Knowledge-Based Systems 91 (2016) 114–125

Contents lists available at ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

Constructive methods of rough approximation operators and multigranulation rough sets

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ARTICLE INFO

Article history:
Received 12 January 2015
Revised 30 August 2015
Accepted 29 September 2015
Available online 8 October 2015

Keywords:
Rough approximation pair
Constructive method
Non-dual multigranulation rough set
Hybrid multigranulation rough set

ABSTRACT

Four kinds of constructive methods of rough approximation operators from existing rough sets are established, and the important conclusion is obtained: some rough sets are essentially direct applications of these constructive methods. Moreover, the new notions of non-dual multigranulation rough sets and hybrid multigranulation rough sets are introduced, and some properties are investigated.

1. Introduction

The Pawlak's rough set model [17] is based on equivalence relations, it has been generalized to arbitrary binary relations based rough sets, tolerance or similarity relations based rough sets, fuzzy rough sets and intuitionistic fuzzy rough sets (see [4,5,44]), etc. Moreover, as one of generalized models, covering rough sets has attracted much attention and induced lots of interesting results [13,19,31,34,38,40,46].

For the above rough set models, we usually only consider a single approximation space. In some directions of research on multiple-source approximation systems (see [6]), multi-agent systems or groups of intelligent agents (see [23–25,29]), multigranulation rough sets (see [7,9–12,20–22,27,36,39]), dynamic spaces and collections of general approximation spaces (see [15,16]), we need to consider multiple approximation spaces. Therefore, the algebraic structures and the relationship between various rough approximation pairs based on different approximation spaces have become an important research topic. In fact, algebra approach is widely applied in the research of rough set theory (see [2,3,8,14,18,26,28,35,37,41–43,45]). In this paper we will investigate basic algebraic operations (union and intersection) of rough approximations pairs based on multiple approximation spaces.

From another point of view, many rough set models (in particular, various multigranulation rough set models) are introduced, for these rough approximation operators, whether there are some inherent regularity or general generation rules? In this paper, we give a novel answer of the question.

The remainder of this paper is organized as follows. The next section deals with some preliminary concepts and properties regarding the Pawlak's rough sets and multigranulation rough sets. In Section 3, we introduce four kinds of constructive methods of rough approximation operations from existing rough sets, and discuss their basic properties and applications. In Section 4, we apply the constructive methods to multigranulation rough sets, and firstly introduce the new notions of non-dual multigranulation rough sets and hybrid multigranulation rough sets. We also discuss multigranulation rough sets based on general binary relations.

2. Basic concepts and properties

2.1. Pawlak's rough sets

Let U denote a non-empty set called the universe. Let $R \subseteq U \times U$ be an equivalence relation on U. The pair $apr = (U, R)$ is called an approximation space. The equivalence relation R partitions the set U into disjoint subsets. Let U/R denote the quotient set consisting of equivalence classes of R, and $[x]_R$ the equivalence class containing x. Given an arbitrary set $A \subseteq U$, in general it may not be possible to describe X precisely in (U, R). One may characterize X by a pair of lower

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http://dx.doi.org/10.1016/j.knosys.2015.09.036
0950-7051/

and upper approximations:

$$\underline{R}(X) = \{x \in U \mid [x]_R \subseteq X\},$$

$$\overline{R}(X) = \{x \in U \mid [x]_R \cap A \neq \emptyset\}.$$

The pair $(\underline{R}(X), \overline{R}(X))$ is referred to as rough set approximation of X.

Let $\sim X = U - X$, we have the following basic properties of Pawlak's rough sets.

(L1) $\underline{R}(X) \subseteq X$	(H1) $X \subseteq \overline{R}(X)$
(L2) $\underline{R}(\emptyset) = \emptyset$	(H2) $\overline{R}(\emptyset) = \emptyset$
(L3) $\underline{R}(U) = U$	(H3) $\overline{R}(U) = U$
(L4) $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$	(H4) $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$
(L5) $X \subseteq Y \Rightarrow \underline{R}(X) \subseteq \underline{R}(Y)$	(H5) $X \subseteq Y \Rightarrow \overline{R}(X) \subseteq \overline{R}(Y)$
(L6) $\underline{R}(X \cup Y) \supseteq \underline{R}(X) \cup \underline{R}(Y)$	(H6) $\overline{R}(X \cap Y) \subseteq \overline{R}(X) \cap \overline{R}(Y)$
(L7) $\underline{R}(\sim X) = \sim \overline{R}(X)$	(H7) $\overline{R}(\sim X) = \sim \underline{R}(X)$
(L8) $\underline{R}(\underline{R}(X)) = \underline{R}(X)$	(H8) $\overline{R}(\overline{R}(X)) = \overline{R}(X)$
(L9) $\underline{R}(\overline{R}(X)) = \overline{R}(X)$	(H9) $\overline{R}(\underline{R}(X)) = \underline{R}(X)$

2.2. Multigranulation rough sets

In recent years, Qian et al. [20–22] have proposed a new rough set model called multigranulation rough set. In this model, a target concept is approximated by multiple binary relations. Next, we briefly outline two definitions of multigranulation rough set models, i.e., optimistic and pessimistic multigranulation rough sets respectively. Detailed descriptions could be found in [20–22].

Definition 2.1. Let $K = (U, \mathbf{R})$ be a knowledge base, where $\mathbf{R}$ is a family of equivalence relations on the universe U. Let $A_1, A_2, \ldots, A_m \in \mathbf{R}$, where m is a natural number. For any $X \subseteq U$, its optimistic lower and upper approximations with respect to $A_1, A_2 \ldots, A_m$ are respectively defined as follows.

$$\underline{\sum_{i=1}^{m} A_i}^{O}(X) = \{x \in U \mid [x]_{A_1} \subseteq X \text{ or } [x]_{A_2} \subseteq X \text{ or } \cdots \text{ or } [x]_{A_m} \subseteq X\};$$

$$\overline{\sum_{i=1}^{m} A_i}^{O}(X) = \sim \underline{\sum_{i=1}^{m} A_i}^{O}(\sim X).$$

$(\underline{\sum_{i=1}^{m} A_i}^{O}(X), \overline{\sum_{i=1}^{m} A_i}^{O}(X))$ is called the optimistic multigranulation rough sets of X. Here, the word "optimistic" means that only one granular structure is needed to satisfy with the inclusion condition between an equivalence class and a target concept when multiple independent granular structures are available in problem processing.

Definition 2.2. Let $K = (U, \mathbf{R})$ be a knowledge base, where $\mathbf{R}$ is a family of equivalence relations on the universe U. Let $A_1, A_2, \ldots, A_m \in \mathbf{R}$, where m is a natural number. For any $X \subseteq U$, its pessimistic lower and upper approximations with respect to $A_1, A_2, \ldots, A_m$ are respectively defined as follows.

$$\underline{\sum_{i=1}^{m} A_i}^{P}(X) = \{x \in U \mid [x]_{A_1} \subseteq X \text{ and } [x]_{A_2} \subseteq X \text{ and } \cdots \text{ and } [x]_{A_m} \subseteq X\};$$

$$\overline{\sum_{i=1}^{m} A_i}^{P}(X) = \sim \underline{\sum_{i=1}^{m} A_i}^{P}(\sim X).$$

$(\underline{\sum_{i=1}^{m} A_i}^{P}(X), \overline{\sum_{i=1}^{m} A_i}^{P}(X))$ is called the pessimistic multigranulation rough sets of X. Here, the word "pessimistic" means that all granular structures are needed to satisfy with the inclusion condition between an equivalence class and a target concept when multiple independent granular structures are available.

3. Constructive methods of rough approximation operators from existing approximation operators

In this section, we establish the constructive methods of rough approximation operators from existing rough approximation operators. At first, we give some notations and preliminary results.

For convenience and unity, we use the following symbols (L1)–(L9) and (H1)–(H9) to denote the basic properties of any operator pair $(\underline{apr}, \overline{apr})$, where $\underline{apr}$ and $\overline{apr}$ are mappings from $P(U)$ to $P(U)$:

(L1) $\underline{apr}(X) \subseteq X$;
(H1) $X \subseteq \overline{apr}(X)$
(L2) $\underline{apr}(\emptyset) = \emptyset$
(H2) $\overline{apr}(\emptyset) = \emptyset$
(L3) $\underline{apr}(U) = U$
(H3) $\overline{apr}(U) = U$
(L4) $\underline{apr}(X \cap Y) = \underline{apr}(X) \cap \underline{apr}(Y)$
(L4′) $\underline{apr}(X \cap Y) \subseteq \underline{apr}(X) \cap \underline{apr}(Y)$
(H4) $\overline{apr}(X \cup Y) = \overline{apr}(X) \cup \overline{apr}(Y)$
(H4′) $\overline{apr}(X \cup Y) \supseteq \overline{apr}(X) \cup \overline{apr}(Y)$
(L5) $X \subseteq Y \Rightarrow \underline{apr}(X) \subseteq \underline{apr}(Y)$
(H5) $X \subseteq Y \Rightarrow \overline{apr}(X) \subseteq \overline{apr}(Y)$
(L6) $\underline{apr}(X \cup Y) \supseteq \underline{apr}(X) \cup \underline{apr}(Y)$
(H6) $\overline{apr}(X \cap Y) \subseteq \overline{apr}(X) \cap \overline{apr}(Y)$
(L7) $\underline{apr}(\sim X) = \sim \overline{apr}(X)$
(H7) $\overline{apr}(\sim X) = \sim \underline{apr}(X)$
(L8) $\underline{apr}(\underline{apr}(X)) = \underline{apr}(X)$
(L8′) $\underline{apr}(\underline{apr}(X)) \supseteq \underline{apr}(X)$
(H8) $\overline{apr}(\overline{apr}(X)) = \overline{apr}(X)$
(H8′) $\overline{apr}(\overline{apr}(X)) \subseteq \overline{apr}(X)$
(L9) $\underline{apr}(\overline{apr}(X)) = \overline{apr}(X)$
(L9′) $\underline{apr}(\overline{apr}(X)) \subseteq \overline{apr}(X)$
(H9) $\overline{apr}(\underline{apr}(X)) = \underline{apr}(X)$
(H9′) $\overline{apr}(\underline{apr}(X)) \supseteq \underline{apr}(X)$

It is easy to prove the following lemma and the proof is omitted.

Lemma 3.1. *Let U be a non-empty set, $\underline{apr}$ and $\overline{apr}$ be mappings from $P(U)$ to $P(U)$. Then*

(1) *If $\underline{apr}$ satisfies (L1) for any $X \in P(U)$, then $\underline{apr}$ satisfies (L2).*
(2) *If $\overline{apr}$ satisfies (H1) for any $X \in P(U)$, then $\overline{apr}$ satisfies (H3).*
(3) *For any $X, Y \in P(U)$, if $\underline{apr}$ satisfies (L4), then $\underline{apr}$ satisfies (L5).*
(4) *For any $X, Y \in P(U)$, if $\overline{apr}$ satisfies (H4), then $\overline{apr}$ satisfies (H5).*
(5) *If $\underline{apr}$ satisfies (L1) and (L5) for any $X, Y \in P(U)$, then $\underline{apr}$ satisfies (L4′).*
(6) *If $\overline{apr}$ satisfies (H1) and (H5) for any $X, Y \in P(U)$, then $\overline{apr}$ satisfies (H4′).*
(7) *For any $X, Y \in P(U)$, if $\underline{apr}$ satisfies (L6), then $\underline{apr}$ satisfies (L5).*
(8) *For any $X, Y \in P(U)$, if $\overline{apr}$ satisfies (H6), then $\overline{apr}$ satisfies (H5).*
(9) *If $\underline{apr}$ satisfies (L5) for any $X, Y \in P(U)$, then $\underline{apr}$ satisfies (L6) for any $X, Y \in P(U)$.*
(10) *If $\overline{apr}$ satisfies (H5) for any $X, Y \in P(U)$, then $\overline{apr}$ satisfies (H6) for any $X, Y \in P(U)$.*
(11) *If $\underline{apr}$ and $\overline{apr}$ satisfy (L7) for any $X \in P(U)$, then $\underline{apr}$ and $\overline{apr}$ satisfy (H7) for any $X \in P(U)$. Moreover, if $\underline{apr}$ and $\overline{apr}$ satisfy (H7) for any $X \in P(U)$, then $\underline{apr}$ and $\overline{apr}$ satisfy (L7) for any $X \in P(U)$.*
(12) *If $\underline{apr}$ and $\overline{apr}$ satisfy (L7) or (H7) for any $X \in P(U)$, then*

$$\text{(L8)} \Longleftrightarrow \text{(H8)}, \ \text{(L8′)} \Longleftrightarrow \text{(H8′)}, \ \text{(L9)} \Longleftrightarrow \text{(H9)}, \ \text{(L9′)} \Longleftrightarrow \text{(H9′)}.$$

(13) *If $\underline{apr}$ satisfies (L1) and* (L8′) *for any $X \in P(U)$, then $\underline{apr}$ satisfies (L8) for any $X \in P(U)$.*
(14) *If $\overline{apr}$ satisfies (H1) and* (H8′) *for any $X \in P(U)$, then $\overline{apr}$ satisfies (H8) for any $X \in P(U)$.*
(15) *If $\underline{apr}$ satisfies (L1) for any $X \in P(U)$, then $\underline{apr}$ satisfies* (L9′) *for any $X \in P(U)$.*
(16) *If $\overline{apr}$ satisfies (H1) for any $X \in P(U)$, then $\overline{apr}$ satisfies* (H9′) *for any $X \in P(U)$.*

3.1. The first constructive method: (∪ , ∩)-type

Definition 3.1. Let U be a non-empty set, $P(U)$ be the power set of U. If $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ are some mappings from $P(U)$ to $P(U)$, $i = 1, 2, \ldots, n$. Then define two new mappings from $P(U)$ to $P(U)$ as follows: for any $X \subseteq U$,

$$\underline{apr}^{(1..n)}_{(\cup,\cap)}(X) = \bigcup_{i=1}^{n} \underline{apr}^{(i)}(X); \ \overline{apr}^{(1..n)}_{(\cup,\cap)}(X) = \bigcap_{i=1}^{n} \overline{apr}^{(i)}(X).$$

We call $(\underline{apr}^{(1..n)}_{(\cup,\cap)}, \overline{apr}^{(1..n)}_{(\cup,\cap)})$ is (∪ , ∩)-type generated pair by operator pairs $(\underline{apr}^{(i)}, \overline{apr}^{(i)})$ $(i = 1, 2, \ldots, n)$, where $(1..n)$ means that from 1 to n. When n=2, denote $(1..n)$ by $(1, 2)$.

Theorem 3.1. *Let U be a non-empty set,* $\underline{apr}^{(i)}$ *and* $\overline{apr}^{(i)}$ *be mappings from* $P(U)$ *to* $P(U)$, $i = 1, 2, \ldots, n$.

(1) *If for all* $X, Y \in P(U)$, $\underline{apr}^{(i)} (i = 1, 2, \ldots, n)$ *satisfy* (Lk), *then* $\underline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfies* (Lk), *where* $k = 1, 2, 3, 5$.
(2) *If for all* $X, Y \in P(U)$, $\overline{apr}^{(i)} (i = 1, 2, \ldots, n)$ *satisfy* (Hk), *then* $\overline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfies* (Hk), *where* $k = 1, 2, 3, 5$.
(3) *If for all* $X, Y \in P(U)$, $\underline{apr}^{(i)} (i = 1, 2, \ldots, n)$ *satisfy (L4), then* $\underline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfies* (L4′).
(4) *If for all* $X, Y \in P(U)$, $\overline{apr}^{(i)} (i = 1, 2, \ldots, n)$ *satisfy (H4), then* $\overline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfies* (H4′).
(5) *If for all* $X, Y \in P(U)$, $\underline{apr}^{(i)} (i = 1, 2, \ldots, n)$ *satisfy (L6), then* $\underline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfies (L6).*
(6) *If for all* $X, Y \in P(U)$, $\overline{apr}^{(i)} (i = 1, 2, \ldots, n)$ *satisfy (H6), then* $\overline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfies (H6).*
(7) *If for all* $X \in P(U)$, $\underline{apr}^{(i)}$ *and* $\overline{apr}^{(i)} (i = 1, 2, \ldots, n)$ *satisfy (L7) or (H7), then* $\underline{apr}^{(i)}$ *and* $\overline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfy (L7) or (H7).*
(8) *If for all* $X \in P(U)$, $\underline{apr}^{(i)} (i = 1, 2, \ldots, n)$ *satisfy (L6) and* (L8′), *then* $\underline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfies* (L8′).
(9) *If for all* $X \in P(U)$, $\overline{apr}^{(i)} (i = 1, 2, \ldots, n)$ *satisfy (H6) and* (H8′), *then* $\overline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfies* (H8′).
(10) *If for all* $X \in P(U)$, $\underline{apr}^{(i)} (i = 1, 2, \ldots, n)$ *satisfy (L1), (L5) or (L6), and (L8), then* $\underline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfies (L8).*
(11) *If for all* $X \in P(U)$, $\overline{apr}^{(i)} (i = 1, 2, \ldots, n)$ *satisfy (H1), (H5) or (H6), and (H8), then* $\overline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfies (H8).*
(12) *If for all* $X \in P(U)$, $\underline{apr}^{(i)} (i = 1, 2, \ldots, n)$ *satisfy (L1), then* $\underline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfies* (L9′).
(13) *If for all* $X \in P(U)$, $\overline{apr}^{(i)} (i = 1, 2, \ldots, n)$ *satisfy (H1), then* $\overline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfies*(H9′).

Proof. It is easy to verify that (L1), (L2), (L3), (L5), (H1), (H2), (H3) and (H5) hold for $\underline{apr}^{(1..n)}_{(\cup,\cap)}$ and $\overline{apr}^{(1..n)}_{(\cup,\cap)}$) when the corresponding conditions are satisfied. We prove other properties as follows.

(3): It follows from (1) and Lemma 3.1 (5).
(4): It follows from (2) and Lemma 3.1 (6).
(5): Applying (1) and Lemma 3.1 (9) we can get (5).
(6): Applying (2) and Lemma 3.1 (10) we can get (6).
(7): For all $X \in P(U)$, if $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L7), from this and Definition 3.1 we have

$$\underline{apr}^{(1..n)}_{(\cup,\cap)}(\sim X) = \bigcup_{i=1}^{n} \underline{apr}^{(i)}(\sim X) = \bigcup_{i=1}^{n} (\sim \overline{apr}^{(i)}(X))$$
$$= \sim \left(\bigcap_{i=1}^{n} \overline{apr}^{(i)}(X)\right) = \sim \overline{apr}^{(1..n)}_{(\cup,\cap)}(X).$$

This means that $\underline{apr}^{(1..n)}_{(\cup,\cap)}$ and $\overline{apr}^{(1..n)}_{(\cup,\cap)}$ satisfy (L7).

Similarly, we can get that $\underline{apr}^{(1..n)}_{(\cup,\cap)}$ and $\overline{apr}^{(1..n)}_{(\cup,\cap)}$ satisfy (H7).

(8) We only prove the case of $n = 2$. For all $X \in P(U)$, if $\underline{apr}^{(i)}$ $(i = 1, 2)$ satisfy (L6) and (L8′), by Definition 3.1 we have

$$\begin{aligned}
\underline{apr}^{(1,2)}_{(\cup,\cap)}(\underline{apr}^{(1,2)}_{(\cup,\cap)}(X)) &= \underline{apr}^{(1,2)}_{(\cup,\cap)}(\underline{apr}^{(1)}(X) \cup \underline{apr}^{(2)}(X)) && \text{(By Definition 3.1)}\\
&\supseteq \underline{apr}^{(1,2)}_{(\cup,\cap)}(\underline{apr}^{(1)}(X)) \cup \underline{apr}^{(1,2)}_{(\cup,\cap)}(\underline{apr}^{(2)}(X)) && \text{(By (L6) for } \underline{apr}^{(1,2)}_{(\cup,\cap)} \text{ from (5))}\\
&= (\underline{apr}^{(1)}(\underline{apr}^{(1)}(X)) \cup \underline{apr}^{(2)}(\underline{apr}^{(1)}(X)))\\
&\quad \cup (\underline{apr}^{(1)}(\underline{apr}^{(2)}(X)) \cup \underline{apr}^{(2)}(\underline{apr}^{(2)}(X))) && \text{(By Definition 3.1)}\\
&\supseteq (\underline{apr}^{(1)}(\underline{apr}^{(1)}(X))) \cup (\underline{apr}^{(2)}(\underline{apr}^{(2)}(X)))\\
&\supseteq \underline{apr}^{(1)}(X) \cup \underline{apr}^{(2)}(X) && \text{(By (L8′) for } \underline{apr}^{(1)} \text{ and } \underline{apr}^{(2)})\\
&= \underline{apr}^{(1,2)}_{(\cup,\cap)}(X).
\end{aligned}$$

This means that $\underline{apr}^{(1,2)}_{(\cup,\cap)}$ satisfies (L8′).

(9) We only prove the case of $n = 2$. For all $X \in P(U)$, if $\overline{apr}^{(i)}$ $(i = 1, 2)$ satisfy (H6) and (H8′), by Definition 3.1 we have

$$\begin{aligned}
\overline{apr}^{(1,2)}_{(\cup,\cap)}(\overline{apr}^{(1,2)}_{(\cup,\cap)}(X)) &= \overline{apr}^{(1,2)}_{(\cup,\cap)}(\overline{apr}^{(1)}(X) \cap \overline{apr}^{(2)}(X)) && \text{(By Definition 3.1)}\\
&\subseteq \overline{apr}^{(1,2)}_{(\cup,\cap)}(\overline{apr}^{(1)}(X)) \cap \overline{apr}^{(1,2)}_{(\cup,\cap)}(\overline{apr}^{(2)}(X)) && \text{(By (H6) for } \overline{apr}^{(1,2)}_{(\cup,\cap)} \text{ from (6))}\\
&= (\overline{apr}^{(1)}(\overline{apr}^{(1)}(X)) \cap \overline{apr}^{(2)}(\overline{apr}^{(1)}(X)))\\
&\quad \cap (\overline{apr}^{(1)}(\overline{apr}^{(2)}(X)) \cap \overline{apr}^{(2)}(\overline{apr}^{(2)}(X))) && \text{(By Definition 3.1)}\\
&\subseteq (\overline{apr}^{(1)}(\overline{apr}^{(1)}(X))) \cap (\overline{apr}^{(2)}(\overline{apr}^{(2)}(X)))\\
&\subseteq \overline{apr}^{(1)}(X) \cap \overline{apr}^{(2)}(X) && \text{(By (H8′) for } \overline{apr}^{(1)} \text{ and } \overline{apr}^{(2)})\\
&= \overline{apr}^{(1,2)}_{(\cup,\cap)}(X).
\end{aligned}$$

This means that $\overline{apr}^{(1,2)}_{(\cup,\cap)}$ satisfies (H8′).

(10) By (8) and Lemma 3.1 (7), (9) and (13) we can get (10).
(11) By (9) and Lemma 3.1 (8), (10) and (14) we can get (11).
(12) It follows from (1) and Lemma 3.1 (15).
(13) It follows from (2) and Lemma 3.1 (16). □

Corollary 3.1. *Let U be a non-empty set,* $(\underline{apr}^{(i)}, \overline{apr}^{(i)})$ $(i = 1, 2, \ldots, n)$ *be Pawlak's rough approximation pairs on U. Then* $\underline{apr}^{(1..n)}_{(\cup,\cap)}$ *and* $\overline{apr}^{(1..n)}_{(\cup,\cap)}$ *satisfy (L1)–(L3), (L4′), (L5)–(L8), (L9′), (H1)–(H3), (H4′), (H5)–(H8), and (H9′).*

In general case, $\underline{apr}^{(1..n)}_{(\cup,\cap)}$ and $\overline{apr}^{(1..n)}_{(\cup,\cap)}$ do not satisfy (L4), (H4), (L9) and (H9). Please see the following examples.

Example 3.1. Let

$U = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$,

$U/R_1 = \{\{e_1, e_7\}, \{e_2, e_3, e_4, e_5, e_6\}, \{e_8\}\}$,

$U/R_2 = \{\{e_1, e_2\}, \{e_3, e_4, e_5\}, \{e_6, e_7, e_8\}\}$.

If $apr^{(1)}$ and $apr^{(2)}$ denote Pawlak's rough approximations by equivalence relations R_1 and R_2, respectively. Putting

$X = \{e_1, e_2, e_3, e_4, e_5, e_6\}$, $Y = \{e_3, e_4, e_5, e_6, e_7, e_8\}$.

Then

$$\underline{apr}^{(1,2)}_{(\cup,\cap)}(X)=\underline{apr}^{(1)}(X)\bigcup\underline{apr}^{(2)}(X)=\{e_1,e_2,e_3,e_4,e_5,e_6\},$$

$$\underline{apr}^{(1,2)}_{(\cup,\cap)}(Y)=\underline{apr}^{(1)}(Y)\bigcup\underline{apr}^{(2)}(Y)=\{e_3,e_4,e_5,e_6,e_7,e_8\},$$

$$\underline{apr}^{(1,2)}_{(\cup,\cap)}(X\cap Y)=\underline{apr}^{(1,2)}_{(\cup,\cap)}(\{e_3,e_4,e_5,e_6\})=\{e_3,e_4,e_5\}.$$

$$\underline{apr}^{(1,2)}_{(\cup,\cap)}(X\cap Y)\neq\underline{apr}^{(1,2)}_{(\cup,\cap)}(X)\cap\underline{apr}^{(1,2)}_{(\cup,\cap)}(Y)=\{e_3,e_4,e_5,e_6\}.$$

That is, $\underline{apr}^{(1,2)}_{(\cup,\cap)}$ does not satisfy (L4).

Similarly, let $X_1=\{e_1,e_2,e_8\},\ Y_1=\{e_7,e_8\}$, then we can get

$$\overline{apr}^{(1,2)}_{(\cup,\cap)}(X\cup Y)\neq\overline{apr}^{(1,2)}_{(\cup,\cap)}(X)\cup\overline{apr}^{(1,2)}_{(\cup,\cap)}(Y).$$

That is,

This means that $\overline{apr}^{(1,2)}_{(\cup,\cap)}$ does not satisfy (H4).

Example 3.2. Let $apr^{(1)}$ and $apr^{(2)}$ be the rough approximations by equivalence relations R_1 and R_2 in Example 3.1, respectively. Putting

$$X=\{e_1,e_2,e_3,e_4,e_5,e_6\},\ X_1=\{e_1,e_2,e_8\}.$$

Then

$$\underline{apr}^{(1,2)}_{(\cup,\cap)}(\overline{apr}^{(1,2)}_{(\cup,\cap)}(X))=\{e_1,e_2,e_3,e_4,e_5,e_6,e_7\}\neq X,$$

$$\overline{apr}^{(1,2)}_{(\cup,\cap)}(\underline{apr}^{(1,2)}_{(\cup,\cap)}(X_1))=\{e_1,e_2,e_6,e_7,e_8\}\neq X_1.$$

That is, $\underline{apr}^{(1,2)}_{(\cup,\cap)}$ and $\overline{apr}^{(1,2)}_{(\cup,\cap)}$ do not satisfy (L9) and (H9).

By Definition 2.1 and 3.1 we have

Proposition 3.1. *Let $K=(U,\mathbf{R})$ be a knowledge base, where $\mathbf{R}$ is a family of equivalence relations on the universe U. Let $A_1,A_2,\ldots,A_m\in\mathbf{R}$, $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ be the lower and upper approximation operators induced by equivalence relation A_i ($i=1,2,\ldots,m$), where m is a natural number. Then, for any $X\subseteq U$,*

$$\underline{\sum_{i=1}^{m}A_i}^{O}(X)=\underline{apr}^{(1..m)}_{(\cup,\cap)}(X);$$

$$\overline{\sum_{i=1}^{m}A_i}^{O}(X)=\overline{apr}^{(1..m)}_{(\cup,\cap)}(X).$$

Remark 3.1. By Proposition 3.1 we know that the main results of Theorem 3 and 4 in [22] (or Proposition 3.2 and 3.3 in [21]) are corollaries of Theorem 3.1 in this paper.

Definition 3.2. ([36]) Let $\mathscr{I}=(U,AT,V,f)$ be an information system, $A_1,A_2,\ldots,A_s\subseteq AT$ be attribute subsets ($s\leq 2^{|AT|}$), and $R_{A_1},R_{A_2},\ldots,R_{A_s}$ be equivalence relations, respectively. The operators $\underline{FR_{\sum_{i=1}^{s}A_i}}$ and $\overline{FR_{\sum_{i=1}^{s}A_i}}:P(U)\to P(U)$ are defined as follows: for any $X\in P(U)$,

$$\underline{FR_{\sum_{i=1}^{s}A_i}}(X)=\left\{u\mid\bigvee_{i=1}^{s}[u]_{A_i}\subseteq X\right\},$$

$$\overline{FR_{\sum_{i=1}^{s}A_i}}(X)=\left\{u\mid\bigwedge_{i=1}^{s}([u]_{A_i}\cap X\neq\emptyset)\right\},$$

where "$\vee$" means "some", and "$\wedge$" means "all". We call them the first type of multiple granulation lower and upper approximation operators, and call $\underline{FR_{\sum_{i=1}^{s}A_i}}(X)$ and $\overline{FR_{\sum_{i=1}^{s}A_i}}(X)$ the first type of multiple granulation lower approximation set and upper approximation set of X, respectively.

By Definitions 3.2 and 3.1 we have

Proposition 3.2. *Let $\mathscr{I}=(U,AT,V,f)$ be an information system, $A_1,A_2,\ldots,A_s\subseteq AT$ be attribute subsets ($s\leq 2^{|AT|}$), and $R_{A_1},R_{A_2},\ldots,R_{A_s}$ be equivalence relations, respectively. If $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ are the lower and upper approximation operators induced by equivalence relation R_{A_i} ($i=1,2,\ldots,s$), then, for any $X\subseteq U$,*

$$\underline{FR_{\sum_{i=1}^{s}A_i}}(X)=\underline{apr}^{(1..s)}_{(\cup,\cap)}(X);$$

$$\overline{FR_{\sum_{i=1}^{s}A_i}}(X)=\overline{apr}^{(1..s)}_{(\cup,\cap)}(X).$$

Remark 3.2. By Proposition 3.2 we know that the main results of Proposition 2, 3, 6 and 7 in [36] are corollaries of Theorem 3.1 in this paper.

Definition 3.3. ([1]) Let $\{R_i:\ i=1,2,\ldots,n\}$ be a finite family of binary relations on non-empty finite set U. We can define n-lower and n-upper approximations of $X\subseteq U$, according to R_i, $i=1,2,\ldots,n$, as follows:

$$_{\sqsubset n\sqsupset}\underline{apr}^{(*)}(X)=\bigcup_{i=1}^{n}\underline{R}_i^{*}(X),\quad {}^{\sqsubset n\sqsupset}\overline{apr}^{(*)}(X)=\bigcap_{i=1}^{n}\overline{R}_i^{*}(X).$$

where $\underline{R}_i^*(X)$ and $\overline{R}_i^*(X)$ are lower and upper approximations (see Definition 2.2 in [1]) based on binary relation R_i, respectively.

By Definitions 3.3 and 3.1 we have

Proposition 3.3. *Let $\{R_i:\ i=1,2,\ldots,n\}$ be a finite family of binary relations on non-empty finite set U, $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ be the lower and upper approximation operators induced by binary relation R_i ($i=1,2,\ldots,n$). Then, for any $X\subseteq U$,*

$$_{\sqsubset n\sqsupset}\underline{apr}^{(*)}(X)=\underline{apr}^{(1..n)}_{(\cup,\cap)}(X);$$

$$^{\sqsubset n\sqsupset}\overline{apr}^{(*)}(X)=\overline{apr}^{(1..n)}_{(\cup,\cap)}(X).$$

Remark 3.3. By Proposition 3.3 we know that the main results of Proposition 4.1 and 4.2 in [1] are corollaries of Theorem 3.1 in this paper.

Remark 3.4. From Propositions 3.1, 3.2 and 3.3 we know that optimistic multigranulation rough sets, the first type of multiple granulation rough sets and n-rough sets are essentially direct applications of the above first constructive method. Moreover, the first constructive method is extensive, because the operators $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ are only mappings from $P(U)$ to $P(U)$, $i=1,2,\ldots,n$. Hence, The constructive method can be applied to hybrid multigranulation rough sets and others (see the following sections).

3.2. The second constructive method: ($\cap$, $\cup$)-type

Definition 3.4. Let U be a non-empty set, $P(U)$ be the power set of U. If $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ are some mappings from $P(U)$ to $P(U)$, $i=1,2,\ldots,n$. Then define two new mappings from $P(U)$ to $P(U)$ as follows: for any $X\subseteq U$,

$$\underline{apr}^{(1..n)}_{(\cap,\cup)}(X)=\bigcap_{i=1}^{n}\underline{apr}^{(i)}(X);\quad \overline{apr}^{(1..n)}_{(\cap,\cup)}(X)=\bigcup_{i=1}^{n}\overline{apr}^{(i)}(X).$$

We call $(\underline{apr}^{(1..n)}_{(\cap,\cup)},\overline{apr}^{(1..n)}_{(\cap,\cup)})$ is ($\cap$, $\cup$)-type generated pair by operator pairs $(\underline{apr}^{(i)},\overline{apr}^{(i)})$ $(i=1,2,\ldots,n)$, where $(1..n)$ means that from 1 to n. When n=2, denote $(1..n)$ by $(1,2)$.

Theorem 3.2. *Let U be a non-empty set, $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ be mappings fromP(U) to $P(U)$, $i=1,2,\ldots,n$.*

(1) If for all X, $Y\in P(U)$, $\underline{apr}^{(i)}$ $(i=1,2,\ldots,n)$ satisfy (Lk), then $\underline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfies (Lk), where $k=1,2,3,5$.

(2) If for all $X, Y \in P(U)$, $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (Hk), then $\overline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfies (Hk), where $k = 1, 2, 3, 5$.

(3) If for all $X, Y \in P(U)$, $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L4), then $\underline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfies (L4).

(4) If for all $X, Y \in P(U)$, $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H4), then $\overline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfies (H4).

(5) If for all $X, Y \in P(U)$, $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L6), then $\underline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfies (L6).

(6) If for all $X, Y \in P(U)$, $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H6), then $\overline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfies (H6).

(7) If for all $X \in P(U)$, $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L7) or (H7), then $\underline{apr}^{(i)}$ and $\overline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfy (L7) or (H7).

(8) If for all $X \in P(U)$, $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L1), (L4), (L8) and X satisfies (C1), then $\underline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfies (L8), where (C1) means that

$$\underline{apr}^{(i)}(\underline{apr}^{(j)}(X)) = \underline{apr}^{(j)}(X), \quad i, j = 1, 2, \ldots, n, i \neq j.$$

(9) If for all $X \in P(U)$, $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H1), (H4), (H8) and X satisfies (C2), then $\overline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfies (H8), where (C2) means that

$$\overline{apr}^{(i)}(\overline{apr}^{(j)}(X)) = \overline{apr}^{(j)}(X) \quad i, j = 1, 2, \ldots, n, i \neq j.$$

(10) If for all $X \in P(U)$, $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L1), then $\underline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfies (L9′).

(11) If for all $X \in P(U)$, $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H1), then $\overline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfies (H9′).

Proof. It is easy to verify that (L1), (L2), (L3), (L5), (H1), (H2), (H3) and (H5) hold for $\underline{apr}^{(1..n)}_{(\cap,\cup)}$ and $\overline{apr}^{(1..n)}_{(\cap,\cup)}$). We prove other properties in follows.

(3): For all $X, Y \in P(U)$, if $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L4), from this and Definition 3.4 we have

$$\underline{apr}^{(1..n)}_{(\cap,\cup)}(X \cap Y) = \bigcap_{i=1}^{n} \underline{apr}^{(i)}(X \cap Y) = \bigcap_{i=1}^{n} (\underline{apr}^{(i)}(X) \cap \underline{apr}^{(i)}(Y))$$
$$= \left(\bigcap_{i=1}^{n} \underline{apr}^{(i)}(X)\right) \cap \left(\bigcap_{i=1}^{n} \underline{apr}^{(i)}(Y)\right) = \underline{apr}^{(1..n)}_{(\cap,\cup)}(X) \cap \underline{apr}^{(1..n)}_{(\cap,\cup)}(Y).$$

This means that $\underline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfies (L4).

(4): Similar to (3).

(5): It follows from (1) and Lemma 3.1 (9).

(6): Applying (2) and Lemma 3.1 (10) we can get (5).

(7): For all $X \in P(U)$, if $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L7), from this and Definition 3.4 we have

$$\underline{apr}^{(1..n)}_{(\cap,\cup)}(\sim X) = \bigcap_{i=1}^{n} \underline{apr}^{(i)}(\sim X) = \bigcap_{i=1}^{n} (\sim \overline{apr}^{(i)}(X))$$
$$=\sim \left(\bigcup_{i=1}^{n} \overline{apr}^{(i)}(X)\right) =\sim \overline{apr}^{(1..n)}_{(\cap,\cup)}(X).$$

This means that $\underline{apr}^{(1..n)}_{(\cap,\cup)}$ and $\overline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfy (L7).

Similarly, we can get that $\underline{apr}^{(1..n)}_{(\cap,\cup)}$ and $\overline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfy (H7).

(8) We only prove the case of $n = 2$. For all $X \in P(U)$, if $\underline{apr}^{(i)}$ $(i = 1, 2)$ satisfy (L1), (L4) and (L8), by Lemma 3.1 (3) we know that $\underline{apr}^{(i)}$ $(i = 1, 2)$ satisfy (L5). By Definition 3.4 we have

$$\underline{apr}^{(1,2)}_{(\cap,\cup)}(\underline{apr}^{(1,2)}_{(\cap,\cup)}(X)) = \underline{apr}^{(1,2)}_{(\cap,\cup)}(\underline{apr}^{(1)}(X) \cap \underline{apr}^{(2)}(X))$$
(By Definition 3.4)

$$= \underline{apr}^{(1,2)}_{(\cap,\cup)}(\underline{apr}^{(1)}(X)) \cap \underline{apr}^{(1,2)}_{(\cap,\cup)}(\underline{apr}^{(2)}(X))$$
(By (L4) for $\underline{apr}^{(1,2)}_{(\cap,\cup)}$, applying (3))

$$= (\underline{apr}^{(1)}(\underline{apr}^{(1)}(X)) \cap \underline{apr}^{(2)}(\underline{apr}^{(1)}(X))) \cap (\underline{apr}^{(1)}(\underline{apr}^{(2)}(X)) \cap \underline{apr}^{(2)}(\underline{apr}^{(2)}(X)))$$
(By Definition 3.4)

$$= \underline{apr}^{(1)}(X) \cap \underline{apr}^{(2)}(\underline{apr}^{(1)}(X)) \cap \underline{apr}^{(1)}(\underline{apr}^{(2)}(X)) \cap \underline{apr}^{(2)}(X)$$
(By (L8) for $\underline{apr}^{(1)}$ and $\underline{apr}^{(2)}$)

$$= (\underline{apr}^{(1)}(X) \cap \underline{apr}^{(1)}(\underline{apr}^{(2)}(X))) \cap (\underline{apr}^{(2)}(X) \cap \underline{apr}^{(2)}(\underline{apr}^{(1)}(X)))$$
(By the associativity of $\cap$)

$$= \underline{apr}^{(1)}(\underline{apr}^{(2)}(X)) \cap \underline{apr}^{(2)}(\underline{apr}^{(1)}(X))$$
(By (L1) and (L5) for $\underline{apr}^{(1)}$ and $\underline{apr}^{(2)}$)

$$= \underline{apr}^{(2)}(X) \cap \underline{apr}^{(1)}(X)$$
(By (C1) for X)

$$= \underline{apr}^{(1)}(X) \cap \underline{apr}^{(2)}(X)$$
$$= \underline{apr}^{(1,2)}_{(\cap,\cup)}(X).$$

This means that $\underline{apr}^{(1,2)}_{(\cap,\cup)}$ satisfies (L8).

(9) It is similar to (8).

(10) Using (1) and Lemma 3.1 (15) we can get (10).

(11) It follows from (2) and Lemma 3.1 (16). □

Corollary 3.2. *Let U be a non-empty set, $(\underline{apr}^{(i)}, \overline{apr}^{(i)})$ $(i = 1, 2, \ldots, n)$ be Pawlak's rough approximation pairs on U. Then $\underline{apr}^{(1..n)}_{(\cap,\cup)}$ and $\overline{apr}^{(1..n)}_{(\cap,\cup)}$ satisfy (L1)–(L8), (L9′), (H1)–(H8) and (H9′).*

In general case, $\underline{apr}^{(1..n)}_{(\cap,\cup)}$ and $\overline{apr}^{(1..n)}_{(\cap,\cup)}$ do not satisfy (L9) and (H9). Please see the following example.

Example 3.3. Let $apr^{(1)}$ and $apr^{(2)}$ be the rough approximations by equivalence relations R_1 and R_2 in Example 3.1, respectively. Putting

$$X = \{e_1, e_2, e_3, e_4, e_5, e_6\}, \ X_1 = \{e_1, e_2, e_8\}.$$

Then

$$\underline{apr}^{(1,2)}_{(\cap,\cup)}(\overline{apr}^{(1,2)}_{(\cap,\cup)}(X)) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\} \neq X,$$

$$\overline{apr}^{(1,2)}_{(\cap,\cup)}(\underline{apr}^{(1,2)}_{(\cap,\cup)}(X_1)) = \emptyset \neq X_1.$$

That is, $\underline{apr}^{(1,2)}_{(\cap,\cup)}$ and $\overline{apr}^{(1,2)}_{(\cap,\cup)}$ do not satisfy (L9) and (H9).

By Definitions 2.2 and 3.4 we have

Proposition 3.4. *Let $K = (U, \mathbf{R})$ be a knowledge base, where $\mathbf{R}$ is a family of equivalence relations on the universe U. Let $A_1, A_2, \ldots, A_m \in \mathbf{R}$, $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ be the lower and upper approximation operators induced by equivalence relation A_i $(i = 1, 2, \ldots, m)$, where m is a natural number. Then, for any $X \subseteq U$,*

$$\underline{\sum_{i=1}^{m} A_i{}^P(X)} = \underline{apr}^{(1..m)}_{(\cap,\cup)}(X);$$

$$\overline{\sum_{i=1}^{m} A_i{}^P(X)} = \overline{apr}^{(1..m)}_{(\cap,\cup)}(X).$$

Remark 3.5. By Proposition 3.4 we know that the main results of Theorem 8 and 9 in [22] are corollaries of Theorem 3.2 in this paper.

Definition 3.5. ([36]) Let $\mathscr{I} = (U, AT, V, f)$ be an information system, $A_1, A_2, \ldots, A_s \subseteq AT$ be attribute subsets ($s \leq 2^{|AT|}$), and $R_{A_1}, R_{A_2}, \ldots, R_{A_s}$ be equivalence relations, respectively. The operators $\underline{SR_{\sum_{i=1}^s A_i}}$ and $\overline{SR_{\sum_{i=1}^s A_i}}: P(U) \to P(U)$ are defined as follows: for any $X \in P(U)$,

$$\underline{SR_{\sum_{i=1}^s A_i}}(X) = \left\{ u \mid \bigwedge_{i=1}^{s} [u]_{A_i} \subseteq X \right\},$$

$$\overline{SR_{\sum_{i=1}^s A_i}}(X) = \left\{ u \mid \bigvee_{i=1}^{s} ([u]_{A_i} \cap X \neq \emptyset) \right\},$$

where "$\vee$" means "some", and "$\wedge$" means "all". We call them the second type of multiple granulation lower and upper approximation operators, and call $\underline{SR_{\sum_{i=1}^s A_i}}(X)$ and $\overline{SR_{\sum_{i=1}^s A_i}}(X)$ the second type of multiple granulation lower approximation set and upper approximation set of X, respectively.

By Definitions 3.4 and 3.5 we have

Proposition 3.5. *Let $\mathscr{I} = (U, AT, V, f)$ be an information system, $A_1, A_2, \ldots, A_s \subseteq AT$ be attribute subsets ($s \leq 2^{|AT|}$), and $R_{A_1}, R_{A_2}, \ldots, R_{A_s}$ be equivalence relations, respectively. If $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ are the lower and upper approximation operators induced by equivalence relation R_{A_i} ($i = 1, 2, \ldots, s$), then, for any $X \subseteq U$,*

$$\underline{SR_{\sum_{i=1}^s A_i}}(X) = \underline{apr}^{(1..s)}_{(\cap,\cup)}(X);$$

$$\overline{SR_{\sum_{i=1}^s A_i}}(X) = \overline{apr}^{(1..s)}_{(\cap,\cup)}(X).$$

Remark 3.6. By Proposition 3.5 we know that the main results of Propositions 15, 16, 19 and 20 in [36] are corollaries of Theorem 3.2 in this paper.

Remark 3.7. From Propositions 3.4 and 3.5 we know that pessimistic multigranulation rough sets and the second type of multiple granulation rough sets are essentially direct applications of the above second constructive method. Moreover, the second constructive method is extensive, because the operators $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ are only mappings from $P(U)$ to $P(U)$, $i = 1, 2, \ldots, n$. Hence, The constructive method can be applied to hybrid multigranulation rough sets and others (see the following sections).

3.3. The third constructive method: ($\cap$, $\cap$)-type

Definition 3.6. Let U be a non-empty set, $P(U)$ be the power set of U. If $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ are some mappings from $P(U)$ to $P(U)$, $i = 1, 2, \ldots, n$. Then define two new mappings from $P(U)$ to $P(U)$ as follows: for any $X \subseteq U$,

$$\underline{apr}^{(1..n)}_{(\cap,\cap)}(X) = \bigcap_{i=1}^{n} \underline{apr}^{(i)}(X); \quad \overline{apr}^{(1..n)}_{(\cap,\cap)}(X) = \bigcap_{i=1}^{n} \overline{apr}^{(i)}(X).$$

We call $(\underline{apr}^{(1..n)}_{(\cap,\cap)}, \overline{apr}^{(1..n)}_{(\cap,\cap)})$ is $(\cap, \cap)-$type generated pair by operator pairs $(\underline{apr}^{(i)}, \overline{apr}^{(i)})$ $(i = 1, 2, \ldots, n)$, where $(1..n)$ means that from 1 to n. When n=2, denote $(1..n)$ by $(1, 2)$.

Theorem 3.3. *Let U be a non-empty set, $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ be mappings fromP(U) to $P(U)$, $i = 1, 2, \ldots, n$.*

(1) If for all $X, Y \in P(U)$, $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (Lk), then $\underline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfies (Lk), where $k = 1, 2, 3, 5$.

(2) If for all $X, Y \in P(U)$, $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (Hk), then $\overline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfies (Hk), where $k = 1, 2, 3, 5$.

(3) If for all $X, Y \in P(U)$, $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L4), then $\underline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfies (L4).

(4) If for all $X, Y \in P(U)$, $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H4′), then $\overline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfies (H4′).

(5) If for all $X, Y \in P(U)$, $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L6), then $\underline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfies (L6).

(6) If for all $X, Y \in P(U)$, $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H6), then $\overline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfies (H6).

(7) If for all $X \in P(U)$, $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L1), (L4), (L8) and X satisfies (C1), then $\underline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfies (L8), where (C1) means that

$$\underline{apr}^{(i)}(\underline{apr}^{(j)}(X)) = \underline{apr}^{(j)}(X), \quad i, j = 1, 2, \ldots, n, i \neq j.$$

(8) If for all $X \in P(U)$, $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H1), (H5), (H8′) and X satisfies (C2), then $\overline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfies (H8′), where (C2) means that

$$\overline{apr}^{(i)}(\overline{apr}^{(j)}(X)) = \overline{apr}^{(j)}(X) \;\; i, j = 1, 2, \ldots, n, i \neq j.$$

(9) If for all $X \in P(U)$, $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L1), then $\underline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfies (L9′).

(10) If for all $X \in P(U)$, $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H1), then $\overline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfies (H9′).

Proof. It is easy to verify that (L1), (L2), (L3), (L5), (H1), (H2), (H3) and (H5) hold for $\underline{apr}^{(1..n)}_{(\cap,\cap)}$ and $\overline{apr}^{(1..n)}_{(\cap,\cap)}$). We prove other properties in follows.

(3): For all $X, Y \in P(U)$, if $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L4), from this and Definition 3.6 we have

$$\underline{apr}^{(1..n)}_{(\cap,\cap)}(X \cap Y) = \bigcap_{i=1}^{n} \underline{apr}^{(i)}(X \cap Y) = \bigcap_{i=1}^{n} (\underline{apr}^{(i)}(X) \cap \underline{apr}^{(i)}(Y))$$
$$= \left(\bigcap_{i=1}^{n} \underline{apr}^{(i)}(X) \right) \cap \left(\bigcap_{i=1}^{n} \underline{apr}^{(i)}(Y) \right) = \underline{apr}^{(1..n)}_{(\cap,\cap)}(X) \cap \underline{apr}^{(1..n)}_{(\cap,\cap)}(Y).$$

This means that $\underline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfies (L4).

(4): For all $X, Y \in P(U)$, if $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H4′), from this and Definition 3.6 we have

$$\overline{apr}^{(1..n)}_{(\cap,\cap)}(X \cup Y) = \bigcap_{i=1}^{n} \overline{apr}^{(i)}(X \cup Y) \supseteq \bigcap_{i=1}^{n} (\overline{apr}^{(i)}(X) \cup \overline{apr}^{(i)}(Y))$$
$$\supseteq \left(\bigcap_{i=1}^{n} \overline{apr}^{(i)}(X) \right) \cup \left(\bigcap_{i=1}^{n} \overline{apr}^{(i)}(Y) \right) = \overline{apr}^{(1..n)}_{(\cap,\cap)}(X) \cup \overline{apr}^{(1..n)}_{(\cap,\cap)}(Y).$$

This means that $\overline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfies (H4′).

(5): For all $X, Y \in P(U)$, if $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L6), from this and Definition 3.6 we have

$$\underline{apr}^{(1..n)}_{(\cap,\cap)}(X \cup Y) = \bigcap_{i=1}^{n} \underline{apr}^{(i)}(X \cup Y) \supseteq \bigcap_{i=1}^{n} (\underline{apr}^{(i)}(X) \cup \underline{apr}^{(i)}(Y))$$
$$\supseteq \left(\bigcap_{i=1}^{n} \underline{apr}^{(i)}(X) \right) \cup \left(\bigcap_{i=1}^{n} \underline{apr}^{(i)}(Y) \right) = \underline{apr}^{(1..n)}_{(\cap,\cap)}(X) \cup \underline{apr}^{(1..n)}_{(\cap,\cap)}(Y).$$

This means that $\underline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfies (L6).

(6): For all $X, Y \in P(U)$, if $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H6), from this and Definition 3.6 we have

$$\overline{apr}^{(1..n)}_{(\cap,\cap)}(X \cap Y) = \bigcap_{i=1}^{n} \overline{apr}^{(i)}(X \cap Y) \subseteq \bigcap_{i=1}^{n} (\overline{apr}^{(i)}(X) \cap \overline{apr}^{(i)}(Y))$$
$$= \left(\bigcap_{i=1}^{n} \overline{apr}^{(i)}(X) \right) \cap \left(\bigcap_{i=1}^{n} \overline{apr}^{(i)}(Y) \right) = \overline{apr}^{(1..n)}_{(\cap,\cap)}(X) \cap \overline{apr}^{(1..n)}_{(\cap,\cap)}(Y).$$

That is, $\overline{apr}^{(1..n)}_{(\cap,\cap)}$) satisfies (H6).

(7), (8), (9) and (10): The proofs are similar to Theorem 3.2 (8), (9), (10) and (11), respectively. □

Corollary 3.3. *Let U be a non-empty set, $(\underline{apr}^{(i)}, \overline{apr}^{(i)})$ $(i = 1, 2, \ldots, n)$ be Pawlak's rough approximation pairs on U. Then $\underline{apr}^{(1..n)}_{(\cap,\cap)}$ and $\overline{apr}^{(1..n)}_{(\cap,\cap)}$ satisfy (L1)–(L6), (L8), (L9′), (H1)–(H3), (H4′), (H5), (H6), (H8′) and (H9′).*

Remark 3.8. In general, $\overline{apr}^{(1..n)}_{(\cap,\cap)})$ and $\overline{apr}^{(1..n)}_{(\cap,\cap)})$ do not satisfy the properties (L7) and (H7), see the following Example 3.4. Thus, the third generated rough approximation operator pairs are not dual. Moreover, $\overline{apr}^{(1..n)}_{(\cap,\cap)})$ does not satisfy the property (H4), see the following Example 3.5.

Example 3.4. Let

$U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$,

$U/R_1 = \{\{e_1, e_2\}, \{e_3\}, \{e_4, e_5, e_6\}\}$,

$U/R_2 = \{\{e_1\}, \{e_2, e_3, e_4\}, \{e_5, e_6\}\}$.

If $apr^{(1)}$ and $apr^{(2)}$ denote Pawlak's rough approximations by equivalence relations R_1 and R_2, respectively. Putting $X = \{e_1, e_2, e_4\}$. Then

$$\underline{apr}^{(1,2)}_{(\cap,\cap)}(X) = \underline{apr}^{(1)}(X) \bigcap \underline{apr}^{(2)}(X) = \{e_1\},$$

$$\sim \overline{apr}^{(1,2)}_{(\cap,\cap)}(\sim X) = \sim \overline{apr}^{(1,2)}_{(\cap,\cap)}(\{e_3, e_5, e_6\}) = \{e_1, e_2\}.$$

That is, $\underline{apr}^{(1,2)}_{(\cap,\cap)}$ and $\overline{apr}^{(1,2)}_{(\cap,\cap)}$ do not satisfy (L7). Similarly,

$$\overline{apr}^{(1,2)}_{(\cap,\cap)}(X) = \overline{apr}^{(1)}(X) \bigcap \overline{apr}^{(2)}(X) = \{e_1, e_2, e_4, e_5, e_6\},$$

$$\sim \underline{apr}^{(1,2)}_{(\cap,\cap)}(\sim X) = \sim \underline{apr}^{(1,2)}_{(\cap,\cap)}(\{e_3, e_5, e_6\}) = U.$$

That is, $\underline{apr}^{(1,2)}_{(\cap,\cap)}$ and $\overline{apr}^{(1,2)}_{(\cap,\cap)}$ do not satisfy (H7).

Example 3.5. Let $apr^{(1)}$ and $apr^{(2)}$ be the rough approximations by equivalence relations R_1 and R_2 in Example 3.4, respectively. Putting $X = \{e_1\},\ Y = \{e_4\}$. Then

$$\overline{apr}^{(1,2)}_{(\cap,\cap)}(X \cup Y) = \{e_1, e_2, e_4, e_5, e_6\} \cap \{e_1, e_2, e_3, e_4\} = \{e_1, e_2, e_4\},$$

$$\overline{apr}^{(1,2)}_{(\cap,\cap)}(X) = \{e_1, e_2\} \cap \{e_1\} = \{e_1\},$$

$$\overline{apr}^{(1,2)}_{(\cap,\cap)}(Y) = \{e_4, e_5, e_6\} \cap \{e_2, e_3, e_4\} = \{e_4\}.$$

$$\overline{apr}^{(1,2)}_{(\cap,\cap)}(X \cup Y) \neq \overline{apr}^{(1,2)}_{(\cap,\cap)}(X) \cup \overline{apr}^{(1,2)}_{(\cap,\cap)}(Y).$$

That is, $\overline{apr}^{(1,2)}_{(\cap,\cap)}$ does notsatisfy (H4).

3.4. The fourth constructive method: (∪, ∪)-type

Definition 3.7. Let U be a non-empty set, $P(U)$ be the power set of U. If $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ are some mappings from $P(U)$ to $P(U)$, $i = 1, 2, \ldots, n$. Then define two new mappings from $P(U)$ to $P(U)$ as follows: for any $X \subseteq U$,

$$\underline{apr}^{(1..n)}_{(\cup,\cup)}(X) = \bigcup_{i=1}^{n} \underline{apr}^{(i)}(X);\ \overline{apr}^{(1..n)}_{(\cup,\cup)}(X) = \bigcup_{i=1}^{n} \overline{apr}^{(i)}(X).$$

We call $(\underline{apr}^{(1..n)}_{(\cup,\cup)}, \overline{apr}^{(1..n)}_{(\cup,\cup)})$ is $(\cup,\cup)-$type generated pair by operator pairs $(\underline{apr}^{(i)}, \overline{apr}^{(i)})$ $(i = 1, 2, \ldots, n)$, where $(1..n)$ means that from 1 to n. When n=2, denote $(1..n)$ by $(1, 2)$.

Theorem 3.4. *Let U be a non-empty set, $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ be mappings from P(U) to P(U), $i = 1, 2, \ldots, n$.*

(1) If for all X, Y ∈ P(U), $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (Lk), then $\underline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (Lk), where $k = 1, 2, 3, 5$.

(2) If for all X, Y ∈ P(U), $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (Hk), then $\overline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (Hk), where $k = 1, 2, 3, 5$.

(3) If for all X, Y ∈ P(U), $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L4), then $\underline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (L4′).

(4) If for all X, Y ∈ P(U), $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H4), then $\overline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (H4).

(5) If for all X, Y ∈ P(U), $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L6), then $\underline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (L6).

(6) If for all X, Y ∈ P(U), $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H6), then $\overline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (H6).

(7) If for all X, Y ∈ P(U), $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L6) and (L8′), then $\underline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (L8′).

(8) If for all X, Y ∈ P(U), $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H1), (H4), (H8′) and X satisfies (C2′), then $\overline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (H8′), where (C2′) means that

$$\overline{apr}^{(i)}(\overline{apr}^{(j)}(X)) \subseteq \overline{apr}^{(j)}(X)\ \ i, j = 1, 2, \ldots, n, i \neq j.$$

(9) If for all X, Y ∈ P(U), $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L1), (L5) and (L8), then $\underline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (L8).

(10) If for all X, Y ∈ P(U), $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H1), (H4), (H8) and X satisfies (C2), then $\overline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (H8), where (C2) means that

$$\overline{apr}^{(i)}(\overline{apr}^{(j)}(X)) = \overline{apr}^{(j)}(X)\ \ i, j = 1, 2, \ldots, n, i \neq j.$$

(11) If for all X ∈ P(U), $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L1), then $\underline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (L9′).

(12) If for all X ∈ P(U), $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H1), then $\overline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (H9′).

Proof. It is easy to verify that (L1), (L2), (L3), (L5), (H1), (H2), (H3) and (H5) hold for $\underline{apr}^{(1..n)}_{(\cup,\cap)}$ and $\overline{apr}^{(1..n)}_{(\cup,\cup)})$ when the corresponding conditions are satisfied. We prove other properties in follows.

(3): For all $X, Y \in P(U)$, if $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L4), from this and Definition 3.7 we have

$$\underline{apr}^{(1..n)}_{(\cup,\cup)}(X \cap Y) = \bigcup_{i=1}^{n} \underline{apr}^{(i)}(X \cap Y) = \bigcup_{i=1}^{n} (\underline{apr}^{(i)}(X) \cap \underline{apr}^{(i)}(Y))$$
$$\subseteq \left(\bigcup_{i=1}^{n} \underline{apr}^{(i)}(X)\right) \cap \left(\bigcup_{i=1}^{n} \underline{apr}^{(i)}(Y)\right) = \underline{apr}^{(1..n)}_{(\cup,\cup)}(X) \cap \underline{apr}^{(1..n)}_{(\cup,\cup)}(Y).$$

This means that $\underline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (L4′).

(4): For all $X, Y \in P(U)$, if $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H4), from this and Definition 3.7 we have

$$\overline{apr}^{(1..n)}_{(\cup,\cup)}(X \cup Y) = \bigcup_{i=1}^{n} \overline{apr}^{(i)}(X \cup Y) = \bigcup_{i=1}^{n} (\overline{apr}^{(i)}(X) \cup \overline{apr}^{(i)}(Y))$$
$$= \left(\bigcup_{i=1}^{n} \overline{apr}^{(i)}(X)\right) \cup \left(\bigcup_{i=1}^{n} \overline{apr}^{(i)}(Y)\right) = \overline{apr}^{(1..n)}_{(\cup,\cup)}(X) \cup \overline{apr}^{(1..n)}_{(\cup,\cup)}(Y).$$

This means that $\underline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (H4).

(5): For all $X, Y \in P(U)$, if $\underline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L6), from this and Definition 3.7 we have

$$\underline{apr}^{(1..n)}_{(\cup,\cup)}(X \cup Y) = \bigcup_{i=1}^{n} \underline{apr}^{(i)}(X \cup Y) \supseteq \bigcup_{i=1}^{n} (\underline{apr}^{(i)}(X) \cup \underline{apr}^{(i)}(Y))$$
$$= \left(\bigcup_{i=1}^{n} \underline{apr}^{(i)}(X)\right) \cup \left(\bigcup_{i=1}^{n} \underline{apr}^{(i)}(Y)\right) = \underline{apr}^{(1..n)}_{(\cup,\cup)}(X) \cup \underline{apr}^{(1..n)}_{(\cup,\cup)}(Y).$$

This means that $\underline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (L6).

(6): For all $X, Y \in P(U)$, if $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (H6), from this and Definition 3.7 we have

$$\overline{apr}^{(1..n)}_{(\cup,\cup)}(X \cap Y) = \bigcup_{i=1}^{n} \overline{apr}^{(i)}(X \cap Y) \subseteq \bigcup_{i=1}^{n} (\overline{apr}^{(i)}(X) \cap \overline{apr}^{(i)}(Y))$$

$$\subseteq \left(\bigcup_{i=1}^{n} \overline{apr}^{(i)}(X)\right) \cap \left(\bigcup_{i=1}^{n} \overline{apr}^{(i)}(Y)\right) = \overline{apr}^{(1..n)}_{(\cup,\cup)}(X) \cap \overline{apr}^{(1..n)}_{(\cup,\cup)}(Y).$$

That is, $\overline{apr}^{(1..n)}_{(\cup,\cup)}$) satisfies (H6).

(7) We only prove the case of $n = 2$. For all $X \in P(U)$, if $\underline{apr}^{(i)}$ $(i = 1, 2)$ satisfy (L6) and (L8′), by Definition 3.7 we have

$$\underline{apr}^{(1,2)}_{(\cup,\cup)}(\underline{apr}^{(1,2)}_{(\cup,\cup)}(X)) = \underline{apr}^{(1,2)}_{(\cup,\cup)}(\underline{apr}^{(1)}(X) \cup \underline{apr}^{(2)}(X))$$

(By Definition 3.7)

$$\supseteq \underline{apr}^{(1,2)}_{(\cup,\cup)}(\underline{apr}^{(1)}(X)) \cup \underline{apr}^{(1,2)}_{(\cup,\cup)}(\underline{apr}^{(2)}(X))$$

(By (L6) for $\underline{apr}^{(1,2)}_{(\cup,\cup)}$, applying (5))

$$= (\underline{apr}^{(1)}(\underline{apr}^{(1)}(X)) \cup \underline{apr}^{(2)}(\underline{apr}^{(1)}(X))) \cup (\underline{apr}^{(1)}(\underline{apr}^{(2)}(X)) \cup \underline{apr}^{(2)}(\underline{apr}^{(2)}(X)))$$

(By Definition 3.7)

$$\supseteq (\underline{apr}^{(1)}(\underline{apr}^{(1)}(X))) \cup (\underline{apr}^{(2)}(\underline{apr}^{(2)}(X)))$$

$$\supseteq \underline{apr}^{(1)}(X) \cup \underline{apr}^{(2)}(X)$$

(By (L8′) for $\underline{apr}^{(1)}$ and $\underline{apr}^{(2)}$)

$$= \underline{apr}^{(1,2)}_{(\cup,\cup)}(X).$$

This means that $\underline{apr}^{(1,2)}_{(\cup,\cup)}$ satisfies (L8′).

(8) For all $X \in P(U)$, if $\overline{apr}^{(i)}$ $(i = 1, 2)$ satisfy (H1), (H4) and (H8′), by Lemma 3.1 (4) we know that $\overline{apr}^{(i)}$ $(i = 1, 2)$ satisfy (H5). By Definition 3.7 we have

$$\overline{apr}^{(1,2)}_{(\cup,\cup)}(\overline{apr}^{(1,2)}_{(\cup,\cup)}(X)) = \overline{apr}^{(1,2)}_{(\cup,\cup)}(\overline{apr}^{(1)}(X) \cup \overline{apr}^{(2)}(X))$$

(By Definition 3.7)

$$= \overline{apr}^{(1,2)}_{(\cup,\cup)}(\overline{apr}^{(1)}(X)) \cup \overline{apr}^{(1,2)}_{(\cup,\cup)}(\overline{apr}^{(2)}(X))$$

(By (H4) for $\overline{apr}^{(1,2)}_{(\cup,\cup)}$, applying (4))

$$= (\overline{apr}^{(1)}(\overline{apr}^{(1)}(X)) \cup \overline{apr}^{(2)}(\overline{apr}^{(1)}(X))) \cup (\overline{apr}^{(1)}(\overline{apr}^{(2)}(X)) \cup \overline{apr}^{(2)}(\overline{apr}^{(2)}(X)))$$

(By Definition 3.7)

$$\subseteq \overline{apr}^{(1)}(X) \cup \overline{apr}^{(2)}(\overline{apr}^{(1)}(X)) \cup \overline{apr}^{(1)}(\overline{apr}^{(2)}(X)) \cup \overline{apr}^{(2)}(X)$$

(By (H8′) for $\overline{apr}^{(1)}$ and $\overline{apr}^{(2)}$)

$$= (\overline{apr}^{(1)}(X) \cup \overline{apr}^{(1)}(\overline{apr}^{(2)}(X))) \cup (\overline{apr}^{(2)}(X) \cup \overline{apr}^{(2)}(\overline{apr}^{(1)}(X)))$$

(By the associativity of $\cup$)

$$= \overline{apr}^{(1)}(\overline{apr}^{(2)}(X)) \cup \overline{apr}^{(2)}(\overline{apr}^{(1)}(X))$$

(By (H1) and (H5) for $\overline{apr}^{(1)}$ and $\overline{apr}^{(2)}$)

$$\subseteq \overline{apr}^{(2)}(X) \cup \overline{apr}^{(1)}(X)$$

(By (C2′) for X)

$$= \overline{apr}^{(1)}(X) \cup \overline{apr}^{(2)}(X)$$

$$= \overline{apr}^{(1,2)}_{(\cup,\cup)}(X).$$

This means that $\overline{apr}^{(1,2)}_{(\cup,\cup)}$ satisfies (H8′).

(9) By (7), $\underline{apr}^{(1,2)}_{(\cup,\cup)}(\underline{apr}^{(1,2)}_{(\cup,\cup)}(X)) \supseteq \underline{apr}^{(1,2)}_{(\cup,\cup)}(X)$. On the other hand, since $\underline{apr}^{(1)}$ and $\underline{apr}^{(2)}$ satisfy (L1), from (1), $\underline{apr}^{(1,2)}_{(\cup,\cup)}$ satisfies (L1), that is, $\underline{apr}^{(1,2)}_{(\cup,\cup)}(X) \subseteq X$. Applying (L5) and (1), $\underline{apr}^{(1,2)}_{(\cup,\cup)}(\underline{apr}^{(1,2)}_{(\cup,\cup)}(X)) \subseteq \underline{apr}^{(1,2)}_{(\cup,\cup)}(X)$. Therefore, $\underline{apr}^{(1..n)}_{(\cup,\cup)}$ satisfies (L8).

(10) It is similar to (8).

(11) It is follows from (1) and Lemma 3.1 (15).

(12) Applying (2) and Lemma 3.1 (16) we can get (12). □

Corollary 3.4. *Let U be a non-empty set,* $(\underline{apr}^{(i)}, \overline{apr}^{(i)})$ $(i = 1, 2, \ldots, n)$ *be Pawlak's rough approximation pairs on U. Then* $\underline{apr}^{(1..n)}_{(\cup,\cup)}$ *and* $\overline{apr}^{(1..n)}_{(\cup,\cup)}$ *satisfy (L1)–(L3), (L4′), (L5), (L6), (L8), (L9′), (H1)–(H6), (H8) and (H9′).*

Remark 3.9. In general, $\overline{apr}^{(1..n)}_{(\cup,\cup)}$) and $\underline{apr}^{(1..n)}_{(\cup,\cup)}$) do not satisfy the properties (L7) and (H7). Thus, the fourth generated rough approximation operator pairs are not dual. Moreover, $\underline{apr}^{(1..n)}_{(\cup,\cup)}$) does not satisfy the property (L4), see the following Example 3.6.

Example 3.6. Let $U = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and

$$U/R_1 = \{\{e_1, e_2\}, \{e_3\}, \{e_4, e_5, e_6\}\},$$
$$U/R_2 = \{\{e_1\}, \{e_2, e_3, e_4\}, \{e_5, e_6\}\}.$$

If $\underline{apr}^{(1)}$ and $\underline{apr}^{(2)}$ denote Pawlak's rough approximations by equivalence relations R_1 and R_2, respectively. Putting $X = \{e_2, e_3, e_4, e_5, e_6\}$, $Y = \{e_1, e_2, e_3, e_5, e_6\}$. Then

$$\underline{apr}^{(1,2)}_{(\cup,\cup)}(X \cap Y) = \{e_3\} \cup \{e_5, e_6\} = \{e_3, e_5, e_6\},$$

$$\underline{apr}^{(1,2)}_{(\cup,\cup)}(X) = \{e_2, e_3, e_4, e_5, e_6\}, \ \underline{apr}^{(1,2)}_{(\cup,\cup)}(Y) = \{e_1, e_2, e_3, e_5, e_6\}.$$

$$\underline{apr}^{(1,2)}_{(\cup,\cup)}(X \cap Y) \neq \underline{apr}^{(1,2)}_{(\cup,\cup)}(X) \cap \underline{apr}^{(1,2)}_{(\cup,\cup)}(Y).$$

That is, $\underline{apr}^{(1,2)}_{(\cup,\cup)}$ does notsatisfy (L4).

3.5. The relationships among four kinds of generated rough approximations

By Definitions 3.1, 3.4, 3.6, 3.7, Theorems 3.1, 3.2, 3.3 and 3.4, we can easy to prove the following proposition.

Proposition 3.6. *Let U be a non-empty set,* $\underline{apr}^{(i)}$ *and* $\overline{apr}^{(i)}$ *be mappings from* $P(U)$ *to* $P(U)$, $i = 1, 2, \ldots, n$. *If for all* $X \in P(U)$, $\underline{apr}^{(i)}$ *and* $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ *satisfy (L1) and (H1), then* $(\forall X \in P(U), \forall i, j \le n)$

(1) $\underline{apr}^{(1..n)}_{(\cap,\cap)}(X) = \underline{apr}^{(1..n)}_{(\cap,\cup)}(X)$, $\underline{apr}^{(1..n)}_{(\cup,\cap)}(X) = \underline{apr}^{(1..n)}_{(\cup,\cup)}(X)$;

(2) $\overline{apr}^{(1..n)}_{(\cap,\cap)}(X) = \overline{apr}^{(1..n)}_{(\cup,\cap)}(X)$, $\overline{apr}^{(1..n)}_{(\cap,\cup)}(X) = \overline{apr}^{(1..n)}_{(\cup,\cup)}(X)$;

(3) $\underline{apr}^{(1..n)}_{(\cap,\cap)}(X) \subseteq \underline{apr}^{(i)}(X) \subseteq \underline{apr}^{(1..n)}_{(\cup,\cap)}(X) \subseteq X \subseteq \overline{apr}^{(1..n)}_{(\cup,\cap)}(X) \subseteq \overline{apr}^{(i)}(X) \subseteq \overline{apr}^{(1..n)}_{(\cap,\cup)}(X)$;

(4) $\underline{apr}^{(1..n)}_{(\cap,\cup)}(X) \subseteq \underline{apr}^{(j)}(X) \subseteq \underline{apr}^{(1..n)}_{(\cup,\cup)}(X) \subseteq X \subseteq \overline{apr}^{(1..n)}_{(\cap,\cap)}(X) \subseteq \overline{apr}^{(j)}(X) \subseteq \overline{apr}^{(1..n)}_{(\cup,\cup)}(X)$.

Proof. We only prove (1) and (3).

(1) By Definitions 3.4 and 3.5, we have

$$\underline{apr}^{(1..n)}_{(\cap,\cap)}(X) = \bigcap_{i=1}^{n} \underline{apr}^{(i)}(X) = \underline{apr}^{(1..n)}_{(\cap,\cup)}(X).$$

By Definitions 3.1 and 3.7, we have

$$\underline{apr}^{(1..n)}_{(\cup,\cap)}(X) = \bigcup_{i=1}^{n} \underline{apr}^{(i)}(X) = \underline{apr}^{(1..n)}_{(\cup,\cup)}(X).$$

(3) Since $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ $(i = 1, 2, \ldots, n)$ satisfy (L1) and (H1), applying Theorem 3.1 (1) and (2), $\underline{apr}^{(1..n)}_{(\cap,\cap)}$ and $\overline{apr}^{(1..n)}_{(\cup,\cap)}$ satisfy (L1) and (H1), that is, for any $X \in P(U)$,

$$\underline{apr}^{(1..n)}_{(\cup,\cap)}(X) \subseteq X \subseteq \overline{apr}^{(1..n)}_{(\cup,\cap)}(X).$$

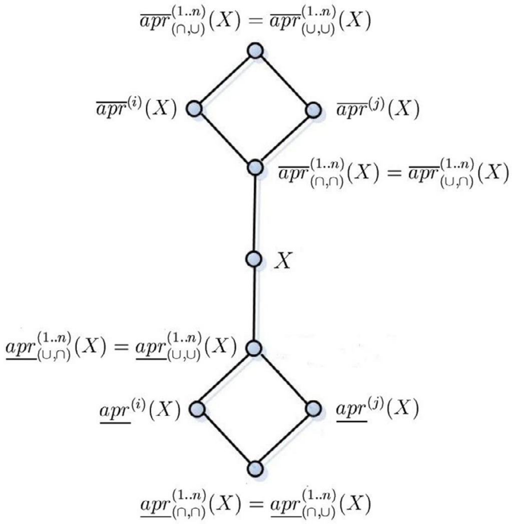

Fig. 1. The relationships among four kinds of generated rough approximations.

And, by Definitions 3.6 and 3.1, we have ($\forall X \in P(U)$ and $\forall i \le n$)

$$\underline{apr}^{(1..n)}_{(\cap,\cap)}(X) = \bigcap_{i=1}^{n} \underline{apr}^{(i)}(X) \subseteq \underline{apr}^{(i)}(X) \subseteq \bigcup_{i=1}^{n} \underline{apr}^{(i)}(X) = \underline{apr}^{(1..n)}_{(\cup,\cap)}(X).$$

Similarly, by Definitions 3.1 and 3.4, we have ($\forall X \in P(U)$ and $\forall i \le n$)

$$\overline{apr}^{(1..n)}_{(\cup,\cap)}(X) = \bigcap_{i=1}^{n} \overline{apr}^{(i)}(X) \subseteq \overline{apr}^{(i)}(X) \subseteq \bigcup_{i=1}^{n} \overline{apr}^{(i)}(X) = \overline{apr}^{(1..n)}_{(\cap,\cup)}(X).$$

This means that (3) holds. □

The above results show that the four kinds of lower and upper approximations equipped with the inclusion relation $\subseteq$ can construct a lattice. This fact can be described by Fig. 1, where $i \ne j$, each node denotes an approximation or a concept, and each diagonal line connects two approximations, the lower node is a subset of the upper node.

4. Some applications to multigranulation rough sets

4.1. Non-dual multigranulation rough sets

The main difference between single-granulation rough sets and multigranulation ones lies in that the approximations of a target concept in multigranulation rough sets are constructed by using multi-distinct sets of information granules. But, until now, multigranulation rough approximation pairs are dual, they are established using the above first and second constructive methods. Now, we introduce two new types of multigranulation rough sets, they are non-dual.

Definition 4.1. Let $K = (U, \mathbf{R})$ be a knowledge base, where $\mathbf{R}$ is a family of equivalence relations on the universe U. Let $A_1, A_2, \ldots, A_m \in \mathbf{R}$, where m is a natural number. For any $X \subseteq U$, its intersection-type lower and intersection-type upper approximations with respect to $A_1, A_2, \ldots, A_n$ are respectively defined as follows.

$$\underline{\sum_{i=1}^{m} A_i^{(\cap)}}(X) = \{x \in U | \forall j \in \{1, 2, \ldots, m\},\ [x]_{A_j} \subseteq X\};$$

$$\overline{\sum_{i=1}^{m} A_i^{(\cap)}}(X) = \{x \in U | \forall j \in \{1, 2, \ldots, m\},\ [x]_{A_j} \cap X \ne \emptyset\}.$$

$(\underline{\sum_{i=1}^{m} A_i^{(\cap)}}(X), \overline{\sum_{i=1}^{m} A_i^{(\cap)}}(X))$ is called the intersection-type multigranulation rough set of X.

Remark 4.1. The intersection-type multigranulation rough set can be regarded that it uses the decision making method: select optimistic decision in upper direction and select pessimistic decision in lower direction.

Example 4.1. Let $K = (U, \mathbf{R})$ be a knowledge base, $X \subseteq U$, $R_1, R_2 \in \mathbf{R}$, where $U = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, $X = \{e_1, e_2, e_6, e_8\}$, and

$U/R_1 = \{\{e_1, e_7\}, \{e_2, e_3, e_4, e_5, e_6\}, \{e_8\}\}$,

$U/R_2 = \{\{e_1, e_2\}, \{e_3, e_4, e_5\}, \{e_6, e_7, e_8\}\}$.

By computing we have

$\underline{(R_1 + R_2)^{(\cap)}}(X) = \{x \in U \mid [x]_{R_1} \subseteq X,\ [x]_{R_2} \subseteq X\} = \emptyset$,

$\overline{(R_1 + R_2)^{(\cap)}}(X) = \{x \in U \mid [x]_{R_1} \cap X \ne \emptyset,$
$[x]_{R_2} \cap X \ne \emptyset\} = \{e_1, e_2, e_6, e_7, e_8\}$.

By Example 1 in [22] we know that

$\underline{(R_1 + R_2)^{(\cap)}}(X) \ne \underline{(R_1 + R_2)^{(O)}}(X)$,

$\overline{(R_1 + R_2)^{(\cap)}}(X) \ne \overline{(R_1 + R_2)^{(O)}}(X)$.

By Definitions 4.1 and 3.6 we have

Proposition 4.1. *Let $K = (U, \mathbf{R})$ be a knowledge base, where $\mathbf{R}$ is a family of equivalence relations on the universe U. Let $A_1, A_2, \ldots, A_m \in \mathbf{R}$, $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ be the lower and upper approximation operators induced by equivalence relation A_i ($i = 1, 2, \ldots, m$), where m is a natural number. Then, for any $X \subseteq U$,*

$$\underline{\sum_{i=1}^{m} A_i^{(\cap)}}(X) = \underline{apr}^{(1..m)}_{(\cap,\cap)}(X);$$

$$\overline{\sum_{i=1}^{m} A_i^{(\cap)}}(X) = \overline{apr}^{(1..m)}_{(\cap,\cap)}(X).$$

By Proposition 4.1 and Corollary 3.3 we have

Proposition 4.2. *The intersection-type multigranulation rough lower and upper approximation operators $\underline{\sum_{i=1}^{m} A_i^{(\cap)}}$ and $\overline{\sum_{i=1}^{m} A_i^{(\cap)}}$ satisfy (L1)–(L6), (L8), (L9′) (H1)–(H3), (H4′), (H5), (H6), (H8′) and (H9′).*

Definition 4.2. Let $K = (U, \mathbf{R})$ be a knowledge base, where $\mathbf{R}$ is a family of equivalence relations on the universe U. Let $A_1, A_2, \ldots, A_m \in \mathbf{R}$, where m is a natural number. For any $X \subseteq U$, its union-type lower and union-type upper approximations with respect to $A_1, A_2, \ldots, A_n$ are respectively defined as follows.

$$\underline{\sum_{i=1}^{m} A_i^{(\cup)}}(X) = \{x \in U | \exists j \in \{1, 2, \ldots, m\},\ [x]_{A_j} \subseteq X\};$$

$$\overline{\sum_{i=1}^{m} A_i^{(\cup)}}(X) = \{x \in U | \exists j \in \{1, 2, \ldots, m\},\ [x]_{A_j} \cap X \ne \emptyset\}.$$

$(\underline{\sum_{i=1}^{m} A_i^{(\cup)}}(X), \overline{\sum_{i=1}^{m} A_i^{(\cup)}}(X))$ is called the union-type multigranulation rough sets of X.

Remark 4.2. The union-type multigranulation rough set can be regarded that it uses the decision making method: select optimistic decision in lower direction and select pessimistic decision in upper direction.

Example 4.2. Let $K = (U, \mathbf{R})$ be a knowledge base, $X \subseteq U$, $R_1, R_2 \in \mathbf{R}$, where $U = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, $X = \{e_1, e_2, e_6, e_8\}$, and

$U/R_1 = \{\{e_1, e_7\}, \{e_2, e_3, e_4, e_5, e_6\}, \{e_8\}\}$,

$U/R_2 = \{\{e_1, e_2\}, \{e_3, e_4, e_5\}, \{e_6, e_7, e_8\}\}$.

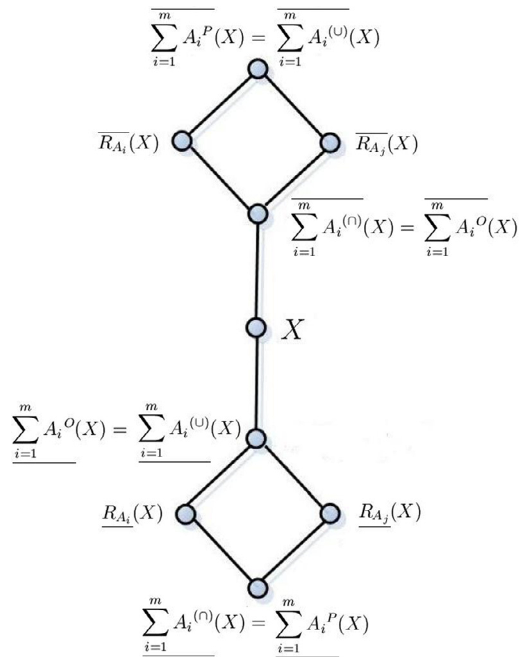

Fig. 2. The relationships among four kinds of multigranulation rough approximations.

By computing we have

$$\underline{(R_1+R_2)^{(\cup)}}(X) = \{x \in U \mid [x]_{R_1} \subseteq X \text{ or } [x]_{R_2} \subseteq X\} = \{e_1, e_2, e_8\},$$

$$\overline{(R_1+R_2)^{(\cup)}}(X) = \{x \in U \mid [x]_{R_1} \cap X \neq \emptyset \text{ or } [x]_{R_2} \cap X \neq \emptyset\} = U.$$

By Definition 2.2 we know that

$$\underline{(R_1+R_2)^{(\cup)}}(X) \neq \underline{(R_1+R_2)^{(P)}}(X),$$

$$\overline{(R_1+R_2)^{(\cup)}}(X) \neq \overline{(R_1+R_2)^{(P)}}(X).$$

By Definitions 4.2 and 3.7 we have

Proposition 4.3. *Let $K = (U, \mathbf{R})$ be a knowledge base, where $\mathbf{R}$ is a family of equivalence relations on the universe U. Let $A_1, A_2, \ldots, A_m \in \mathbf{R}$, $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ be the lower and upper approximation operators induced by equivalence relation A_i ($i = 1, 2, \ldots, m$), where m is a natural number. Then, for any $X \subseteq U$,*

$$\underline{\sum_{i=1}^{m}} A_i^{(\cup)}(X) = \underline{apr}^{(1..m)}_{(\cup,\cup)}(X);$$

$$\overline{\sum_{i=1}^{m}} A_i^{(\cup)}(X) = \overline{apr}^{(1..m)}_{(\cup,\cup)}(X).$$

By Proposition 4.3 and Corollary 3.4 we have

Proposition 4.4. *The union-type multigranulation rough lower and upper approximation operators $\underline{\sum_{i=1}^{m}} A_i^{(\cup)}$ and $\overline{\sum_{i=1}^{m}} A_i^{(\cup)}$ satisfy satisfy (L1)–(L3), (L4′), (L5), (L6), (L8), (L9′), (H1)–(H6), (H8) and (H9′).*

Applying Proposition 3.6, we can obtain the relationships among four kinds of multigranulation rough approximations as described in Fig. 2, where $\underline{R_{A_i}}(X)$ and $\overline{R_{A_i}}(X)$ are Pawlak's rough approximation operators.

4.2. Hybrid multigranulation rough sets

Recently, many multigranulation rough set models are introduced, include multigranulation covering rough sets (see [9–11]), multigranulation rough sets based on binary relations (see [1]), intuitionistic fuzzy multigranulation rough sets (see [5]), multigranulation rough sets based on ordered information systems, etc. But, these models are all based on the same kind granulations, that is, the granulations are either generated by the equivalence relations, or are generated by the coverings, or are generated by the fuzzy relations. In fact, data usually exists with hybrid formats in real-world applications (see [32,33]), hence, we need to study general multigranulation rough set models whose granulations have different types (for example, some ones induced by equivalence relations, some ones induced by coverings, other ones induced by fuzzy binary relations). We call them hybrid multigranulation rough set models. In this section, we only discuss two kinds of hybrid multigranulation rough sets.

Definition 4.3. Let $R_1, R_2, \ldots, R_m$ be binary relations on the universe U, where m is a natural number. For any $X \subseteq U$, its optimistic lower and upper approximations with respect to $R_1, R_2, \ldots, R_m$ are respectively defined as follows.

$$\underline{\sum_{i=1}^{m}} R_i^{O}(X) = \{x \in U \mid \exists j \in \{1, 2, \ldots, m\},\ r_{R_j}(x) \subseteq X\};$$

$$\overline{\sum_{i=1}^{m}} R_i^{O}(X) = \{x \in U \mid \forall j \in \{1, 2, \ldots, m\},\ r_{R_j}(x) \cap X \neq \emptyset\},$$

where $r_{R_j}(x) = \{y \mid y \in U,\ xRy\}$. $(\underline{\sum_{i=1}^{m}} R_i^{O}(X), \overline{\sum_{i=1}^{m}} R_i^{O}(X))$ is called the optimistic multigranulation rough sets of X.

Remark 4.3. In fact, the above multigranulation rough sets are hybrid, because some ones in $\{R_i \mid i = 1, 2, \ldots, m\}$ may be equivalence relations, some ones in $\{R_i \mid i = 1, 2, \ldots, m\}$ may be reflexive relations, and others may be general binary relations.

Example 4.3. Let R_1, R_2 be two binary relations on $U = \{e_1, e_2, e_3\}$, where

$$R_1 = \{(e_1, e_2), (e_2, e_1), (e_2, e_3), (e_3, e_3)\},$$

$$R_2 = \{(e_1, e_1), (e_2, e_2), (e_3, e_3), (e_2, e_3), (e_3, e_2)\}.$$

It is easy to verify that R_2 is an equivalence relation and R_1 is a general binary relation on U. Then

$$r_{R_1}(e_1) = \{e_2\},\ r_{R_1}(e_2) = \{e_1, e_3\},\ r_{R_1}(e_3) = \{e_3\},$$

$$r_{R_2}(e_1) = \{e_1\},\ r_{R_2}(e_2) = r_{R_2}(e_3) = \{e_2, e_3\}.$$

Putting $X = \{e_1, e_2\}$, by computing we have

$$\underline{(R_1+R_2)^{(O)}}(X) = \{x \in U \mid r_{R_1}(x) \subseteq X \text{ or } r_{R_2}(x) \subseteq X\} = \{e_1\},$$

$$\overline{(R_1+R_2)^{(O)}}(X) = \{x \in U \mid r_{R_1}(x) \cap X \neq \emptyset \text{ and } r_{R_2}(x) \cap X \neq \emptyset\} = \{e_1, e_2\}.$$

By Definitions 4.3 and 3.1 we have

Proposition 4.5. *Let $R_1, R_2, \ldots, R_m$ be binary relations on the universe U, and $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ be the lower and upper approximation operators induced by binary relation R_i ($i = 1, 2, \ldots, m$), where m is a natural number. Then, for any $X \subseteq U$,*

$$\underline{\sum_{i=1}^{m}} R_i^{O}(X) = \underline{apr}^{(1..m)}_{(\cup,\cap)}(X);$$

$$\overline{\sum_{i=1}^{m}} R_i^{O}(X) = \overline{apr}^{(1..m)}_{(\cup,\cap)}(X).$$

Remark 4.4. (1) For the lower and upper approximation operators induced by binary relations, there are two different definitions, please see [12,37] and [1]. In this section, we use the definition in [12,37].

That is, $\underline{R}(X)=\{x \mid r_{R_j}(x)\subseteq X\}$, $\overline{R}(X)=\{x \mid r_{R_j}(x)\cap X\neq\emptyset\}$, for arbitrary relation R. (2) Definition 4.3 is different from Definition 4.1 in [1], because the definition of lower and upper approximation operators induced by binary relations is another one (see Definition 2.2 in [1]).

By Proposition 1 in [12], using Proposition 4.5 and Theorem 3.1 we have

Proposition 4.6. *The optimistic lower and upper approximation operators with respect to binary relations $R_1, R_2, \ldots, R_m$ satisfy (L2), (L3), (L4′), (H4′), (L5), (H5), (L6), (H6), (L7) and (H7).*

Similarly, we can also give the definitions of pessimistic, intersection-type and union-type lower and upper approximation operators based on general binary relations, they are omitted here.

Definition 4.4. Let $R_1, R_2, \ldots, R_m$ be equivalence relations on the universe U, and $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ be coverings of the universe U, where m and n are natural number. For any $X\subseteq U$, its type-1 optimistic lower and upper approximations with respect to $R_1, R_2, \ldots, R_m$ and $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ are respectively defined as follows.

$$\underline{FR_{\sum_{i,j=1}^{m,n} R_i\mathcal{C}_j}}(X)=\{x\in U|\exists i\in\{1,2,\ldots,m\}\,\textit{or}\,\exists j\in\{1,2,\ldots,n\}$$

$$\textit{such that}\ [x]_{R_i}\subseteq X\ \textit{or}\ \cap md(\mathcal{C}_j,x)\subseteq X\},$$

$$\overline{FR_{\sum_{i,j=1}^{m,n} R_i\mathcal{C}_j}}(X)=\{x\in U|\forall i\in\{1,2,\ldots,m\}\,\textit{and}\,\forall j\in\{1,2,\ldots,n\}$$

$$\textit{such that}\ [x]_{R_i}\cap X\neq\emptyset\ \textit{and}\ \cap md(\mathcal{C}_j,x)\cap X\neq\emptyset\},$$

where $md(\mathcal{C}_j, x)$ is the minimal descriptors of x with respect to $\mathcal{C}_j$.

Remark 4.5. (1) The above definition can be regarded as a common generalization of Definition 2.1 and Definition 2.7 in [11], this model is a hybrid multigranulation rough set model with respect to equivalence relations and coverings. (2) For the definition of the minimal descriptors, please see [38,46].

Example 4.4. Let R_1 be an equivalence relation and $\mathcal{C}_1$ a covering on $U=\{e_1,e_2,e_3,e_4\}$, where

$$U/R_1=\{\{e_1\},\{e_2,e_3\},\{e_4\}\},$$

$$\mathcal{C}_1=\{\{e_1,e_2\},\{e_2,e_3,e_4\},\{e_3,e_4\}\}.$$

Then

$$md(\mathcal{C}_1,e_1)=\{\{e_1,e_2\}\},\ md(\mathcal{C}_1,e_2)=\{\{e_1,e_2\},\{e_2,e_3,e_4\}\},$$

$$md(\mathcal{C}_1,e_3)=md(\mathcal{C}_1,e_4)=\{\{e_3,e_4\}\}.$$

Putting $X=\{e_1,e_3\}$, by computing we have

$$\underline{FR_{R_1+\mathcal{C}_1}}(X)=\{x\in U\mid [x]_{R_1}\subseteq X\ \textit{or}\ \cap md(\mathcal{C}_1,x)\subseteq X\}=\{e_1\},$$

$$\overline{FR_{R_1+\mathcal{C}_1}}(X)=\{x\in U\mid [x]_{R_1}\cap X\neq\emptyset\ \textit{and}\ \cap md(\mathcal{C}_1,x)\cap X\neq\emptyset\}=\{e_1,e_3\}.$$

By Definitions 4.4 and 3.1 we have

Proposition 4.7. *Let $R_1, R_2, \ldots, R_m$ be equivalence relations on the universe U, and $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_n$ be coverings of the universe U. $\underline{apr}^{(i)}$ and $\overline{apr}^{(i)}$ denote Pawlak's lower and upper approximation operators induced by R_i $(i=1,2,\ldots,m)$, $\underline{apr}^{(m+j)}$ and $\overline{apr}^{(m+j)}$ denote the lower and upper approximation operators induced by $md(\mathcal{C}_j,x)$ $(j=1,2,\ldots,n)$, that is,*

$$\underline{apr}^{(m+j)}(X)=\{x\in U\mid \cap md(\mathcal{C}_j,x)\subseteq X\};$$

$$\overline{apr}^{(m+j)}(X)=\{x\in U\mid \cap md(\mathcal{C}_j,x)\cap X\neq\emptyset\}.$$

Then, for any $X\subseteq U$,

$$\underline{FR_{\sum_{i,j=1}^{m,n} R_i\mathcal{C}_j}}(X)=\underline{apr}^{(1..m+n)}_{(\cup,\cap)}(X);$$

$$\overline{FR_{\sum_{i,j=1}^{m,n} R_i\mathcal{C}_j}}(X)=\overline{apr}^{(1..m+n)}_{(\cup,\cap)}(X).$$

Applying Theorem 3.1 and Proposition 4.6 we can get basic properties of type-1 optimistic lower and upper approximation operators with respect to equivalence relations and coverings. Moreover, similar to Definition 4.4, we can introduce type-2, type-3 and type-4 optimistic lower and upper approximation operators with respect to equivalence relations and coverings; furthermore, using Definitions 2.2, 4.1 and 4.2, we can define many types hybrid multigranulation rough set models. These topics are omitted here.

Remark 4.6. The models of multigranulation rough sets in this paper can be applied to multi attribute group decision making problems. The main idea is as following: firstly, from some cases data sets, we can constitute a multigranulation approximate space; next, calculate the importance degree of each attribute by using the models of multigranulation rough sets; Finally, the weight of each attribute can be obtained by the above importance degree of each attribute, and then it can be applied to the new object set using the traditional multiple attribute decision making method. Moreover, this method can be compared with the previous methods such as the methods introduced in the literature [30]. For these questions, we will discuss in another paper.

5. Summary

From the view point of the union and intersection operations of rough approximation pairs, we investigated the general generation rules of approximation operators. We established four kinds of constructive methods of rough approximation operators from existing rough sets, and showed that many rough sets (include all of multigranulation rough sets) are essentially direct applications of these constructive methods. From these studies, we know that some results in the literatures can be regarded as the corollaries of the main conclusions of this paper.

The constructive methods of rough approximation operators proposed in this paper are not only used in multi-granularity rough sets, and can be used in other ways. For example, from existing covering rough approximation operators, applying the constructive methods, we can get new covering rough set models. This topic will be discussed in forthcoming paper.

Acknowledgments

This work is supported by National Natural Science Foundation of China (Nos. 61175044, 61473239), China National Natural Science Foundation of Youth Science Foundation (No. 61305052) and Innovation Program of Shanghai Municipal Education Commission (No. 13ZZ122).

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