presented the notion of discernibility matrix and discernibility function for calculating positive region reducts. Clearly, the discernibility matrix and discernibility function are effective and feasible to construct attribute reducts of decision information systems. In practice, there are several types of discernibility matrixes and discernibility functions, and we need to propose the generalized discernibility matrix and discernibility function for constructing relative reducts of decision information systems.

The rest of this paper is organized as follows: Sect. 2 briefly reviews the basic concepts of rough set theory. In Sect. 3, we investigate the relationship between relative reducts and present the generalized relative reduct. In Sect. 4, we review several discernibility matrixes and discernibility functions and present the generalized discernibility matrix and discernibility function. We conclude the paper in Sect. 5.

2 Preliminaries

In this section, we review some concepts of rough set theory.

Definition 1. *[20] An information system is a 4-tuple $S = (U, A, V, f)$, where $U = \{x_1 x_2, ..., x_n\}$ is a finite set of objects, A is a finite set of attributes, $V = \{V_a | a \in A\}$, where V_a is the set of values of attribute a, and $card(V_a) > 1$, f is a function from $U \times A$ into V.*

For convenience, we take $S = (U, C \cup D)$ and $f(x, a) = a(x)$ for $x \in U$ and $a \in C \cup D$ in this work, where C and D denote a non-empty finite set of conditional attributes and a non-empty finite set of decision attributes, respectively.

Definition 2. *[20] Let $S = (U, C \cup D)$ be an information system, and $B \subseteq C$. Then an indiscernibility relation $IND(B) \subseteq U \times U$ is defined as $IND(B) = \{(x, y) \in U \times U | \forall a \in B, a(x) = a(y)\}$, where $a(x)$ and $a(y)$ denote the values of objects x and y on a, respectively.*

If $(x, y) \in IND(B)$ for $x, y \in U$, then x and y are indiscernible based on the attribute set B. We can receive a quotient set which is called the family of equivalence classes or the blocks of the universe U by $IND(B)$, denoted as $U/B = \{[x]_B | x \in U\}$. Moreover, if $IND(C) \subseteq IND(D)$, then S is called a consistent information system. Otherwise, it is inconsistent.

Definition 3. *[20] Let $S = (U, C \cup D)$ be an information system, and $B \subseteq C$. For any $X \subseteq U$, the lower and upper approximations of X with respect to B are defined as $\underline{R}_B(X) = \bigcup\{[x]_B | [x]_B \subseteq X\}$ and $\overline{R}_B(X) = \bigcup\{[x]_B | [x]_B \cap X \neq \emptyset\}$.*

By Definition 3, we have the positive region, boundary region and negative region of $X \subseteq U$ as follows: $POS_R(X) = \underline{R}(X), BND_R(X) = \overline{R}(X) - \underline{R}(X)$ and $NEG_R(X) = U - \overline{R}(X)$. Furthermore, we have the positive region and boundary region of D as follows: $\underline{R}(D) = POS_R(D) = \bigcup_{i=1}^{|U/D|} \underline{R}(D_i), \overline{R}(D) = \bigcup_{i=1}^{|U/D|} \overline{R}(D_i)$ and $BND_R(D) = \overline{R}(D) - \underline{R}(D)$, and the Confidence of the rule $[x]_B \rightarrow D_i$ is defined as $Confidence([x]_B \rightarrow D_i) = \frac{|[x]_B \cap D_i|}{|[x]_B|} = P(D_i | [x]_B)$, where $U/D = \{D_i | 1 \leq i \leq r\}$ and $| \bullet |$ denotes the cardinality of $\bullet$.

In a consistent decision information system, the boundary region is empty, but it is not empty in an inconsistent decision information system.

Table 1. An inconsistent decision information system.

	c_1	c_2	c_3	c_4	c_5	c_6	d
x_1	0	0	0	1	1	0	0
x_2	1	0	0	1	1	0	1
x_3	1	1	0	1	1	0	1
x_4	1	1	0	1	1	0	1
x_5	1	1	0	1	1	0	0
x_6	1	1	0	1	0	1	0
x_7	1	1	0	1	0	1	1
x_8	1	1	0	1	0	1	1
x_9	1	1	0	0	0	1	0
x_{10}	1	1	0	0	0	1	1
x_{11}	1	1	0	0	0	1	1
x_{12}	1	1	0	0	0	1	1
x_{13}	1	1	0	0	0	1	1
x_{14}	1	1	1	0	0	1	1
x_{15}	1	1	1	0	0	1	2

Example 1. Let Table 1 be an inconsistent decision information system $S = (U, C \cup \{d\})$, $C = \{c_1, c_2, ..., c_6\}$, $U/d = \{D_1, D_2, D_3\}$, $D_1 = \{x_1, x_5, x_6, x_9\}$, $D_2 = \{x_2, x_3, x_4, x_7, x_8, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}\}$, $D_3 = \{x_{15}\}$. By Definition 3, we have $\overline{R}(D) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, ..., x_{13}, x_{14}, x_{15}\}$ and $\underline{R}(D) = \{x_1, x_2\}$. Therefore, we obtain $BND_R(D) = \{x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, ..., x_{13}, x_{14}, x_{15}\}$ and the partition $\{\{x_3, x_4, x_5\}, \{x_6, x_7, x_8\}, \{x_9, x_{10}, ..., x_{13}\}, \{x_{14}, x_{15}\}\}$ of $BND_R(D)$.

Definition 4. *[18] Let $S = (U, C \cup D)$ be an information system, and $B \subseteq C$. Then the relative indiscernibility and discernibility relations defined by B with respect to D are defined as follows:*

$$IND(B|D) = \{(x, y) \in U \times U | \forall c \in B, c(x) = c(y) \vee d(x) = d(y)\};$$
$$DIS(B|D) = \{(x, y) \in U \times U | \exists c \in B, c(x) \neq c(y) \wedge d(x) \neq d(y)\},$$

where $c(x)$ and $c(y)$ denote the values of objects x and y on c, respectively.

By Definition 4, the relative indiscernibility relation is reflexive, symmetric, but not transitive; the relative discernibility relation is irreflexive, symmetric, but not transitive. By Definitions 2 and 4, we observe that if $(x, y) \in IND(C)$ and $(x, y) \in IND(D)$ for $(x, y) \in U \times U$, then S is inconsistent; S is consistent, otherwise. In other words, all objects in an equivalence class $[x]_C$ satisfying one and only one class in a consistent decision information system; while all objects in an equivalence class $[x]_C$ may satisfy different classes in an inconsistent decision information system.

Definition 5. *[18] Let $S = (U, C \cup D)$ be an information system, $A \subseteq C$, and $\tau(x|A) = \{d(y)|y \in [x]_A\}$ for any $x \in U$. Then the set of generalized decisions $\Delta(A)$ of all objects in U is denoted as $\Delta(A) = \{\tau(x_1|A), \tau(x_2|A), ..., \tau(x_n|A)\}$.*

If there exists x such that $\tau(x|C) > 1$, then the decision information system is inconsistent; otherwise, it is inconsistent.

Definition 6. *[32] Let $S = (U, C \cup D)$ be a decision information system, $U/P = \{X_1, X_2, ..., X_n\}$, $U/Q = \{Y_1, Y_2, ..., Y_m\}$, $p(Y_i) = \frac{|Y_i|}{|U|}$, and $p(Y_i|X_i) = \frac{|Y_i \cap X_i|}{|X_i|}$. Then the entropy $H(Q)$ and conditional entropy $H(Q|P)$ are defined as follows;*

$$H(Q) = -\sum_{j=1}^{m} p(Y_i)log(p(Y_i)), H(Q|P) = -\sum_{i=1}^{n} p(X_i) \sum_{j=1}^{m} p(Y_j|X_i)log(p(Y_j|X_i)).$$

The entropy $H(Q)$ can be interpreted as a measure of the information content of the uncertainty about knowledge Q, which reaches a maximum value $log(|U|)$ and the minimum value 0 when the knowledge Q becomes finest and the distribution of the knowledge Q focuses on a particular value, respectively.

Definition 7. *[32] Let $S = (U, C \cup D)$ be a decision information system. Then the mutual information $I(P; Q)$ is defined as $I(P; Q) = H(Q) - H(Q|P)$.*

The mutual information $I(P; Q)$ measures the decrease of uncertainty about Q caused by P. Furthermore, the amount of information contained in P about itself is $H(P)$ obviously.

3 Relative Reducts

In this section, we investigate the relationship between relative reducts.

Definition 8. *Let $S = (U, C \cup D)$ be a decision information system, and $B \subseteq C$. Then*

(1) If $POS_B(D) = POS_C(D)$, then B is called a positive consistent set of C with respect to D.
(2) If $IND(B|D) = IND(C|D)$, then B is called a boundary consistent set of C with respect to D.
(3) If $\tau(x|B) = \tau(x|C)$ for any $x \in U$, then B is called a decision consistent set of C with respect to D.
(4) If $I(B; D) = I(C; D)$, then B is called a mutual information consistent set of C with respect to D.
(5) If $\mu_B(x) = \mu_C(x)$ for any $x \in U$, then B is called a distribution consistent set of C with respect to D, where $\mu_B(x) = (\frac{|[x]_B \cap D_1|}{|[x]_B|}, \frac{|[x]_B \cap D_2|}{|[x]_B|}, ..., \frac{|[x]_B \cap D_{|U/D|}|}{|[x]_B|})$.
(6) If $\gamma_B(x) = \gamma_C(x)$ for any $x \in U$, then B is called a maximum distribution consistent set of C with respect to D, where $\gamma_B(x) = max\{\frac{|[x]_B \cap D_i|}{|[x]_B|} | 1 \leq i \leq |U/D|\}$.

Definition 9. *[7,20,23,32,39] Let $S = (U, C \cup D)$ be a decision information system, and $B \subseteq C$. Then*

(1) *B is called a positive region reduct of C with respect to D if*
(I) $POS_B(D) = POS_C(D)$; (II) $POS_{B'}(D) \neq POS_B(D)$ *for any* $B' \subset B$.
(2) *B is called a boundary region reduct of C with respect to D if*
(I) $IND(B|D) = IND(C|D)$; (II) $\forall B' \subset B, IND(B'|D) \neq IND(B|D)$.
(3) *B is called a decision reduct of C with respect to D if*
(I) $\forall x \in U, \tau(x|B) = \tau(x|C)$; (II) $\forall B' \subset B, \forall x \in U, \tau(x|B') \neq \tau(x|B)$.
(4) *B is called a mutual information reduct of C with respect to D if*
(I) $I(B; D) = I(C; D)$; (II) $\forall B' \subset B, I(B'; D) \neq I(B; D)$.
(5) *B is called a distribution reduct of C with respect to D if*
(I) $\forall x \in U, \mu_B(x) = \mu_C(x)$; (II) $\forall B' \subset B, \forall x \in U, \mu_{B'}(x) \neq \mu_B(x)$.
(6) *B is called a maximum distribution reduct of C with respect to D if*
(I) $\forall x \in U, \gamma_B(x) = \gamma_C(x)$; (II) $\forall B' \subset B, \forall x \in U, \gamma_{B'}(x) \neq \gamma_B(x)$.

In consistent information systems, the positive region reducts which are the minimum attribute sets remaining the positive regions are equivalent to the boundary region reducts which are the minimum attribute sets keeping the boundary region, but they are not equivalent to each other in inconsistent information systems; the decision reduct is the minimum attribute set remaining the decision sets of equivalence classes; the mutual information reduct is the minimum attribute set keeping the mutual information; the distribution reduct is the minimum attribute set remaining the distributions of all objects with respect to the decision equivalence classes; the maximum distribution reduct is the minimum attribute set keeping the maximum distributions of all objects with respect to the decision equivalence classes. Therefore, we should consider the corresponding concepts of reducts for solving the problems in practical situations.

Theorem 1. *Let $S = (U, C \cup D)$ be a decision information system, and $B \subseteq C$. Then*

(1) *If* $IND(B|D) = IND(C|D)$, *then* $\tau(x|B) = \tau(x|C)$ *for any* $x \in U$.
(2) *If* $I(B; D) = I(C; D)$, *then* $\mu_B(x) = \mu_C(x)$ *for any* $x \in U$.
(3) *If* $\mu_B(x) = \mu_C(x)$ *for any* $x \in U$, *then* $I(B; D) = I(C; D)$.
(4) *If* $\mu_B(x) = \mu_C(x)$ *for any* $x \in U$, *then* $\tau(x|B) = \tau(x|C)$ *for any* $x \in U$.
(5) *If* $\tau(x|B) = \tau(x|C)$ *for any* $x \in U$, *then* $POS_B(D) = POS_C(D)$.
(6) *If* $IND(B|D) = IND(C|D)$, *then* $\tau(x|B) = \tau(x|C)$ *for any* $x \in U$.
(7) *If* $\mu_B(x) = \mu_C(x)$ *for any* $x \in U$, *then* $\gamma_B(x) = \gamma_C(x)$ *for any* $x \in U$.

Proof: The results can be proved by Definitions 8 and 9. □

By Theorem 1, the boundary consistent set is a distribution consistent set; the mutual information consistent set is a distribution consistent set, and vice versa; the distribution consistent set is a decision consistent set; the boundary region is a positive consistent set; the boundary consistent set is a decision consistent set; the distribution consistent set is a maximum distribution consistent set.

By Definition 9, we can obtain several reducts of the inconsistent decision information system shown in Table 1 as follows.

Example 2 (Continued from Example 1). By Definition 9, we have that

(1) positive region reduct $\{c_1, c_2\}$ and the partition of the boundary region $\{x_3, x_4, ..., x_{15}\}$;
(2) decision reduct $\{c_1, c_2, c_3\}$ and the partition of the boundary region $\{\{x_3, x_4, ..., x_{13}\}, \{x_{14}, x_{15}\}\}$;
(3) distribution reduct $\{c_1, c_2, c_3, c_4\}$ and the partition of the boundary region $\{\{x_3, x_4, ..., x_8\}, \{x_9, x_{10}, ..., x_{13}\}, \{x_{14}, x_{15}\}\}$;
(4) boundary region reduct $\{c_1, c_2, c_3, c_4, c_5\}$ and the partition of the boundary region $\{\{x_3, x_4, x_5\}, \{x_6, x_7, x_8\}, \{x_9, x_{10}\}, \{x_{14}, x_{15}\}\}$;

In Example 2, we obtain the same boundary region by using different reducts, but there are different partitions for the same boundary regions of different reducts. Therefore, it is of interest to investigate the properties of boundary regions with respect to different reducts for inconsistent information systems.

Example 3 (Continued from Examples 1 and 2).

(1) For $B = \{c_1, c_2\}$, we have that

$$Confidence([x_1]_B \rightarrow D_1) = 1; Confidence([x_2]_B \rightarrow D_2) = 1;$$
$$Confidence([x_3]_B \rightarrow D_1) = \frac{3}{13}; Confidence([x_3]_B \rightarrow D_2) = \frac{9}{13};$$
$$Confidence([x_3]_B \rightarrow D_3) = \frac{1}{13}.$$

(2) For $B = \{c_1, c_2, c_3\}$, we have that

$$Confidence([x_1]_B \rightarrow D_1) = 1; Confidence([x_2]_B \rightarrow D_2) = 1;$$
$$Confidence([x_3]_B \rightarrow D_1) = \frac{3}{11}; Confidence([x_3]_B \rightarrow D_2) = \frac{8}{11};$$
$$Confidence([x_{14}]_B \rightarrow D_2) = \frac{1}{2}; Confidence([x_{14}]_B \rightarrow D_3) = \frac{1}{2}.$$

(3) For $B = \{c_1, c_2, c_3, c_4\}$, we have that

$$Confidence([x_1]_B \rightarrow D_1) = 1; Confidence([x_2]_B \rightarrow D_2) = 1;$$
$$Confidence([x_3]_B \rightarrow D_1) = \frac{1}{3}; Confidence([x_3]_B \rightarrow D_2) = \frac{2}{3};$$
$$Confidence([x_9]_B \rightarrow D_1) = \frac{1}{5}; Confidence([x_9]_B \rightarrow D_2) = \frac{4}{5};$$
$$Confidence([x_{14}]_B \rightarrow D_2) = \frac{1}{2}; Confidence([x_{14}]_B \rightarrow D_3) = \frac{1}{2}.$$

(4) For $B = \{c_1, c_2, c_3, c_4, c_5\}$, we have that

$$Confidence([x_1]_B \rightarrow D_1) = 1; Confidence([x_2]_B \rightarrow D_2) = 1;$$
$$Confidence([x_3]_B \rightarrow D_1) = \frac{1}{3}; Confidence([x_3]_B \rightarrow D_2) = \frac{2}{3};$$
$$Confidence([x_6]_B \rightarrow D_1) = \frac{1}{3}; Confidence([x_6]_B \rightarrow D_2) = \frac{2}{3};$$

$$Confidence([x_9]_B \to D_1) = \frac{1}{5}; Confidence([x_9]_B \to D_2) = \frac{4}{5};$$
$$Confidence([x_{14}]_B \to D_2) = \frac{1}{2}; Confidence([x_{14}]_B \to D_3) = \frac{1}{2}.$$

By Definition 8, we present the concept of generalized consistent set as follows:

Definition 10. *Let $S = (U, C \cup D)$ be a decision information system, and a function $e : 2^C \to L$, which evaluates the property $\mathcal{P}$, where L is a poset. If $e(B) = e(C)$ for $B \subseteq C$, then B is called a generalized consistent set of C with respect to D.*

By Definition 10, we observe that a positive consistent set, a boundary consistent set, a decision consistent set, a mutual information consistent set, a distribution consistent set and a maximum distribution consistent set are special cases of the generalized consistent sets.

Definition 11 *[18]. Let $S = (U, C \cup D)$ be a decision information system, and a function $e : 2^C \to L$, which evaluates the property $\mathcal{P}$, where L is a poset. If $e(B) = e(C)$ for $B \subseteq C$ and $e(B') \neq e(B)$ for any $B' \subset B$, then B is called a generalized relative reduct of C with respect to D.*

By Definition 11, we see that a positive region reduct, a boundary region reduct, a decision reduct, a mutual information reduct, a distribution reduct and a maximum distribution reduct are special cases of the generalized relative reducts.

Definition 12 *[18]. Let $S = (U, C \cup D)$ be a decision information system. Then $CORE_{\mathcal{P}} = \{a \in C | g(C - \{a\}) \neq g(C)\}$ is called the generalized relative core of C with respect to D.*

By Definitions 10, 11 and 12, Miao et al. presented the following proposition for the generalized relative core.

Proposition 1 *[18]. Let $S = (U, C \cup D)$ be a decision information system. Then the generalized relative core $CORE_{\mathcal{P}} = \bigcap RED_{\mathcal{P}}(C|D)$.*

4 Discernibility Matrix and Discernibility Function

In this section, we present several discernibility matrixes and discernibility functions.

4.1 Classical Discernibility Matrix and Discernibility Function

To construct relative reducts, we present the typical discernibility matrix and discernibility function.

Definition 13 *[25]. Let $S = (U, C \cup D)$ be an information system. Then the discernibility matrix $M = (M(x, y))$ is defined as follows:*

$$M(x,y) = \begin{cases} \{c \in C | c(x) \neq c(y)\}, \text{ if } d(x) \neq d(y); \\ \emptyset, \qquad\qquad\qquad\quad otherwise. \end{cases}$$

By Definition 13, we obtain the discernibility function as follows.

Definition 14 *[25]. Let $S = (U, C \cup D)$ be an information system, and $M = (M(x, y))$ the discernibility matrix. Then the discernibility function is defined as follows:*

$$f(M) = \bigwedge\{\bigvee\{(M(x,y)) | \forall x, y \in U | M(x,y) \neq \emptyset\}\}.$$

The expression $\bigvee\{(M(x,y))|\forall x, y \in U|M(x,y) \neq \emptyset\}$ is the disjunction of all attributes in $M(x, y)$, which indicates that the object pair (x, y) can be distinguished by any attribute in $M(x, y)$; the expression $\bigwedge\{\bigvee\{(M(x,y))|\forall x, y \in U|M(x,y) \neq \emptyset\}\}$ is the conjunction of all attributes in $\bigvee M(x, y)$, which indicates that the family of discernible object pairs can be distinguished by a set of attributes satisfying $\bigwedge\{\bigvee\{(M(x,y))|\forall x, y \in U|M(x,y) \neq \emptyset\}\}$.

4.2 Elements-Based Discernibility Matrixes and Discernibility Functions

In this section, we provide several elements-based discernibility matrixes for constructing attribute reducts of decision information systems.

Definition 15 *[7, 20, 23]. Let $S = (U, C \cup D)$ be an information system. Then*

(1) $M_{positive} = (M_{positive}(x, y))$*, where*

$$M(x,y) = \begin{cases} \{c \in C | c(x) \neq c(y)\}, \text{ if } [x]_C \vee [y]_C \in POS_C(D); \\ \emptyset, \qquad\qquad\qquad\qquad\qquad otherwise. \end{cases}$$

(2) $M_{boundary} = (M_{boundary}(x, y))$*, where*

$$M_{boundary}(x,y) = \begin{cases} \{c \in C | c(x) \neq c(y)\}, \text{ if } (x, y) \in IND(C|D); \\ \emptyset, \qquad\qquad\qquad\qquad\qquad otherwise. \end{cases}$$

(3) $M_{decision} = (M_{decision}(x, y))$*, where*

$$M_{decision}(x,y) = \begin{cases} \{c \in C | c(x) \neq c(y)\}, \text{ if } \tau_C(x) \neq \tau_C(y); \\ \emptyset, \qquad\qquad\qquad\qquad otherwise. \end{cases}$$

In Definition 15, $M_{positive}$, $M_{boundary}$ and $M_{decision}$ denote the discernibility matrixes for constructing the positive region reduct, boundary region reduct and decision reduct, respectively.

Subsequently, Miao et al. proposed the elements-based discernibility matrix and discernibility function for constructing relative reducts.

Definition 16 *[18]. Let $S = (U, C \cup D)$ be an information system, and $D = \{d\}$. Then the elements-based discernibility matrix $M_{\mathcal{P}} = (M_{\mathcal{P}}(x, y))$ is defined as follows:*

$$M_{\mathcal{P}}(x, y) = \begin{cases} \{c \in C | c(x) \neq c(y)\}, \text{ if } & x, y \textit{ are distinguishable w.r.t. } \mathcal{P}; \\ \emptyset, & \textit{otherwise.} \end{cases}$$

By Definition 16, we see that $M_{positive}, M_{boundary}$ and $M_{decision}$ are special cases of the elements-based discernibility matrix $M_{\mathcal{P}}$. Furthermore, Miao et al. presented the concept of the elements-based discernibility function as follows.

Definition 17 *[18]. Let $S = (U, C \cup D)$ be an information system, and $M_{\mathcal{P}}$ the discernibility matrix. Then the elements-based discernibility function $f(M_{\mathcal{P}})$ is defined as follows:*

$$f(M_{\mathcal{P}}) = \bigwedge\{\bigvee\{(M_{\mathcal{P}}(x, y)) | \forall x, y \in U | M_{\mathcal{P}}(x, y) \neq \emptyset\}\}.$$

4.3 Blocks-Based Discernibility Matrixes and Discernibility Functions

In this section, we provide several blocks-based discernibility matrixes for constructing attribute reducts of decision information systems.

Definition 18 *[39]. Let $S = (U, C \cup D)$ be a decision information system. (1) If $D_1^{\bullet} = \{([x]_C, [y]_C) | \mu_A(x) \neq \mu_A(y)\}$. Then the discernibility matrix $M_{DR} = (M_{DR}(X, Y))$ for constructing distribution reducts is defined as follows:*

$$M_{DR}(X, Y) = \begin{cases} \{c \in C | c(X) \neq c(Y)\}, \text{ if } & (X, Y) \in D_1^{\bullet}; \\ C, & (X, Y) \notin D_1^{\bullet}. \end{cases}$$

(2) If $D_2^{\bullet} = \{([x]_C, [y]_C) | \gamma_A(x) \neq \gamma_A(y)\}$. Then the discernibility matrix $M_{MD} = (M_{MD}(X, Y))$ for constructing the maximum distribution reducts is defined as follows:

$$M_{MD}(X, Y) = \begin{cases} \{c \in C | c(X) \neq c(Y)\}, \text{ if } & (X, Y) \in D_2^{\bullet}; \\ C, & (X, Y) \notin D_2^{\bullet}. \end{cases}$$

In Definition 18, M_{DR} and M_{MD} denote the discernibility matrixes for constructing the distribution reduct and the maximum distribution reduct, respectively. Subsequently, we presented the concept of blocks-based discernibility matrixes as follows:

Definition 19. *Let $S = (U, C \cup D)$ be a decision information system, and $D^{\mathcal{Q}} = \{(X, Y) \in U/C \times U/C | X \text{ and } Y \text{ satisfy the property } \mathcal{Q}\}$. Then the blocks-based discernibility matrix $M_{\mathcal{Q}} = (M_{\mathcal{Q}}(X, Y))$ is defined as follows:*

$$M_{\mathcal{Q}}(X, Y) = \begin{cases} \{c \in C | c(X) \neq c(Y)\}, \text{ if } & (X, Y) \in D^{\mathcal{Q}}; \\ C, & (X, Y) \notin D^{\mathcal{Q}}. \end{cases}$$

Definition 20. *Let $S = (U, C \cup D)$ be a decision information system, and $M_{\mathcal{Q}} = (M_{\mathcal{Q}}(X, Y))$ the discernibility matrix. Then the blocks-based discernibility functions $f(M_{\mathcal{Q}})$ is defined as follows:*

$$f(M_{\mathcal{Q}}) = \bigwedge\{\bigvee\{(M_{\mathcal{Q}}(X, Y))|\forall X, Y \in U/C|M_{\mathcal{Q}}(X, Y) \neq \emptyset\}\}.$$

By Definitions 16 and 19, we present the concept of the generalized discernibility matrix.

Definition 21. *Let $S = (U, C \cup D)$ be a decision information system, $\mathcal{X}, \mathcal{Y} \in F(U, C)$, where $F(U, C)$ is a quotient set of U with respect to C, and $\mathcal{D} = \{(\mathcal{X}, \mathcal{Y})|\mathcal{X}$ and $\mathcal{Y}$satisfy the property $\mathcal{Q}\}$. Then the generalized discernibility matrix $M_G = (M_G(\mathcal{X}, \mathcal{Y}))$ is defined as follows:*

$$M_G(\mathcal{X}, \mathcal{Y}) = \begin{cases} \{c \in C|c(\mathcal{X}) \neq c(\mathcal{Y})\}, \text{ if } & (\mathcal{X}, \mathcal{Y}) \in \mathcal{D}; \\ C, & (\mathcal{X}, \mathcal{Y}) \notin \mathcal{D}. \end{cases}$$

5 Conclusions

In this paper, we have investigated the relationship between relative reducts and presented the generalized relative reduct. We have studied the relationship between discernibility matrixes and presented the generalized discernibility matrix. We have employed several examples to illustrate the related results simultaneously.

In the future, we will further investigate the relationship between different relative reducts. Additionally, we will present the generalized relative reducts for information systems and construct a foundation for attribute reduction of information systems.

Acknowledgments. This work is supported by the National Natural Science Foundation of China (NO. 61273304), the Scientific Research Fund of Hunan Provincial Education Department (No. 14C0049).

References

1. Chen, D.G., Wang, C.Z., Hu, Q.H.: A new approach to attribute reduction of consistent and inconsistent covering decision systems with covering rough sets. Inf. Sci. **177**, 3500–3518 (2007)
2. Chen, Y.M., Miao, D.Q., Wang, R.Z., Wu, K.S.: A rough set approach to feature selection based on power set tree. Knowl. Based Syst. **24**, 275–281 (2011)
3. Du, F.W., Qin, K.Y.: Analysis of several reduction standards and their relationship for inconsistent decision table. Acta Electronica Sinica **39**(6), 1336–1340 (2011)
4. Dai, J.H., Wang, W.T., Tian, H.W., Liu, L.: Attribute selection based on a new conditional entropy for incomplete decision systems. Knowl. Based Syst. **39**, 207–213 (2013)

5. Hu, Q.H., Yu, D.R., Xie, Z.X.: Information-preserving hybrid data reduction based on fuzzy-rough techniques. Pattern Recogn. Lett. **27**(5), 414–423 (2006)
6. Jia, X.Y., Shang, L., Zhou, B., Yao, Y.Y.: Generalized attribute reduct in rough set theory. Knowledge-Based Systems (2015). doi:10.1016/j.knosys.2015.05.017
7. Kryszkiewicz, M.: Comparative study of alternative types of knowledge reduction in inconsistent systems. Int. J. Intell. Syst. **16**, 105–120 (2001)
8. Lang, G.M., Li, Q.G., Cai, M.J., Yang, T.: Characteristic matrices-based knowledge reduction in dynamic covering decision information systems. Knowl. Based Syst. (2015). doi:10.1016/j.knosys.2015.03.021
9. Lang, G.M., Li, Q.G., Cai, M.J., Yang, T., Xiao, Q.M.: Incremental approaches to knowledge reduction based on characteristic matrices. Int. J. Mach. Learn. Cybern. (2014) http://dx.doi:101007/s13042-014-0315-4
10. Lang, G.M., Li, Q.G., Yang, T.: An incremental approach to attribute reduction of dynamic set-valued information systems. Int. J. Mach. Learn. Cybern. **5**, 775–788 (2014)
11. Leung, Y., Ma, J.M., Zhang, W.X., Li, T.J.: Dependence-space-based attribute reductions in inconsistent decision information systems. Int. J. Approximate Reasoning **49**, 623–630 (2008)
12. Li, J.H., Mei, C.L., Lv, Y.J.: Knowledge reduction in real decision formal contexts. Inf. Sci. **189**, 191–207 (2012)
13. Li, T.R., Ruan, D., Song, J.: Dynamic maintenance of decision rules with rough set under characteristic relation. Wireless Communications, Networking and Mobile Computing, pp. 3713–3716 (2007)
14. Li, M., Shang, C.X., Feng, S.Z., Fan, J.P.: Quick attribute reduction in inconsistent decision tables. Inf. Sci. **254**, 155–180 (2014)
15. Liang, J.Y., Mi, J.R., Wei, W., Wang, F.: An accelerator for attribute reduction based on perspective of objects and attributes. Knowl. Based Syst. **44**, 90–100 (2013)
16. Liu, D., Li, T.R., Zhang, J.B.: A rough set-based incremental approach for learning knowledge in dynamic incomplete information systems. Int. J. Approximate Reasoning **55**(8), 1764–1786 (2014)
17. Mi, J.S., Wu, W.Z., Zhang, W.X.: Approaches to knowledge reduction based on variable precision rough set model. Inf. Sci. **159**(3–4), 255–272 (2004)
18. Miao, D.Q., Zhao, Y., Yao, Y.Y., Li, H.X., Xu, F.F.: Relative reducts in consistent and inconsistent decision tables of the Pawlak rough set model. Inf. Sci. **179**, 4140–4150 (2009)
19. Meng, Z.Q., Shi, Z.Z.: Extended rough set-based attribute reduction in inconsistent incomplete decision systems. Inf. Sci. **204**, 44–69 (2012)
20. Pawlak, Z.: Rough sets. Int. J. Comput. Inf. Sci. **11**(5), 341–356 (1982)
21. Qian, J., Lv, P., Yue, X.D., Liu, C.H., Jing, Z.J.: Hierarchical attribute reduction algorithms for big data using MapReduce. Knowl. Based Syst. **73**, 18–31 (2015)
22. Qian, Y.H., Liang, J.Y., Li, D.Y., Wang, F., Ma, N.N.: Approximation reduction in inconsistent incomplete decision tables. Knowl. Based Syst. **23**, 427–433 (2010)
23. Slezak, D.: Normalized decision functions and measures for inconsistent decision tables analysis. Fundamenta Informaticae **44**, 291–319 (2000)
24. Shao, M.W., Leung, Y.: Relations between granular reduct and dominance reduct in formal contexts. Knowl. Based Syst. **65**, 1–11 (2014)
25. Skowron, A., Rauszer, C.: The discernibility matrices and functions in information systems. In: Slowinski, R. (ed.) Handbook of Applications and Advances of the Rough Sets Theory, pp. 331–362. Kluwer Academic Publishers, Dordrecht (1992)

26. Skowron, A., Rauszer, C.: The discernibility matrices and functions in information systems. In: Słowiński, R. (ed.) Intelligent Decision Support Handbook of Applications and Advances of the Rough Sets Theory, pp. 331–362. Kluwer Academic Publishers, Dordrecht (1992)
27. Tan, A.H., Li, J.J., Lin, Y.J., Lin, G.P.: Matrix-based set approximations and reductions in covering decision information systems. Int. J. Approximate Reasoning **59**, 68–80 (2015)
28. Tan, A.H., Li, J.J.: A kind of approximations of generalized rough set model. Int. J. Mach. Learn. Cybern. **6**(3), 455–463 (2015)
29. Wu, W.Z.: Attribute reduction based on evidence theory in incomplete decision systems. Inf. Sci. **178**(5), 1355–1371 (2008)
30. Wang, F., Liang, J.Y., Dang, C.Y.: Attribute reduction for dynamic data sets. Appl. Soft Comput. **13**, 676–689 (2013)
31. Wang, G.Y., Ma, X.A., Yu, H.: Monotonic uncertainty measures for attribute reduction in probabilistic rough set model. Int. J. Approximate Reasoning **59**, 41–67 (2015)
32. Xu, F.F., Miao, D.Q., Wei, L.: Fuzzy-rough attribute reduction via mutual information with an application to cancer classification. Comput. Math. Appl. **57**, 1010–1017 (2009)
33. Xu, W.H., Zhang, W.X.: Measuring roughness of generalized rough sets induced by a covering. Fuzzy Sets Syst. **158**(22), 2443–2455 (2007)
34. Yang, T., Li, Q.G.: Reduction about approximation spaces of covering generalized rough sets. Int. J. Approximate Reasoning **51**(3), 335–345 (2010)
35. Yang, Y.Y., Chen, D.G., Dong, Z.: Novel algorithms of attribute reduction with variable precision rough set model. Neurocomputing **139**, 336–344 (2014)
36. Yang, X.B., Qi, Y., Yu, H.L., Song, X.N., Yang, J.Y.: Updating multigranulation rough approximations with increasing of granular structures. Knowl. Based Syst. **64**, 59–69 (2014)
37. Yao, Y.Y., Zhao, Y.: Discernibility matrix simplification for constructing attribute reducts. Inf. Sci. **179**(7), 867–882 (2009)
38. Ye, D.Y., Chen, Z.J.: A new discernibility matrix and the computation of a core. Acta Electronica Sinica **30**(7), 1086–1088 (2002)
39. Zhang, W.X., Mi, J.S., Wu, W.Z.: Knowledge reduction in inconsistent information systems. Chin. J. Comput. **26**(1), 12–18 (2003)
40. Zhang, W.X., Mi, J.S., Wu, W.Z.: Approaches to knowledge reductions in inconsistent systems. Int. J. Intell. Syst. **18**(9), 989–1000 (2003)
41. Zhang, J.B., Li, T.R., Ruan, D., Gao, Z.Z., Zhao, C.B.: A parallel method for computing rough set approximations. Inf. Sci. **194**, 209–223 (2012)
42. Zhao, Y., Luo, F., Wong, S.K.M., Yao, Y.: A general definition of an attribute reduct. In: Yao, J.T., Lingras, P., Wu, W.-Z., Szczuka, M.S., Cercone, N.J., Ślęzak, D. (eds.) RSKT 2007. LNCS (LNAI), vol. 4481, pp. 101–108. Springer, Heidelberg (2007)
43. Zhu, W.: Generalized rough sets based on relations. Inf. Sci. **177**(22), 4997–5011 (2007)