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Three-way attribute reducts ☆

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ABSTRACT

Three-way decisions are a fundamental methodology with extensive applications, while attribute reducts play an important role in data analyses. The combination of both topics has theoretical significance and applicable prospects, but rarely gains direct research at present. In this paper, three-way decisions are introduced into attribute reducts and thus three-way attribute reducts are systematically investigated. Firstly, classical qualitative reducts are reviewed by the dependency degree. Then, the dependency degree implements approximation analyses to be improved to a controllable measure: the relative dependency degree, which is monotonic to relatively measure the attribute dependency. Given an approximate bar, the relative dependency degree defines the applicable quantitative reducts, which approach, expand, and weaken the classical qualitative reducts. This type of quantitative reducts is actually the positive quantitative reducts for three-way reducts. Thus, three-way quantitative reducts are established by the relative dependency degree and dual thresholds. The positive, boundary, and negative quantitative reducts divide the power set of the condition attribute set and thus gain acceptance, noncommitment, and rejection decisions, respectively; they exhibit the potential derivation from the higher level to the lower level. Furthermore, three-way qualitative reducts are established by degeneration to implement three-way decisions, and three-way quantitative and qualitative reducts exhibit the approximation, expansion, and strength; by virtue of superiority analyses, three-way reducts improve the latent two-way reducts with only acceptance and rejection decisions. Finally, three-way reducts are practically illustrated by observing an example of decision tables. By developing the relative dependency degree with controllability, three-way reducts implement both a quantitative generalization for qualitative reducts and a structural completion for attribute reducts. The relevant study provides a new insight into both three-way decisions and attribute reducts.

1. Introduction

Three-way decisions are an extension of the commonly used binary decisions with an added third option. They play a key role in everyday decision-making and thus establish a novel and important theory in knowledge discovery, manage-

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ment, and utilization [47]. The idea of three-way decisions is first introduced in the rough set theory by Yao, especially in the decision-theoretic rough sets [4,16,17,46,49]. At present, the theory of three-way decisions moves to a more general trisecting-and-acting framework based on a generic tri-partition of the universe that can assume different interpretations and requires different decision strategies [8,9,20,29–31,33,42,54,55,64]. Herein, we return to the rough set theory and focus on its basis: data tables with objects and attributes. According to a condition classification, the positive, negative, and boundary regions of a concept divide the universe of objects, and they gain three-way decisions [46,47,49]. According to a type of attribute reducts, the set of condition attributes is divided into three-way classes, and they have similar semantics of three-way decisions but slightly different names [6,25,39,53]. Herein, a group of relevant names proposed by Yao and Zhang [53] is adopted. Let RED denote the set of all reducts and let $R \in \mathrm{RED}$. Thus, the set C of condition attributes exhibits a three-way classification:

$$\begin{aligned} \mathrm{CORE} &= \cap\mathrm{RED}, \\ \mathrm{MARGINAL} &= \cup\mathrm{RED} - \cap\mathrm{RED}, \\ \mathrm{NONUSEFUL} &= C - \cup\mathrm{RED}. \end{aligned} \tag{1}$$

The three-way classification provides a three-level characterization of attributes, from the important to the marginal important and to the unimportant, and it can represent a pair of lower and upper bounds for any reduct R:

$$\mathrm{CORE} \subseteq R \subseteq \mathrm{USEFUL} = \cup\mathrm{RED} = \mathrm{CORE} \cup \mathrm{MARGINAL}. \tag{2}$$

However, in terms of the basic space structure, attribute reducts actually exist in the power set of the condition attribute set, i.e., 2^C. There are rarely direct reports on the division of attribute reducts in 2^C, especially from the fundamental viewpoint of three-way decisions. The relevant research can probe systematic reduct structures to establish practical reduction decisions, so it has theoretical significance and applicable prospects. It becomes our main topic in this paper to implement the novel combination of three-way decisions and attribute reducts.

As a subset of C, an attribute reduct satisfies two conditions of joint sufficiency and individual necessity, so it can vastly gain the same ability with respect to C. Attribute reducts play an important role in data table analyses and thus have extensive studies [3,10,14,19,21,24,32,41,56]. The classical reducts for decision tables, proposed by Pawlak [28], are equivalently established on the positive region and dependency degree to become a type of qualitative reducts. The qualitative reducts carry absoluteness to lack a quantitative mechanism, while quantitative reducts implement applicable improvements. For a main approach, quantitative models are constructed to produce quantitative regions, and then quantitative regions produce quantitative reducts. There are multiple quantitative models (especially the probabilistic rough sets) [5,7,36,44,50,52] and corresponding quantitative reducts [12,13,23,45,60,63]. In particular, Chen et al. [2] establish the three-way decision reduction in neighborhood systems, where the positive, boundary, and negative quantitative regions are parallelly utilized. To simulate the classical reduct pattern, the model way mainly quantifies the antecedent region and thus has the indirect quantification function for attribute reducts. In contrast, uncertainty measures (including informational measures) can implement the direct quantification [11,15,18,38,48,61,62] and thus are deeply used in attribute reducts [2,22,37]. As a result, several measures quantify attribute reduction to produce the approximate reducts with tolerance conditions, and Slezak [34,35], Wroblewski [40], Zhang and Miao [59] utilize the informational entropy, quality measure, and double-quantitative measure, respectively. In this paper, we aim to directly quantify the final reduction action. As a key process, the underlying reduct target needs to be quantified to produce the reduct approximation, and a superior measure with reasoning and controllability is especially required. Note that the dependency degree γ of attributes serves as an initial and fundamental measure in knowledge base [1,26,28,45]. In fact, γ closely adheres to the positive region and qualitative reduct, so it has the absolute semantics of attribute dependency to underlie rough reasoning. We will resort to the dependency degree and its quantitative mechanism to develop an improved measure: the relative dependency degree $\gamma_{\mathrm{relative}}$, which has the approximate function and relative semantics of attribute dependency in interval $[0, 1]$. The monotonic and controllable measure $\gamma_{\mathrm{relative}}$ and its quantitative bar α further produce the quantitative target and reduct. The $\gamma_{\mathrm{relative}}$–$\alpha$ quantitative reducts approach, expand, and weaken the classical qualitative reducts and thus exhibit application significance of optimization and generalization.

Usually, only attribute reducts themselves are concerned for applications, in both the qualitative and quantitative patterns. This strategy implies two-way reducts and decisions for 2^C. That is, RED gains acceptance for the reduction action, and surplus $2^C - \mathrm{RED}$ becomes the other part which is negative to be not considered. By following the idea of three-way decisions, a third class and decision are worth adding to represent transitional uncertainty between the positive and negative certainty. In RED, usual reducts correspond to the positive choice and thus can be named as positive reducts. In contrast, $2^C - \mathrm{RED}$, which currently has no in-depth descriptions, can be further divided into two parts with respect to reduction potentiality and impossibility. In this paper, the boundary reducts – a new and variational type of attribute reducts – are proposed in the measure style to describe uncertain reducts in $2^C - \mathrm{RED}$, and the surplus subsets in $2^C - \mathrm{RED}$ generally constitute the negative reducts; with the addition of the positive reducts related to RED, three-way reducts are thereby established by the structural improvements and connotative development. In the qualitative pattern, that monotonic γ reaches maximum γ_{max} based on C produces the positive reducts; interval value $\gamma \in (0, \gamma_{max})$ contains the possibility of reduct targets, and relevant reduct conditions lead to the boundary reducts; attribute subsets which are neither the positive nor boundary reducts have complete impossibility and thus become the negative reducts. Similarly, we will construct three-way reducts in the quantitative pattern, by using the monotonic measure $\gamma_{\text{relative}} \in [0, 1]$ and dual bars α, β. Concretely, the above γ_{relative}–α quantitative reducts utilize reduct target $\gamma_{\text{relative}} \geq \alpha$ to become the positive reducts; reduct target $\gamma_{\text{relative}} \in (\beta, \alpha)$ produces the boundary reducts; the surplus subsets in 2^C become the negative reducts. As a result, three-way reducts are established in both the qualitative and quantitative patterns to divide 2^C, and they can be unified by γ_{relative} which effectively replaces γ. In fact, the positive reducts correspond to the usual reducts which are currently used, the boundary reducts extract the potential reducts which have the theoretical possibility for practical applications, while the negative reducts collect the surplus subsets which need only negation. For the reduction action, three-way reducts correspond to acceptance, noncommitment, and rejection of three-way decisions, respectively. For three-way reducts, after their establishment at the 2^C level, we will depend on the C level to systematically discuss their internal derivation, as well as the expansion and strength between the qualitative and quantitative patterns. Moreover, the superiority of three-way reducts is revealed by the comparison to the two-way reducts.

Finally, the motivation of three-way attribute reducts is summarized from the applied view. In the structural mechanism, the usual reducts pursue the systematic accuracy to induce the two-way reducts. However, the real-world environment inevitably contains data noise, so the fault tolerance needs to be introduced for the systematic structure. In fact, real-life examples with data noise are easily found and their accompanying data tables hold practical deviation, so the two-way reducts adopt the two-way decisions to may lead to systematic overfitting. Thus, the three-way reducts adhere to the three-way decisions to gain the structure improvement and application robustness, and the quantitative and qualitative patterns also provide more application spaces. In summary, three-way reducts aim to offer a better structural and systematic mechanism to apply to noise scenarios, and supporting examples extensively exist in practice.

Against the above background and thought, this paper introduces three-way decisions into attribute reducts to make three basic contributions, i.e., the improvement of dependency degree, generalization of quantitative reducts, and construction of three-way reducts. The relevant contents are organized as follows. Section 2 reviews the classical qualitative reducts by the dependency degree γ. Section 3 develops the relative dependency degree γ_{relative} and constructs the γ_{relative}-α quantitative reducts, i.e., the positive quantitative reducts. Section 4 establishes three-way reducts (including the quantitative and qualitative patterns) and analyzes their internal and mutual relationships; moreover, we reveal the superiority of three-way reducts in contrast to two-way reducts. Section 5 illustrates three-way reducts by observing an example of decision tables. Finally, Section 6 concludes this paper.

2. Qualitative attribute reducts

In this section, we review the qualitative attribute reducts, which are the classical notions [27,28].

The rough set theory focuses on data represented in an information table:

$$T = (OB, AT, \{V_{at} \mid at \in AT\}, \{I_{at} \mid at \in AT\}),$$

where OB is a finite nonempty set of objects called the universe, AT is a finite nonempty set of attributes, V_{at} is the domain of values for $at \in AT$, and $I_{at} : OB \to V_{at}$ is an information function. Each object x takes a value $I_{at}(x)$ on attribute at.

A decision table is a special type of information tables with $AT = C \cup D$ and $C \cap D = \emptyset$, where C and D are the sets of condition and decision attributes, respectively. Given a subset of condition attributes $A \subseteq C$, we define an equivalence relation by:

$$E_A = \{(x, y) \in OB \times OB \mid \forall a \in A(I_a(x) = I_a(y))\}.$$

This equivalence relation induces a classification of OB, i.e., $\pi_A = \{[x]_A \mid x \in OB\}$, where $[x]_A = \{y \mid yE_Ax\}$ is the equivalence class containing object x. The family of equivalence classes in π_A serves as building blocks to construct regions.

Definition 1. For a subset of objects $X \subseteq OB$, the positive, negative, and boundary regions of X given π_A are defined by:

$$\begin{aligned} &\text{POS}(X|\pi_A) = \{x \mid [x]_A \subseteq X\},\\ &\text{NEG}(X|\pi_A) = \{x \mid [x]_A \subseteq OB - X\},\\ &\text{BND}(X|\pi_A) = OB - \text{POS}(X|\pi_A) \cup \text{NEG}(X|\pi_A). \end{aligned} \tag{3}$$

For the set D of decision attributes, we can similarly produce a partition $\pi_D = \{[x]_D \mid x \in OB\}$, which usually has two or more decision classes. On the basis of Definition 1, the positive region of decision classification π_D is defined by taking the union of the positive regions of all decision classes. Furthermore, the relevant measurement is produced to represent the rough dependency.

Definition 2. The positive region of classification π_D given π_A is defined by:

$$\text{POS}(\pi_D|\pi_A) = \bigcup_{X \in \pi_D} \text{POS}(X|\pi_A). \tag{4}$$

The dependency degree of classification π_D given π_A is defined by:

$$\gamma(\pi_D|\pi_A) = \frac{|\mathrm{POS}(\pi_D|\pi_A)|}{|OB|}. \tag{5}$$

The positive region $\mathrm{POS}(\pi_D|\pi_A)$ collects the object which can be properly classified to decision classes of π_D by employing condition classification π_A. $\gamma(\pi_D|\pi_A)$ is the cardinality ratio of the positive region and the universe and thus represents the relevant ability to classify objects. As a result, $\gamma(\pi_D|\pi_A)$ measures the attribute dependency of decision classification π_D from condition classification π_A, which underlies rough reasoning. The positive region and its dependency degree have monotonicity (with respect to the set inclusion of attributes):

$$\begin{aligned} &A_1 \subseteq A_2 \subseteq C \Longrightarrow \mathrm{POS}(\pi_D|\pi_{A_1}) \subseteq \mathrm{POS}(\pi_D|\pi_{A_2}), \\ &A_1 \subseteq A_2 \subseteq C \Longrightarrow \gamma(\pi_D|\pi_{A_1}) \leq \gamma(\pi_D|\pi_{A_2}). \end{aligned} \tag{6}$$

That is, a subset of attributes normally produces the smaller positive region and dependency degree than its superset's. Hence, the dependency degree has range boundedness:

$$\gamma(\pi_D|\pi_A) \in [0, \gamma(\pi_D|\pi_C)],\ \forall A \subseteq C, \tag{7}$$

where $\gamma(\pi_D|\pi_C)$ becomes the maximum. Furthermore, a classical attribute reduct is a minimal set of C to produce the same positive region or dependency degree with respect to classification π_C, and the relevant form with dependency degree γ is mainly adopted in this paper.

Definition 3. A subset of condition attributes $R \subseteq C$ is an attribute reduct of C if it satisfies two conditions:

(S) $\gamma(\pi_D|\pi_R) = \gamma(\pi_D|\pi_C)$,

(N) $\forall c \in R(\gamma(\pi_D|\pi_{R-\{c\}}) < \gamma(\pi_D|\pi_R))$.

The set of all corresponding reducts is denoted by $\mathrm{RED}(\pi_D)$.

Definition 3 provides the classical attribute reducts. The reduct target is to preserve maximum $\gamma(\pi_D|\pi_C)$ to acquire the same classification ability of initial C. Condition (S) is called the joint sufficiency that all attributes in R are jointly sufficient to keep the dependency degree with respect to C. Condition (N) is called the individual necessity that each attribute in R is necessary for keeping the dependency degree with respect to R. Condition (N) reflects the minimality of an attribute reduct, where the removal of any attribute would result in the smaller γ value. Condition (N) can be equivalently replaced by:

(N′) $\forall R' \subset R(\gamma(\pi_D|\pi_{R'}) < \gamma(\pi_D|\pi_R))$.

Although the classical reducts adopt the uncertainty measure γ, they are essentially a type of qualitative reducts because they are related to a sort of qualitative absoluteness.

Definition 4. The sets of core and useful attributes are defined by the intersection and union of all reducts, respectively [53]. That is,

$$\begin{aligned} \mathrm{CORE}(\pi_D) &= \cap \mathrm{RED}(\pi_D), \\ \mathrm{USEFUL}(\pi_D) &= \cup \mathrm{RED}(\pi_D). \end{aligned} \tag{8}$$

As two fundamental notions, the sets of core and useful attributes restrict any reduct in the form of lower and upper bounds (Eq. (2)) and can further underlie the three-way division at the C level (Eq. (1)). They are mainly concerned in this paper. For the classical qualitative reducts, they are defined in Definition 4 by all reducts. $\mathrm{CORE}(\pi_D)$ and $\mathrm{USEFUL}(\pi_D)$ provide the bounds of an arbitrary qualitative reduct R:

$$\mathrm{CORE}(\pi_D) \subseteq R \subseteq \mathrm{USEFUL}(\pi_D), \tag{9}$$

and they correspond to the three-way division of C. $\mathrm{CORE}(\pi_D)$ can be computed by:

$$\mathrm{CORE}(\pi_D) = \{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) < \gamma(\pi_D|\pi_C)\}, \tag{10}$$

so an attribute is a core attribute if it is needed to preserve $\gamma(\pi_D|\pi_C)$. From the two sides of the rough set theory argued by Yao [51], the above Eqs. (8) and (10) refer to the conceptual and computational formulations, respectively, for the set of core attributes. In terms of the intersection of the family of reducts, Eq. (8) provides an explicit link to the reduct notion, which leads to a better understanding of classical reducts and core attributes. In terms of the inequality condition from C, Eq. (10) explicitly provides a computationally efficient method and never requires computing the set of all reducts. Working together, the two formulations offer both an in-depth connotation and a basic computation. In contrast, $\mathrm{USEFUL}(\pi_D)$ only has its conceptual formulation in Eq. (8) but never has its computational formulation.

3. Relative dependency degree and quantitative attribute reducts

The classical qualitative reducts, where the dependency degree makes only its qualitative function, have a basic limitation, i.e., they become somewhat strict and cannot apply to the quantitative environment. Relevant quantitative reducts based on uncertainty measures have practical improvements and applicable significance. Zhang and Miao [59] mine a monotonic double-quantitative measure to establish the approximate and tolerant reducts. In this section, the dependency degree makes full use of its quantitative function to be improved to the relative dependency degree, and this novel measure further produces quantitative reducts.

3.1. Relative dependency degree based on lattice bases and measure mechanisms

As a mathematical preliminary, we first analyze the lattice bases and measure mechanisms. Based on the tolerance and approximation, the dependency degree γ and its monotonicity are utilized to further develop the relative dependency degree γ_{relative}. By virtue of approximate descriptions and quantitative improvements, γ_{relative} is a controllable and monotonic measure to relatively measure the attribute dependency and to firmly underlie quantitative reducts.

A decision table contains three lattices.

(1) The first lattice $(2^C, \subseteq)$ concerns subsets of condition attributes, where $\emptyset$ and C are the least and greatest elements, respectively.
(2) Define $\pi_\emptyset = \{OB\}$, $2^{\pi_C} = \{\pi_A \mid A \subseteq C\}$, and $\pi_{A_1} \Leftarrow \pi_{A_2}$ if $E_{A_1} \supseteq E_{A_2}$. Then, the second lattice $(2^{\pi_C}, \Leftarrow)$ concerns condition classifications, where $\pi_\emptyset$ and π_C are the least and greatest elements, respectively.
(3) The third one $(\{\gamma(\pi_D|\pi_A) \mid A \subseteq C\}, \leq)$ concerns the dependency degree, and it has maximum $\gamma(\pi_D|\pi_C)$ and minimum 0, where $\gamma(\pi_D|\pi_\emptyset) = 0$. $(\{\gamma(\pi_D|\pi_A) \mid A \subseteq C\}, \leq)$ becomes a sublattice of lattice $([0, \gamma(\pi_D|\pi_C)], \leq)$.

These three lattices have relevant homomorphic mappings. (1) Define mapping

$$\pi : 2^C \longrightarrow 2^{\pi_C},\ \pi(A) = \pi_A,\ \forall A \in 2^C.$$

Surjective π is an order-preservation mapping to become a homomorphic mapping, so lattices $(2^C, \subseteq)$ and $(2^{\pi_C}, \Leftarrow)$ correspond to a homomorphism. (2) Define mapping

$$\gamma_\pi : 2^{\pi_C} \longrightarrow \{\gamma(\pi_D|\pi_A) \mid A \subseteq C\},\ \gamma_\pi(\pi_A) = \gamma(\pi_D|\pi_A),\ \forall \pi_A \in 2^{\pi_C}.$$

Similarly, surjective γ_π is an order-preservation mapping, and lattices $(2^{\pi_C}, \Leftarrow)$ and $(\{\gamma(\pi_D|\pi_A) \mid A \subseteq C\}, \leq)$ correspond to a homomorphism. (3) The dependency degree essentially determines surjective mapping

$$\gamma : 2^C \longrightarrow \{\gamma(\pi_D|\pi_A) \mid A \subseteq C\},\ \gamma = \gamma(A) = \gamma(\pi_D|\pi_A),\ \forall A \in 2^C.$$

γ is actually the composite mapping of above π and γ_π, i.e.,

$$\gamma = \gamma_\pi \circ \pi :\ \gamma(A) = (\gamma_\pi \circ \pi)(A) = \gamma_\pi(\pi(A)) = \gamma_\pi(\pi_A) = \gamma(\pi_D|\pi_A),\ \forall A \in 2^C.$$

As a result, the dependency degree with respect to attribute subsets takes effect mainly by the middle level of condition classifications, and homomorphic mapping γ induces the homomorphism between lattices $(2^C, \subseteq)$ and $(\{\gamma(\pi_D|\pi_A) \mid A \subseteq C\}, \leq)$.

Three lattices and their homomorphic mappings are constructed for the condition attribute, condition classification, and dependency degree. From the algebraic viewpoint, the relevant homomorphism results effectively describe the fundamental structural relationship between systematic hierarchies. They are particularly exhibited in Fig. 1. There, the three lattices and their representative elements (including the least and greatest elements) are located at three different levels, and the three homomorphic mappings connect these levels.

In mathematics, the above lattice bases highlight the monotonicity and boundedness of the dependency degree, i.e., Eqs. (6) and (7). On these bases, we further reveal the relevant mechanism of error tolerance, and we mainly focus on the dependency degree and its micro quantitative change.

Lemma 1. *Dependency degree γ has an absolute description with respect to error bar ε. That is,*

$$\forall \varepsilon \in (0, \gamma(\pi_D|\pi_C)),\ \exists A_\varepsilon \in 2^C,\ \text{if } A \supseteq A_\varepsilon,\ \text{then } \gamma(\pi_D|\pi_C) - \gamma(\pi_D|\pi_A) < \varepsilon. \tag{11}$$

In Lemma 1, quantitative threshold ε aims to restrict the absolute error $\gamma(\pi_D|\pi_C) - \gamma(\pi_D|\pi_A)$ in lattice $(\{\gamma(\pi_D|\pi_A) \mid A \subseteq C\}, \leq)$, so

$$\gamma(\pi_D|\pi_C) - \gamma(\pi_D|\pi_A) < \varepsilon$$

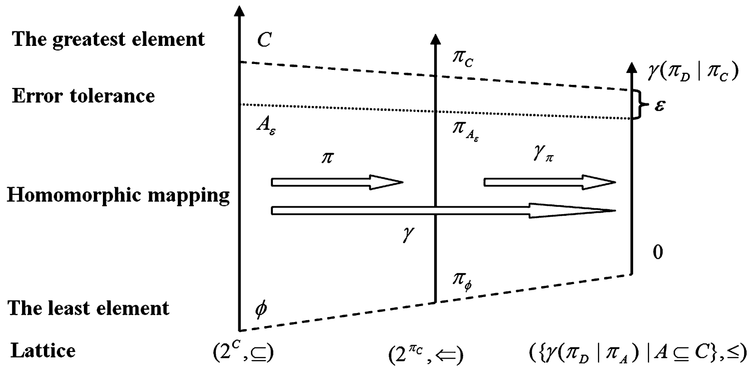

Fig. 1. Three lattices and their homomorphic mappings, error tolerance.

represents the absolute tolerance for the dependency degree. To satisfy this tolerance condition, A_ε acts as a lower bound to provide the feasible condition range $\{A \mid A \supseteq A_\varepsilon\}$ in lattice $(2^C, \subseteq)$. The relevant ε–A_ε description is also reflected by Fig. 1. In fact,

$$\{A \mid \gamma(\pi_D|\pi_C) - \gamma(\pi_D|\pi_A) < \varepsilon\} \tag{12}$$

is the necessary and sufficient range, and it includes at least C and thus is nonempty. This fact shows both the existence of parameter A_ε and the correctness of Lemma 1. Therefore, we can seek the attribute subset whose dependency degree approaches the ideal maximum $\gamma(\pi_D|\pi_C)$, and Eq. (12) provides the tolerant range to construct quantitative reducts, where qualitative reducts correspond to the degeneration with a stricter range:

$$\{A \mid \gamma(\pi_D|\pi_A) = \gamma(\pi_D|\pi_C)\}.$$

For the dependency degree, Lemma 1 depends on its monotonicity and boundedness to underlie the quantitative reducts with the absolute error. Because the relative error/accuracy has more controllability and applicability, our perspective is shifted to the relative measure based on γ. By divided by $\gamma(\pi_D|\pi_C)$ (which is temporarily supposed to be nonzero), Lemma 1 can derive two corollaries.

Corollary 1. *Dependency degree γ has a relative description with respect to error bar δ. That is,*

$$\forall\delta \in (0,1),\ \exists A_\delta \in 2^C,\ \textit{if } A \supseteq A_\delta,\ \textit{then } \frac{\gamma(\pi_D|\pi_C) - \gamma(\pi_D|\pi_A)}{\gamma(\pi_D|\pi_C)} < \delta. \tag{13}$$

Corollary 2. *Dependency degree γ has a relative description with respect to accuracy bar α. That is,*

$$\forall\alpha \in (0,1),\ \exists A_\alpha \in 2^C,\ \textit{if } A \supseteq A_\alpha,\ \textit{then } \frac{\gamma(\pi_D|\pi_A)}{\gamma(\pi_D|\pi_C)} > \alpha. \tag{14}$$

By virtue of the dependency degree, Corollaries 1 and 2 adopt the reverse relative descriptions. The former concerns the relative error and its tolerant threshold, while the latter concerns the accuracy and its required threshold. Only the latter is next utilized to construct quantitative reducts, and nonempty set

$$\{A \mid \frac{\gamma(\pi_D|\pi_A)}{\gamma(\pi_D|\pi_C)} > \alpha\}$$

provides the feasible range for approximate reducts. As a basis, we first extract the relevant measure in Eq. (14).

Definition 5. Define mapping $\gamma_{\text{relative}}: 2^C \longrightarrow [0,1]$, $\gamma_{\text{relative}} = \gamma_{\text{relative}}(\emptyset) = 0$, and

$$\forall\emptyset \neq A \in 2^C,\ \gamma_{\text{relative}} = \gamma_{\text{relative}}(A) = \begin{cases} \frac{\gamma(\pi_D|\pi_A)}{\gamma(\pi_D|\pi_C)}, & \text{if } \gamma(\pi_D|\pi_C) \neq 0; \\ 1, & \text{if } \gamma(\pi_D|\pi_C) = 0. \end{cases} \tag{15}$$

$\gamma_{\text{relative}}(A)$ is called the relative dependency degree of A, with respect to dependency degree $\gamma(\pi_D|\pi_C)$.

For γ of all attribute subsets, $\gamma(\pi_D|\pi_C)$ is the maximum to serve as the ideal and referential value, so $\gamma_{\text{relative}}(A)$ proposed in Definition 5 mainly aims to measure the unidirectional approximation of $\gamma(\pi_D|\pi_A)$ with respect to $\gamma(\pi_D|\pi_C)$.

(1) Critical $\emptyset$ has $\gamma(\emptyset)=0$ to exhibit the smallest approximation, so $\gamma_{\text{relative}}(\emptyset)=0$ is firmly defined despite of the $\gamma(\pi_D|\pi_C)$ value.
(2) When $\gamma(\pi_D|\pi_C)=0$, nonempty set A has $\gamma(A)=0$ to reach the equality, so $\gamma_{\text{relative}}(A)=1$ is defined to represent the largest approximation. For the $\gamma_{\text{relative}}(A)$ stipulation, $\gamma_{\text{relative}}=1$ becomes more reasonable than $\gamma_{\text{relative}}=0$ for the approximate description, especially when considering the further reduction requirement.
(3) When $\gamma(\pi_D|\pi_C)\neq 0$, $\gamma_{\text{relative}}(A)$ adopts the factor form of $\gamma(\pi_D|\pi_A)$ and $\gamma(\pi_D|\pi_C)$, and the relative ratio effectively represents the approximate degree.

Theorem 1. *The relative dependency degree has monotonicity. That is,*

$$A_1\subseteq A_2 \Longrightarrow \gamma_{\text{relative}}(A_1)\le\gamma_{\text{relative}}(A_2), \tag{16}$$

and nonempty A_1 and A_2 lead to the following equality constraint:

$$\gamma_{\text{relative}}(A_1)=\gamma_{\text{relative}}(A_2)\Longleftrightarrow \gamma(\pi_D|\pi_{A_1})=\gamma(\pi_D|\pi_{A_2}). \tag{17}$$

Theorem 2. *The relative dependency degree has controllability with respect to boundedness range* $[0,1]$. *That is,*

$$\gamma_{\text{relative}}(A)\in[0,1],\ \forall A\subseteq C. \tag{18}$$

In particular,

$$\gamma_{\text{relative}}=1\Longleftrightarrow \gamma=\gamma(\pi_D|\pi_C). \tag{19}$$

The γ monotonicity (Eq. (6)) produces the γ_{relative} monotonicity, i.e., Theorem 1. Furthermore, the γ_{relative} monotonicity induces the γ_{relative} range, i.e., Theorem 2, and the relevant range $[0,1]$ manifests the γ_{relative} controllability. As a result, $\gamma_{\text{relative}}(A)$ exactly measures the approximate degree between $\gamma(\pi_D|\pi_A)$ and $\gamma(\pi_D|\pi_C)$. If $\gamma_{\text{relative}}(A)$ is greater, then the approximate degree becomes greater, and $\gamma_{\text{relative}}(A)=1$ reaches the ideal equality according to Eq. (19). For measure γ_{relative}, its approximate measurement function essentially underlies the follow-up approximate/quantitative reducts based on the fundamental dependency degree.

The above approximate mechanism of $\gamma_{\text{relative}}(A)$ places emphasis on the comparability of $\gamma(\pi_D|\pi_A)$ and $\gamma(\pi_D|\pi_C)$. In fact, $\gamma(\pi_D|\pi_C)$ originates from the cardinality ratio and thus becomes a dimensionless constant without a unit, so we can relatively focus on $\gamma(\pi_D|\pi_A)$ to mine the dependency semantics of $\gamma_{\text{relative}}(A)$. $\gamma_{\text{relative}}(A)$ can be viewed to indirectly describe $\gamma(\pi_D|\pi_A)$ by referring to its background constant $\gamma(\pi_D|\pi_C)$, i.e., $\gamma_{\text{relative}}(A)$ relatively represents the dependency degree of attribute subset A with respect to entire attribute set C. In the relativity sense, $\gamma_{\text{relative}}(A)$ is generally named as the relative dependency degree for subset A. In contrast, the previous $\gamma(A)$ can be called as the absolute dependency degree to directly describe $\gamma(\pi_D|\pi_A)$ without reference. By virtue of the fundamental dependency semantics of γ, we can say that $\gamma_{\text{relative}}(A)$ relatively measures the attribute dependency while that the absolute dependency degree γ absolutely measures the attribute dependency. As an example with simplification, we can analyze the main case $\gamma(\pi_D|\pi_C)\neq 0$ in Eq. (15). Thus, $\gamma_{\text{relative}}(A)$ mainly introduces constant factor $1/\gamma(\pi_D|\pi_C)$ to relatively represent $\gamma(\pi_D|\pi_A)$, and the latter exactly measures the attribute dependency with A.

Except for the above explanation based on γ, the approximate mechanism and dependency semantics of $\gamma_{\text{relative}}(A)$ can be exactly clarified by the more essential form with respect to the positive region. For this purpose, $\gamma_{\text{relative}}(A)$ first exhibits the regional style by relevant definitions.

Theorem 3. *Measure γ_{relative} can be equivalently represented by the positive region cardinality. That is, $\gamma_{\text{relative}}(\emptyset)=0$, and*

$$\forall\emptyset\neq A\in 2^C,\ \gamma_{\text{relative}}(A)=\begin{cases}\frac{|\text{POS}(\pi_D|\pi_A)|}{|\text{POS}(\pi_D|\pi_C)|}, & \textit{if } |\text{POS}(\pi_D|\pi_C)|\neq 0;\\ 1, & \textit{if } |\text{POS}(\pi_D|\pi_C)|=0.\end{cases} \tag{20}$$

In Theorem 3, γ_{relative} mainly adopts the factor form to represent the relative cardinality ratio of $\text{POS}(\pi_D|\pi_A)$ and $\text{POS}(\pi_D|\pi_C)$, so γ_{relative} can be interpreted by the positive region cardinality. Herein, the reference of a maximum constant becomes $|\text{POS}(\pi_D|\pi_C)|$. Aiming at the ideal target regarding C, $\gamma_{\text{relative}}(A)$ measures the unidirectional approximation degree between $|\text{POS}(\pi_D|\pi_A)|$ and $|\text{POS}(\pi_D|\pi_C)|$, i.e., the internal approximation degree between $\text{POS}(\pi_D|\pi_A)$ and $\text{POS}(\pi_D|\pi_C)$, where $\text{POS}(\pi_D|\pi_A)\subseteq\text{POS}(\pi_D|\pi_C)$. This regional approximation is equivalent to the above metrical approximation with respect to γ. By comparing Eqs. (5) and (20), $\gamma(\pi_D|\pi_A)$ and $\gamma_{\text{relative}}(A)$ usually have the same numerator $|\text{POS}(\pi_D|\pi_A)|$ but different denominators $|OB|$ and $|\text{POS}(\pi_D|\pi_C)|$, respectively. Both measures contrastively represent the attribute dependency which essentially originates from the positive region, but γ and γ_{relative} place emphasis on the absoluteness from universe OB and the relativity from reference $\text{POS}(\pi_D|\pi_C)$, respectively. In contrast to the γ_{relative} name, γ can be also named as the absolute dependency degree. By virtue of the semantics of positive regions, $\gamma_{\text{relative}}(A)$ measures a relative ability to classify objects to decision classes (by employing condition classification π_A) in contrast to $|POS(\pi_D|\pi_C)|$. In other words, we can also conclude that $\gamma_{\text{relative}}(A)$ relatively measures the attribute dependency.

In mathematics, measure γ_{relative} and its monotonicity/boundedness are related to lattice $(\{\gamma_{\text{relative}}(A) \mid A \subseteq C\}, \leq)$ with minimum 0 and maximum 1, which is a sublattice of lattice $([0,1], \leq)$. According to Eq. (16), surjective γ_{relative} is an order-preservation mapping to induce the homomorphism between lattices $(2^C, \subseteq)$ and $(\{\gamma_{\text{relative}}(A) \mid A \subseteq C\}, \leq)$. From the lattice viewpoint, we next uncover an important connection between measures γ_{relative} and γ, and the previous lattice $(\{\gamma(\pi_D|\pi_A) \mid A \subseteq C\}, \leq)$ is replaced by simpler symbol $(\{\gamma(A) \mid A \subseteq C\}, \leq)$.

In the main case with $\gamma(\pi_D|\pi_C) \neq 0$, define mapping

$$f : \{\gamma(A) \mid A \subseteq C\} \longrightarrow \{\gamma_{\text{relative}}(A) \mid A \subseteq C\},\ f(\gamma(A)) = \gamma_{\text{relative}}(A),\ \forall A \subseteq C.$$

This definition reflects the mapping commutativity, i.e., $f \circ \gamma = \gamma_{\text{relative}}$. Surjective f further becomes a one-to-one mapping because of the injection feature:

$$\gamma(A_1) \neq \gamma(A_2) \Longrightarrow \gamma_{\text{relative}}(A_1) \neq \gamma_{\text{relative}}(A_2). \tag{21}$$

Meanwhile, f is an order-preservation mapping because of

$$\gamma(A_1) \leq \gamma(A_2) \Longrightarrow \gamma_{\text{relative}}(A_1) \leq \gamma_{\text{relative}}(A_2). \tag{22}$$

As a result, f constructs an isomorphism between $(\{\gamma(A) \mid A \subseteq C\}, \leq)$ and $(\{\gamma_{\text{relative}}(A) \mid A \subseteq C\}, \leq)$. In the specific case $\gamma(\pi_D|\pi_C) = 0$, $\{\gamma(A) \mid A \subseteq C\} = \{0\}$ degenerates into a single element set, while $\{\gamma_{\text{relative}}(A) \mid A \subseteq C\} = \{0, 1\}$ degenerates into a dual element set, where $\gamma_{\text{relative}} = 0$ and $\gamma_{\text{relative}} = 1$ correspond to the empty and nonempty sets, respectively. These analyses offer the measure relationships as follows.

Theorem 4. *$(\{\gamma = \gamma(A) \mid A \subseteq C\}, \leq)$ and $(\{\gamma_{\text{relative}} = \gamma_{\text{relative}}(A) \mid A \subseteq C\}, \leq)$ are two sublattices of lattice $([0,1], \leq)$. They construct an isomorphism when $\gamma(\pi_D|\pi_C) \neq 0$. When $\gamma(\pi_D|\pi_C) = 0$, they degenerate into $(\{0\}, \leq)$ and $(\{0,1\}, \leq)$, respectively, where only $\gamma_{\text{relative}} = 0$ from all nonempty attribute subsets corresponds to $\gamma_{\text{relative}} = 1$. The trivial non-isomorphism has only a sole root:*

$$\gamma(\pi_D|\pi_C) = 0 \Longrightarrow \forall A \neq \emptyset,\ \begin{cases} \gamma(A) = 0, \\ \gamma_{\text{relative}}(A) = 1. \end{cases} \tag{23}$$

According to Theorem 4, the algebraic isomorphism in most cases effectively reflects the structure equivalence of measures γ and γ_{relative}, while $\gamma = 0$ for all attribute subsets is extended to $\gamma_{\text{relative}} = 0$ and $\gamma_{\text{relative}} = 1$ to discriminate the empty and nonempty attribute sets, respectively. In view of

$$\gamma(\pi_D|\pi_C) \leq 1 \Longrightarrow \gamma_{\text{relative}}(A) \geq \gamma(A),$$

the necessary interval $[0, \gamma(\pi_D|\pi_C)]$ is extended to $[0, 1]$ by the linear transformation with extension factor $1/\gamma(\pi_D|\pi_C)$, or the exact set $\{0\}$ is extended to $\{0, 1\}$ with two points by the reasonable discrimination stipulation. In the isomorphism sense, the sole difference between measures γ and γ_{relative} is clarified by Eq. (23). The fourth lattice $(\{\gamma_{\text{relative}}(A) \mid A \subseteq C\}, \leq)$ and its relevant structure mapping and approximation description can be appended to Fig. 1, which already presents three lattices and their descriptions.

Finally, we summarize the improvement of γ_{relative} for γ. γ_{relative} is established to implement the approximation description for ideal maximum $\gamma(\pi_D|\pi_C)$ or $|POS(\pi_D|\pi_C)|$, and it depends on the dependency degree or positive region to exhibit fundamental semantics of the relative measure of the attribute dependency. As a result, γ_{relative} also has a core function for dependency reasoning and attribute reducts. When compared to γ, γ_{relative} has the same or even more measurement efficiency due to the main isomorphism equivalence or the point expansion. However, γ_{relative} has better controllability based on maximum constant 1 and minimum constant 0; in contrast, γ has uncontrollable range $[0, \gamma(\pi_D|\pi_C)]$ because $\gamma(\pi_D|\pi_C)$ is a variable for different decision tables. Therefore, γ_{relative} positively improves γ and thus can effectively replaces the latter. The relevant improvement and replacement can be clarified by the later contents of three-way reducts. In fact, the successful mining of γ_{relative} benefits from the idea of double quantification with respect to relativity and absoluteness [43,57,59].

3.2. Quantitative reducts based on the relative dependency degree

By improving the dependency degree γ, the relative dependency degree γ_{relative} is mined to represent the approximation degree of $\gamma(\pi_D|\pi_A)$ with respect to $\gamma(\pi_D|\pi_C)$, and it relatively measures the attribute dependency. γ_{relative} has controllability and monotonicity, and its ideal value is 1 because $\gamma(\pi_D|\pi_C)$ serves as an initial criterion. To implement the quantitative approximation, we introduce an approximate parameter $\alpha \in (0, 1]$, where α usually approaches to 1 while $\alpha = 1$ mainly applies to the completion. In practice, α can be determined by the expert experience or user requirement. Next, γ_{relative} and α are utilized to construct a type of approximate reducts, which is denoted by γ_{relative}–α quantitative reducts.

The approximate mechanism is provided by measure γ_{relative} and relevant Corollary 2. On this basis, $\gamma_{\text{relative}}(A) \geq \alpha$ is focused on. According to Eq. (15), $\gamma_{\text{relative}}(A) \geq \alpha$ means that $\gamma(\pi_D|\pi_A)$ is not smaller than $\alpha \times \gamma(\pi_D|\pi_C)$ to approach ideal $\gamma(\pi_D|\pi_C)$, so $\gamma_{\text{relative}}(A) \geq \alpha$ represents the relative approximation for the dependency degree. According to Eq. (20), $\gamma_{\text{relative}}(A) \geq \alpha$ means that $|\text{POS}(\pi_D|\pi_A)|$ is not smaller than $\alpha \times |\text{POS}(\pi_D|\pi_C)|$ to approach ideal $|\text{POS}(\pi_D|\pi_C)|$. On the premise of $\text{POS}(\pi_D|\pi_A) \subseteq \text{POS}(\pi_D|\pi_C)$, $\gamma_{\text{relative}}(A) \geq \alpha$ quantifies and takes the internal approximation of $\text{POS}(\pi_D|\pi_A)$ with respect to $\text{POS}(\pi_D|\pi_C)$. For this approximate strategy of $\gamma_{\text{relative}} \geq \alpha$, the necessary and sufficient range of attribute subsets becomes:

$$\{A \mid \gamma_{\text{relative}}(A) \geq \alpha\} = \{A \mid \gamma(\pi_D|\pi_A) \geq \alpha \times \gamma(\pi_D|\pi_C)\} = \{A \mid |\text{POS}(\pi_D|\pi_A)| \geq \alpha \times |\text{POS}(\pi_D|\pi_C)|\}. \quad (24)$$

Therefore, $\gamma_{\text{relative}} \geq \alpha$ resorts to the fundamental function of the dependency degree and positive region to correspond to approximate reasoning based on the attribute dependency, and it reasonably becomes a direct approximate target to construct approximate/quantitative reducts. The $\gamma_{\text{relative}} \geq \alpha$ monotonicity is next derived by the γ_{relative} monotonicity (Theorem 1).

Corollary 3. *The approximate target $\gamma_{\text{relative}} \geq \alpha$ has monotonicity. That is, if $A_1 \subseteq A_2$, then*

$$\gamma_{\text{relative}}(A_1) \geq \alpha \Longrightarrow \gamma_{\text{relative}}(A_2) \geq \alpha. \quad (25)$$

Based on the above approximation analyses, γ_{relative} becomes an objective measure while α provides a subjective bar, so approximate demand $\gamma_{\text{relative}} \geq \alpha$ becomes a target of approximate/quantitative reducts. The $\gamma_{\text{relative}} \geq \alpha$ monotonicity is beneficial to relevant reduct studies. A γ_{relative}–α quantitative reduct is a minimal set of C to satisfy approximate requirement $\gamma_{\text{relative}} \geq \alpha$.

Definition 6. A subset of condition attributes $R \subseteq C$ is a γ_{relative}–α quantitative reduct of C if it satisfies two conditions:

$$(\text{S}_\alpha)\ \gamma_{\text{relative}}(R) \geq \alpha,$$
$$(\text{N}_\alpha)\ \forall c \in R(\gamma_{\text{relative}}(R - \{c\}) < \alpha).$$

The set of all γ_{relative}–α quantitative reducts is denoted by $\text{RED}_\alpha(\pi_D)$.

According to approximate target $\gamma_{\text{relative}} \geq \alpha$, Conditions (S_α) and (N_α) naturally concern the joint sufficiency and individual necessity, respectively. By virtue of the target monotonicity (Eq. (25)), γ_{relative}–α quantitative reducts fall into the framework of generalized reducts with a monotonicity target, which is established by Zhang and Miao [59]. As a result, the individual necessity (N_α) can be equivalently described by:

$$(\text{N}'_\alpha)\ \forall R' \subset R(\gamma_{\text{relative}}(R') < \alpha),$$

and the relevant equivalence proof is also proved in Appendix A.

Definition 7. For γ_{relative}–α quantitative reducts, the sets of core and useful attributes are defined by the intersection and union of all reducts, respectively. That is,

$$\text{CORE}_\alpha(\pi_D) = \cap\text{RED}_\alpha(\pi_D),$$
$$\text{USEFUL}_\alpha(\pi_D) = \cup\text{RED}_\alpha(\pi_D). \quad (26)$$

The core and useful attributes are normally defined, and they provide the lower and upper bounds of a quantitative reduct R, i.e.,

$$\text{CORE}_\alpha(\pi_D) \subseteq R \subseteq \text{USEFUL}_\alpha(\pi_D). \quad (27)$$

Moreover, the set of core attributes has the computational formulation:

$$\text{CORE}_\alpha(\pi_D) = \{c \mid \gamma_{\text{relative}}(C - \{c\}) < \alpha\}, \quad (28)$$

which is proved in Appendix B. According to Eq. (28), an attribute is a core attribute if it is necessarily needed to preserve target $\gamma_{\text{relative}} \geq \alpha$. In fact, the deletion of a core attribute leads to that an arbitrary and relevant subset cannot reach $\gamma_{\text{relative}} \geq \alpha$ in view of the γ_{relative} monotonicity. Thus, Algorithm 1 provides the relevant calculation. Step 1 makes the initialization, and Steps 2–6 circularly search the single attribute which satisfies the computable condition. If the γ_{relative} calculation and discrimination as well as the set renewal related to $|C| = m$ are viewed as basic operations, then the time complexity changes from $T(m) = 2m + 0$ to $T(m) = 2m + m$; thus, the time complexity $T(m) = o(m)$ becomes feasible,

Algorithm 1 A basic algorithm of the core attribute set regarding γ_{relative}–α quantitative reducts.

Input: Decision table T with threshold α;
Output: The core attribute set $\text{CORE}_\alpha(\pi_D)$.
$\text{CORE}_\alpha(\pi_D) = \emptyset$.
for each $c \in C$ **do**
  if $\gamma_{\text{relative}}(C - \{c\}) < \alpha$ **then**
    $\text{CORE}_\alpha(\pi_D) \leftarrow \text{CORE}_\alpha(\pi_D) \cup \{c\}$;
  end if
end for
return $\text{CORE}_\alpha(\pi_D)$.

while the space complexity is similarly effective. In a word, $\text{CORE}_\alpha(\pi_D)$ is computable, and it further underlies γ_{relative}–α quantitative reducts by its role of the lower bound.

γ_{relative}–α quantitative reducts mainly originate from the relative dependency degree and thus can be calculated by this bearing measure. Furthermore, monotonic γ_{relative} constructs heuristic information:

$$Sig(A, c) = \gamma_{\text{relative}}(A \cup \{c\}) - \gamma_{\text{relative}}(A); \tag{29}$$

$Sig(A, c)$ represents the deference degree of the γ_{relative} change when subset A is added by attribute c. Furthermore, $Sig(A, c)$ and $\text{CORE}_\alpha(\pi_D)$ can be utilized to develop a heuristic algorithm (i.e., Algorithm 2) to gain a γ_{relative}–α quantitative reduct.

Algorithm 2 A heuristic algorithm for a γ_{relative}–α quantitative reduct.

Input: Decision table T with threshold α;
Output: A γ_{relative}–α quantitative reduct $R_\alpha \in \text{RED}_\alpha(\pi_D)$.
1: Compute $\text{CORE}_\alpha(\pi_D)$ by Algorithm 1.
2: $R_\alpha = \text{CORE}_\alpha(\pi_D)$.
3: **while** $\gamma_{\text{relative}}(R_\alpha) < \alpha$ **do**
4:   $\forall c \in C - R_\alpha$, calculate $Sig(R_\alpha, c)$; choose $c_0 = \arg \max_{c \in C - R_\alpha} Sig(R_\alpha, c)$, and let $R_\alpha \leftarrow R_\alpha \cup \{c_0\}$.
5: **end while**
6: **return** R_α.

Algorithm 2 adopts both an addition strategy based on $\text{CORE}_\alpha(\pi_D)$ and a rapid calculation based on $Sig(A, c)$. In Steps 1 and 2, $\text{CORE}_\alpha(\pi_D)$ is calculated and thus becomes a reduct basis from the lower bound. In Steps 3–5, a superset of $\text{CORE}_\alpha(\pi_D)$ is sought to satisfy the reduct target $\gamma_{\text{relative}} \geq \alpha$, and the added attribute c_0 in the "while" loop is chosen by the highest heuristic information $Sig(R_\alpha, c)$ inside $C - R_\alpha$ to gain a fast reduct way. In Step 6, the superset based on the attribute addition is output. The final result depends on the choice order of the maximum of heuristic information. For the computational complexity, the γ_{relative} calculation as well as the relevant comparison, subtraction, renewal related to $|C| = m$ are viewed as basic operations. In the best case, the core is non-empty to exactly become the sole reduct, and then Algorithm 2 has time complexity $T(m) = (2m + c) + 2 = o(m)$, where $2m + c$ comes from Algorithm 1. The worst case corresponds to $\text{CORE}_\alpha(\pi_D) = \emptyset$ and $\text{Red}_\alpha(\pi_D) = \{C\}$; thus, Algorithm 1 in Step 1 concerns $2m$, the judgement and loop of Step 3 need m times, and Step 4 first relates the Sig determination and comparison, as well as the set assignment to $2m$, $m - 1$, 1, respectively; therefore, the total time complexity becomes $T(m) = 2m + [2m + 3m(m-1)/2] = o(m^2)$. The space complexity can be similarly analyzed. Algorithm 2 is convergent and effective to yield a γ_{relative}–α quantitative reduct.

Next, the γ_{relative}–α quantitative reducts are analyzed based on the classical qualitative reducts. For the qualitative reduct target,

$$\gamma = \gamma(\pi_D|\pi_C) \Longleftrightarrow \gamma_{\text{relative}} = 1. \tag{30}$$

The equivalent target is a part of the quantitative target: $\gamma_{\text{relative}} \geq \alpha$. Therefore, the qualitative reducts (Definition 3) have a new style based on γ_{relative}. For the joint sufficiency and individual necessity, Conditions (S), (N), and (N′) can be equivalently expressed by following forms, respectively, i.e.,

$$(\text{S}_1)\ \gamma_{\text{relative}}(R) = 1,$$
$$(\text{N}_1)\ \forall c \in R(\gamma_{\text{relative}}(R - \{c\}) < 1),$$
$$(\text{N}'_1)\ \forall R' \subset R(\gamma_{\text{relative}}(R') < 1).$$

Conditions (S_1), (N_1), and (N'_1) are compared to Conditions (S_α), (N_α), and (N'_α) in the quantitative pattern, respectively. These analyses provide the following reduct expansion.

Theorem 5. *The γ_{relative}–α quantitative reducts expand the classical qualitative reducts and can degenerate into the latter by setting up $\alpha = 1$.*

The γ_{relative}–α quantitative reducts exhibit the theoretical expansion to include the classical qualitative reducts. The former reducts are more viewed as a novel type of approximate reducts to be practically applied. In usual cases, the quantitative target approaches the ideal qualitative target, so the γ_{relative}–α quantitative reducts mainly approximate the latter, where α approaches value 1.

Zhang and Miao [58] define the strength relationship for two types of attribute reducts. Suppose I and II denote two type of reducts, and relevant reduct notions are labeled by subscripts I and II. If the reduct target of I derives the target of II, i.e., the former is stronger than the latter, then the strength between reducts I and II is defined and denoted by $\text{I} \succeq \text{II}$, which means that reduct I is stronger than reduct II or that reduct II is weaker than reduct I. Strength relation $\succeq$ constructs a partial order to describe a basic relationship between reduct types. By virtue of the reduct conditions, $\text{I} \succeq \text{II}$ produces a reduct relationship:

$$\forall R_{\text{I}} \in \text{RED}_{\text{I}},\ \exists R_{\text{II}} \in \text{RED}_{\text{II}},\ s.t.,\ R_{\text{II}} \subseteq R_{\text{I}}. \tag{31}$$

That is, a strong reduct includes at least a weak reduct; however, a weak reduct can exist beyond this inclusion relationship. By virtue of the reduct intersection, Eq. (31) derives a relationship of core attributes:

$$\text{CORE}_{\text{I}} \supseteq \text{CORE}_{\text{II}}. \tag{32}$$

That is, the set of core attributes with respect to the stronger reduct necessarily includes the set of core attributes with respect to the weaker reduct; however, the opposite usually does not hold. Unfortunately, similar certain relationships cannot be established for the sets of useful attributes, and $\text{USEFUL}_{\text{I}}(\pi_D)$ and $\text{USEFUL}_{\text{II}}(\pi_D)$ can exhibit usual set relationships. The reduct strength is next used to describe the strength connection between the quantitative and qualitative reducts.

For reduct targets,

$$\gamma = \gamma(\pi_D|\pi_C) \Longleftrightarrow \gamma_{\text{relative}} = 1 \Longrightarrow \gamma_{\text{relative}} \geq \alpha. \tag{33}$$

That is, the qualitative target derives the quantitative target; however, the opposite usually does not holds. The target strength produces the condition strength of joint sufficiency and individual necessity, and the latter underlies the further reduct strength. Relevant strength is described as follows according to the above reduct strength theory [58].

Lemma 2. *Consider the γ_{relative}–α quantitative reducts and the classical qualitative reducts. For the joint sufficiency, the quantitative condition weakens the qualitative condition; for the individual necessity, the quantitative condition strengthens the qualitative condition. Concretely, that R satisfies Condition* (S_α) *can be realized by that R satisfies Condition* (S)*, while that R satisfies Condition* $(\text{N}_\alpha)/(\text{N}'_\alpha)$ *deduces that R satisfies Condition* $(\text{N})/(\text{N}')$.

Theorem 6. *The γ_{relative}–α quantitative reducts are weaker than the classical qualitative reducts. They have relationships:*

$$\begin{aligned} &\forall R \in \text{RED}(\pi_D),\ \exists R_\alpha \in \text{RED}_\alpha(\pi_D),\ s.t.,\ R_\alpha \subseteq R; \\ &\text{CORE}(\pi_D) \supseteq \text{CORE}_\alpha(\pi_D). \end{aligned} \tag{34}$$

Theorem 6 exhibits the strength between the quantitative and qualitative reduct types, as well as their relevant relationships. When compared to the qualitative reducts, the quantitative reducts have more reduction optimization by tending to less attributes. In fact, a qualitative reduct can internally derive a quantitative reduct according to Eq. (34), and deleting unnecessary attributes related to (S_α) and (N_α) becomes key in view of Eq. (33). According to a deletion approach from a qualitative reduct, a constructional algorithm is developed to seek a quantitative reduct.

Algorithm 3 A constructional algorithm of a γ_{relative}–α quantitative reduct from a classical qualitative reduct.

Input: A decision table T with threshold α, and a classical qualitative reduct $R \in \text{RED}(\pi_D)$;
Output: A γ_{relative}–α quantitative reduct $R_\alpha \in \text{RED}_\alpha(\pi_D)$ to satisfy $R_\alpha \subseteq R$.

```
Compute CORE_α(π_D) by Algorithm 1.
R_α = R;
for each c ∈ R − CORE_α(π_D) do
    if γ_relative(R_α − {c}) ≥ α then
        R_α ← R_α − {c};
    end if
end for
return R_α.
```

In Algorithm 3, core set $\text{CORE}_\alpha(\pi_D)$ and qualitative reduct R provide the lower and upper bounds to seek quantitative reduct R_α, i.e.,

$$\text{CORE}_\alpha(\pi_D) \subseteq R_\alpha \subseteq R,$$

and a deletion strategy is adopted in the operational range $R - \mathrm{CORE}_{\alpha}(\pi_D)$. In Step 1, $\mathrm{CORE}_{\alpha}(\pi_D)$ is computed to determine the lower bound. In Step 2, the upper bound R is chosen as the searching starting-point. In Steps 3–7, the "for" loop sequentially checks attribute $c \in R - \mathrm{CORE}_{\alpha}(\pi_D)$ by the "if" condition. If the attribute deletion preserves the quantitative target $\gamma_{\text{relative}} \geq \alpha$, then the attribute is deleted; otherwise, it has the necessity to be retained. By sequentially checking all attributes in $R - \mathrm{CORE}_{\alpha}(\pi_D)$, R_{α} satisfies target $\gamma_{\text{relative}} \geq \alpha$ (with respect to Condition (S_{α})) and never contains any unnecessary attributes (with respect to Condition (N_{α})), so it becomes a quantitative reduct in R. In Step 8, R_{α} is output. Herein, the measurement, comparison, and renewal are viewed as basic operations, and suppose $|R - \mathrm{CORE}_{\alpha}(\pi_D)| = m'$. According to Algorithm 1, Step 1 becomes relatively stable and is related to $T(m) = o(m)$, where m means $|C|$. Furthermore, main Steps 3–7 concern the time complexity range $[2m', 3m']$. The total time complexity becomes $T(m, m') = o(m) + o(m') = o(m)$. Algorithm 3 is convergent and effective, and the final quantitative reduct depends on the sequences of attributes in the "for" loop.

According to γ_{relative}, the strength between the quantitative and qualitative reducts benefits from $\alpha \leq 1$. Relevant strength results can be generalized to the quantitative reducts based on two thresholds. Theorem 6 derives following Corollary 4, which underlies an algorithm similar to Algorithm 3.

Corollary 4. *If $0 < \alpha_1 \leq \alpha_2 \leq 1$, then the γ_{relative}–α_1 quantitative reducts are weaker than the γ_{relative}–α_2 qualitative reduct. They have relationships:*

$$\begin{aligned} &\forall R_{\alpha_2} \in \mathrm{RED}_{\alpha_2}(\pi_D),\ \exists R_{\alpha_1} \in \mathrm{RED}_{\alpha_1}(\pi_D),\ s.t.,\ R_{\alpha_1} \subseteq R_{\alpha_2}; \\ &\mathrm{CORE}_{\alpha_2}(\pi_D) \supseteq \mathrm{CORE}_{\alpha_1}(\pi_D). \end{aligned} \tag{35}$$

Finally, the γ_{relative}–α quantitative reducts are summarized. By improving the dependency degree, the relative dependency degree represents the relative attribute dependency to bear the approximate mechanism, and $\gamma_{\text{relative}} \leq \alpha$ becomes the approximate target. According to the conditions of joint sufficiency and individual necessity, the quantitative reducts are naturally defined and studied by simulating the qualitative reducts. With respect to the classical qualitative reducts, the γ_{relative}–α quantitative reducts exhibit three fundamental features: the expansion, approach, and weakening, which correspond to $\alpha = 1$, $\alpha \to 1$, and $\alpha < 1$, respectively. Therefore, the quantitative reducts exhibit the development and improvement, when compared to the qualitative reducts, and they have applicable significance by the quantitative approximation. In fact, the γ_{relative}–α quantitative reducts closely follow the qualification thought of approximate reducts [34,35,40,59], especially the approximate pattern established by Zhang and Miao [59], but they utilize a completely different measure γ_{relative} by improving the basic measure γ.

4. Three-way attribute reducts

According to the approximate mechanism, the relative dependency degree γ_{relative} is a superior measure with semantics fundamentality and measure controllability. As two special cases, the classical qualitative reducts adopt qualitative target $\gamma_{\text{relative}} = 1$ to implement the qualitative absoluteness, while the γ_{relative}–α quantitative reducts adopt approximate target $\gamma_{\text{relative}} \geq \alpha$ to make the quantitative expansion and improvement. In this section, the γ_{relative}–α quantitative reducts are first extended to three-way quantitative reducts: a complete quantitative pattern, by using fundamental γ_{relative} and dual thresholds α, β. Then, three-way quantitative reducts degenerate into three-way qualitative reducts: a complete qualitative pattern. Furthermore, three-way reducts are summarized by combing the quantitative and qualitative patterns. Finally, the superiority of three-way reducts is revealed by the comparison to two-way reducts. In other words, this section thoroughly discusses three-way attribute reducts by four parts: three-way quantitative reducts, three-way qualitative reducts, their summary of qualitative and quantitative patterns, and their superiority in contrast to two-way reducts.

4.1. Three-way quantitative reducts

To clarify the complete quantitative extension, three-way quantitative reducts are illustrated by two subsections of definitions and relationships.

4.1.1. Definitions of three-way quantitative reducts

Three-way quantitative reducts can be established by the quantitative mechanism. To expand and approximate the classical qualitative reducts, the γ_{relative}–α quantitative reducts mainly apply to the high measure γ_{relative} by using high threshold α, so they correspond to an affirmation of quantitative reducts. Generally, the low γ_{relative} corresponds to a negation of quantitative reducts, while the moderate γ_{relative} corresponds to an uncertain boundary between the affirmation and negation. In other words, γ_{relative} and its three degree levels can be utilized to produce three parts of quantitative reducts. For this purpose, we first introduce quantitative bars and their usual range:

$$0 \leq \beta < \alpha \leq 1. \tag{36}$$

Measure γ_{relative} and dual thresholds α, β constitute a quantitative system, denoted as γ_{relative}–(α, β). For the quantitative system γ_{relative}–(α, β), the γ_{relative}–α quantitative reducts provide only a main case, and the other patterns of quantitative reducts can be developed according to the systematic completion.

For the γ_{relative}–α quantitative reducts, $\gamma_{\text{relative}} \geq \alpha$ serves as the reduct target and exhibits the monotonicity (Corollary 3). In the system γ_{relative}–(α, β), we naturally add two measure targets:

$$\gamma_{\text{relative}} \in (\beta, \alpha), \ \gamma_{\text{relative}} \leq \beta. \tag{37}$$

For quantitative reducts, $\gamma_{\text{relative}} \in (\beta, \alpha)$ serves as a moderate reduct target because of the theoretical uncertainty and practical possibility, while $\gamma_{\text{relative}} \leq \beta$ needs the complete negation because of its low γ_{relative} value. The moderate target usually does not have the monotonicity. In other words, if $A_1 \subseteq A_2$, then $\gamma_{\text{relative}}(A_1) \in (\beta, \alpha)$ and $\gamma_{\text{relative}}(A_2) \in (\beta, \alpha)$ cannot make mutual deduction. In contrast, $\gamma_{\text{relative}} \leq \beta$ has the monotonicity, i.e., if $A_1 \subseteq A_2$, then

$$\gamma_{\text{relative}}(A_1) \leq \beta \Longleftarrow \gamma_{\text{relative}}(A_2) \leq \beta. \tag{38}$$

γ_{relative}–α quantitative reducts act as a normal pattern to achieve the approximate target; the boundary pattern can be defined by maintaining the moderate target $\gamma_{\text{relative}} \in (\beta, \alpha)$; and the negation pattern needs to finally collect both all attribute subsets with $\gamma_{\text{relative}} \leq \beta$ and surplus attribute subsets with $\gamma_{\text{relative}} > \beta$. Along this idea, three-way quantitative reducts are established as follows.

Definition 8. Three-way γ_{relative}–(α, β) quantitative reducts concern the positive, boundary, and negative quantitative reducts. (1) First, the γ_{relative}–(α, β) positive quantitative reducts are exactly the γ_{relative}–α quantitative reducts. For systematic consistency, new symbols ($\mathrm{S}^{\mathrm{POS}}_{\alpha,\beta}$), ($\mathrm{N}^{\mathrm{POS}}_{\alpha,\beta}$), and ($\mathrm{N}'^{\mathrm{POS}}_{\alpha,\beta}$) replace previous ($\mathrm{S}_\alpha$), ($\mathrm{N}_\alpha$), and ($\mathrm{N}'_\alpha$), respectively; the sets of attribute reducts and core/useful attributes are updated by:

$$\begin{aligned} \mathrm{RED}^{\mathrm{POS}}_{\alpha,\beta}(\pi_D) &= \mathrm{RED}_\alpha(\pi_D), \\ \mathrm{CORE}^{\mathrm{POS}}_{\alpha,\beta}(\pi_D) &= \mathrm{CORE}_\alpha(\pi_D), \\ \mathrm{USEFUL}^{\mathrm{POS}}_{\alpha,\beta}(\pi_D) &= \mathrm{USEFUL}_\alpha(\pi_D). \end{aligned} \tag{39}$$

(2) Then, a subset of condition attributes $R \subseteq C$ is a γ_{relative}–(α, β) boundary quantitative reduct of C if it satisfies two conditions:

$$\begin{aligned} &(\mathrm{S}^{\mathrm{BND}}_{\alpha,\beta}) \ \gamma_{\text{relative}}(R) \in (\beta, \alpha), \\ &(\mathrm{N}^{\mathrm{BND}}_{\alpha,\beta}) \ \forall c \in R(\gamma_{\text{relative}}(R - \{c\}) \leq \beta. \end{aligned}$$

The set of all boundary reducts is denoted by $\mathrm{RED}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D)$. Accordingly, the sets of core and useful attributes are denoted and defined by:

$$\begin{aligned} \mathrm{CORE}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D) &= \cap\mathrm{RED}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D), \\ \mathrm{USEFUL}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D) &= \cup\mathrm{RED}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D). \end{aligned} \tag{40}$$

(3) Finally, a subset of condition attributes $R \subseteq C$ is a γ_{relative}–(α, β) negative reduct of C if it is neither the γ_{relative}–(α, β) positive nor boundary quantitative reducts. The set of all negative reducts is denoted by $\mathrm{RED}^{\mathrm{NEG}}_{\alpha,\beta}(\pi_D)$, and

$$\mathrm{RED}^{\mathrm{NEG}}_{\alpha,\beta}(\pi_D) = 2^C - (\mathrm{RED}^{\mathrm{POS}}_{\alpha,\beta}(\pi_D) \cup \mathrm{RED}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D)). \tag{41}$$

In Definition 8, three-way quantitative reducts are formally defined based on system γ_{relative}–(α, β).

(1) The γ_{relative}–(α, β) positive quantitative reducts are exactly the γ_{relative}–α quantitative reducts, which are proposed in Definition 6.
(2) The boundary quantitative reducts simulate the γ_{relative}–α quantitative reducts but adopt moderate target $\gamma_{\text{relative}} \in (\beta, \alpha)$.
(3) To collect the surplus subsets with negation, the negative quantitative reducts are defined by the complementarity of the positive and boundary quantitative reducts.

For γ_{relative}–(α, β) reducts, the positive quantitative reducts have been discussed; the boundary quantitative reducts become a new type of quantitative reducts and thus are next focused on; and the negative quantitative reducts actually become a generalized notion and thus have no exact relevant elements, such as the joint sufficiency and core attributes.

The γ_{relative}–(α, β) boundary quantitative reducts are related to moderate γ_{relative} and interval (β, α). In view of the reduction uncertainty and possibility, $\gamma_{\text{relative}} \in (\beta, \alpha)$ becomes the reduct target, and its nonmonotonicity will lead to less properties. The joint sufficiency is presented by Condition ($\mathrm{S}^{\mathrm{BND}}_{\alpha,\beta}$). For the individual necessity,

$$\forall c \in R(\gamma_{\text{relative}}(R - \{c\}) \notin (\beta, \alpha).$$

is simplified to Condition $(\text{N}^{\text{BND}}_{\alpha,\beta})$ by virtue of the $(\text{S}^{\text{BND}}_{\alpha,\beta})$ premise and γ_{relative} monotonicity. Moreover, $(\text{N}^{\text{BND}}_{\alpha,\beta})$ is equivalent to the condition:

$$(\text{N}'^{\text{BND}}_{\alpha,\beta}) \ \forall R' \subset R(\gamma_{\text{relative}}(R') \le \beta).$$

The equivalence benefits from the $\gamma_{\text{relative}} \le \beta$ monotonicity (Eq. (38)) and thus its proof is similar to the proof in Appendix A. The set of core attributes $\text{CORE}^{\text{BND}}_{\alpha,\beta}(\pi_D)$ has the lower and upper bounds:

$$\begin{aligned} & \{c \mid \gamma_{\text{relative}}(C - \{c\}) \le \beta\} \\ \subseteq\ & \text{CORE}^{\text{BND}}_{\alpha,\beta}(\pi_D) \\ \subseteq\ & \{c \mid \gamma_{\text{relative}}(C - \{c\}) \notin (\beta, \alpha)\} \\ =\ & \{c \mid \gamma_{\text{relative}}(C - \{c\}) \le \beta\} \cup \{c \mid \gamma_{\text{relative}}(C - \{c\}) \ge \alpha\}. \end{aligned} \tag{42}$$

The bound proof is provided in Appendix C and is similar to the proof of $\text{CORE}_{\alpha}(\pi_D)$ in Appendix B. This bound conclusion originates from the restriction of dual thresholds and implies that $\text{CORE}^{\text{BND}}_{\alpha,\beta}(\pi_D)$ cannot be directly computed. By simulating Algorithm 2 but deleting the set basis of core attributes, a heuristic algorithm of the γ_{relative}–(α, β) boundary quantitative reduct can be similarly developed by heuristic information $Sig(A, c)$.

The γ_{relative}–(α, β) positive and boundary quantitative reducts act as two fundamental types of quantitative reducts and have some difference. To describe the approximation and possibility of quantitative reduction, they use the high target $\gamma_{\text{relative}} \ge \alpha$ and moderate target $\gamma_{\text{relative}} \in (\beta, \alpha)$, respectively. The two targets exhibit the monotonicity and nonmonotonicity, respectively, and thus lead to different descriptions. $(\text{S}^{\text{POS}}_{\alpha,\beta})$ and $(\text{S}^{\text{BND}}_{\alpha,\beta})$ with respect to the joint sufficiency have the same structure but piecewise γ_{relative} values, while $(\text{N}^{\text{POS}}_{\alpha,\beta})/(\text{N}'^{\text{POS}}_{\alpha,\beta})$ and $(\text{N}^{\text{BND}}_{\alpha,\beta})/(\text{N}'^{\text{BND}}_{\alpha,\beta})$ with respect to the individual necessity exhibit the similar analyses as above. For the core attributes, $\text{CORE}^{\text{POS}}_{\alpha,\beta}(\pi_D)$ exhibits the exactness and computability, while $\text{CORE}^{\text{BND}}_{\alpha,\beta}(\pi_D)$ exhibits the bounds and incomputability.

Three-way γ_{relative}–(α, β) quantitative reducts complete the γ_{relative}–α quantitative reducts by introducing the boundary quantitative reducts and by collecting the negative quantitative reducts. In the whole calculation, the positive and boundary quantitative reducts are mainly concerned, and the negative quantitative reducts become the supplementarity.

4.1.2. Relationships of three-way quantitative reducts

Based on the above definitions, the relationships of three-way quantitative reducts are analyzed, mainly at two levels of 2^C and C.

By virtue of system γ_{relative}–(α, β), 2^C is divided into three pairwise parts.

(1) Part $\{A \mid \gamma_{\text{relative}}(A) \ge \alpha\}$ has the high measure degree and refers to the joint sufficiency $(\text{S}^{\text{POS}}_{\alpha,\beta})$. Furthermore, the attribute subsets which satisfy the individual necessity $(\text{N}^{\text{POS}}_{\alpha,\beta})$ constitute a subpart: $\text{RED}^{\text{POS}}_{\alpha,\beta}(\pi_D)$, while the surplus subpart refers to the individual unnecessity to belong to $\text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D)$.
(2) Part $\{A \mid \gamma_{\text{relative}}(A) \in (\beta, \alpha)\}$ has the moderate measure degree and refers to the joint sufficiency $(\text{S}^{\text{BND}}_{\alpha,\beta})$. Furthermore, the attribute subsets which satisfy the individual necessity $(\text{N}^{\text{BND}}_{\alpha,\beta})$ constitute a subpart: $\text{RED}^{\text{BND}}_{\alpha,\beta}(\pi_D)$, while the surplus subpart refers to the individual unnecessity to belong to $\text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D)$.
(3) Part $\{A \mid \gamma_{\text{relative}}(A) \le \beta\}$ has the low measure degree and directly belongs to $\text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D)$.

The above partitional mechanism is illustrated in Fig. 2 by the measure range, joint sufficiency, and individual necessity/unnecessity. As a result, the systematic structure of three-way quantitative reducts is revealed in 2^C.

2^C exhibits five blocks. The negative quantitative reducts involve three blocks, i.e.,

$$\begin{aligned} & \text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D) = \text{Block}^{\text{i}}_{\alpha,\beta} \cup \text{Block}^{\text{ii}}_{\alpha,\beta} \cup \text{Block}^{\text{iii}}_{\alpha,\beta}, \\ & \begin{cases} \text{Block}^{\text{i}}_{\alpha,\beta} = \{A \mid \gamma_{\text{relative}}(A) \ge \alpha\} - \text{RED}^{\text{POS}}_{\alpha,\beta}(\pi_D), \\ \text{Block}^{\text{ii}}_{\alpha,\beta} = \{A \mid \gamma_{\text{relative}}(A) \in (\beta, \alpha)\} - \text{RED}^{\text{BND}}_{\alpha,\beta}(\pi_D), \\ \text{Block}^{\text{iii}}_{\alpha,\beta} = \{A \mid \gamma_{\text{relative}}(A) \le \beta\}. \end{cases} \end{aligned} \tag{43}$$

(1) $\text{Block}^{\text{i}}_{\alpha,\beta}$ collects the subset which reaches the joint sufficient $(\text{S}^{\text{POS}}_{\alpha,\beta})$ rather than individual necessity $(\text{N}^{\text{POS}}_{\alpha,\beta})$ (with respect to the positive quantitative reducts).
(2) $\text{Block}^{\text{ii}}_{\alpha,\beta}$ is similar for Conditions $(\text{S}^{\text{BND}}_{\alpha,\beta})$ and $(\text{N}^{\text{BND}}_{\alpha,\beta})$ (with respect to the boundary quantitative reducts).
(3) $\text{Block}^{\text{iii}}_{\alpha,\beta}$ collects the subset with the low measure degree $\gamma_{\text{relative}} \le \beta$, which cannot satisfy the joint sufficient of both the positive and boundary quantitative reducts.

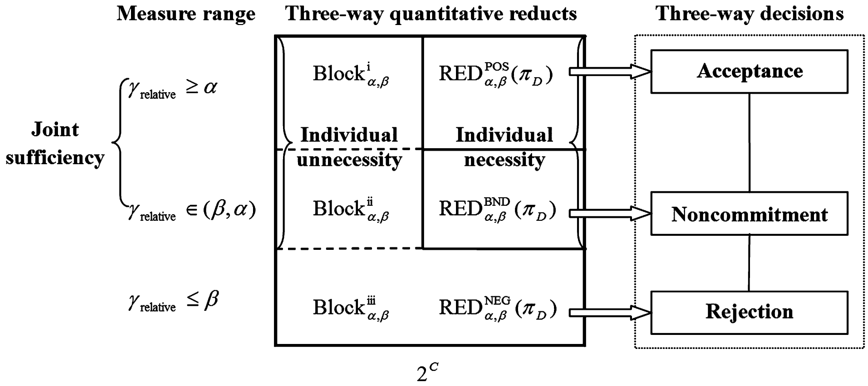

Fig. 2. The partitional mechanism, systematic structure, and three-way decisions of three-way quantitative reducts.

The three blocks are labeled in Fig. 2. Moreover, the positive and boundary quantitative reducts include only one block, respectively. The structure of three-way quantitative reducts is exactly described as follows.

Theorem 7. *Three-way γ_{relative}–(α, β) quantitative reducts are pairwise of entire 2^C. That is,*

$$\text{RED}^{\text{POS}}_{\alpha,\beta}(\pi_D) \cup \text{RED}^{\text{BND}}_{\alpha,\beta}(\pi_D) \cup \text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D) = 2^C;$$
$$\begin{cases} \text{RED}^{\text{POS}}_{\alpha,\beta}(\pi_D) \cap \text{RED}^{\text{BND}}_{\alpha,\beta}(\pi_D) = \emptyset, \\ \text{RED}^{\text{POS}}_{\alpha,\beta}(\pi_D) \cap \text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D) = \emptyset, \\ \text{RED}^{\text{BND}}_{\alpha,\beta}(\pi_D) \cap \text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D) = \emptyset. \end{cases} \tag{44}$$

Based on Theorem 7, three-way γ_{relative}–(α, β) quantitative reducts exhibit a three-way division for their existing space 2^C. According to relevant constructional mechanisms, they naturally correspond to three-way decisions for the reduction action.

(1) The positive quantitative reduct $R^{\text{POS}}_{\alpha,\beta} \in \text{RED}^{\text{POS}}_{\alpha,\beta}(\pi_D)$ adopts the acceptance decision. That is, $R^{\text{POS}}_{\alpha,\beta}$ is viewed as a normal reduct to be quantitatively utilized, as is analyzed for the γ_{relative}–α quantitative reduct (in Section 3.2).
(2) The negative quantitative reduct $R^{\text{NEG}}_{\alpha,\beta} \in \text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D)$ adopts the rejection decision. That is, $R^{\text{NEG}}_{\alpha,\beta}$ is not a normal reduct to be quantitatively used.
(3) The boundary quantitative reduct $R^{\text{BND}}_{\alpha,\beta} \in \text{RED}^{\text{BND}}_{\alpha,\beta}(\pi_D)$ adopts the noncommitment decision. In fact, $R^{\text{BND}}_{\alpha,\beta}$ has the uncertainty and possibility for the affirmation and negation of the normal quantitative reduct, and its nonjudgement leads to that $R^{\text{BND}}_{\alpha,\beta}$ cannot be accepted or rejected.

In a word, three-way quantitative reducts need to reasonably adopt three-way decisions in the quantitative environment, and this fundamental conclusion is also marked in Fig. 2.

Although three-way γ_{relative}–(α, β) quantitative reducts are piecewise in 2^C, they can have common attributes in C. Next, their relationships, mainly those of the positive and boundary quantitative reducts, are analyzed at the C level.

Theorem 8. *With respect to three-way γ_{relative}–(α, β) quantitative reducts, a positive reduct with at least two attributes properly includes a boundary reduct or a nonempty negative reduct, while a boundary reduct with at least two attributes properly includes a nonempty negative reduct. That is,*

$$(1)\ \forall R^{\text{POS}}_{\alpha,\beta} \in \text{RED}^{\text{POS}}_{\alpha,\beta}(\pi_D),\ |R^{\text{POS}}_{\alpha,\beta}| \geq 2,\ \exists R^{\text{BND}}_{\alpha,\beta} \in \text{RED}^{\text{BND}}_{\alpha,\beta}(\pi_D),\ s.t.,\ R^{\text{BND}}_{\alpha,\beta} \subset R^{\text{POS}}_{\alpha,\beta},$$
$$or,\ \exists \emptyset \neq R^{\text{NEG}}_{\alpha,\beta} \in \text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D),\ s.t.,\ R^{\text{NEG}}_{\alpha,\beta} \subset R^{\text{POS}}_{\alpha,\beta};$$
$$(2)\ \forall R^{\text{BND}}_{\alpha,\beta} \in \text{RED}^{\text{BND}}_{\alpha,\beta}(\pi_D),\ |R^{\text{BND}}_{\alpha,\beta}| \geq 2,\ \exists \emptyset \neq R^{\text{NEG}}_{\alpha,\beta} \in \text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D),\ s.t.,\ R^{\text{NEG}}_{\alpha,\beta} \subset R^{\text{BND}}_{\alpha,\beta}.$$

Proof. The joint sufficiency and individual necessity are first considered. Suppose $R \in \text{RED}^{\text{POS}}_{\alpha,\beta}(\pi_D)$.

Condition $(\text{S}^{\text{POS}}_{\alpha,\beta})$ implies $R \notin \text{RED}^{\text{BND}}_{\alpha,\beta}(\pi_D)$ and $R \notin \text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D)$, but Condition $(\text{N}^{\text{POS}}_{\alpha,\beta})$ implies $\gamma_{\text{relative}}(R - \{c\}) < \alpha$ for arbitrary $c \in R$. If R has only one attribute c, then $R - \{c\} = \emptyset$ is not a reduct. Otherwise, nonempty subset $R - \{c\}$ exhibits two cases by considering β.

(1) If $\gamma_{\text{relative}}(R-\{c\}) \in (\beta,\alpha)$, then $R-\{c\}$ satisfies Condition $(\text{S}^{\text{BND}}_{\alpha,\beta})$, so $R-\{c\}$ directly becomes or properly includes a boundary quantitative reduct.

(2) If $\gamma_{\text{relative}}(R-\{c\}) \le \beta$, then $R-\{c\} \in \text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D)$. Case $R \in \text{RED}^{\text{BND}}_{\alpha,\beta}(\pi_D)$ is similar. Concretely, $(\text{S}^{\text{BND}}_{\alpha,\beta})$ implies $R \notin \text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D)$; however, Condition $(\text{N}^{\text{BND}}_{\alpha,\beta})$ implies $\gamma_{\text{relative}}(R-\{c\}) \le \beta$ for arbitrary $c \in R$, so $R-\{c\} \in \text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D)$ if $R-\{c\} \neq \emptyset$.

Therefore, a multiple-attribute quantitative reduct with the higher γ_{relative} level is not a quantitative reduct with the lower γ_{relative} level, but the former can derive the latter by deleting at least an attribute. In other words, the above theorem holds. □

Based on Theorem 8, a positive quantitative reduct with at least two attributes may derive only the boundary quantitative reduct, only the negative quantitative reduct, or both the boundary and negative quantitative reducts. The three results depend on basic relationships between $\gamma_{\text{relative}}(R-\{c\})$ and β, as well as concrete cases of R and c. Whether a positive quantitative reduct derives a boundary quantitative reduct can be judged by the attribute existence for requirement $\gamma_{\text{relative}}(R-\{c\}) \in (\beta,\alpha)$.

Theorem 9. *If positive quantitative reduct $R^{\text{POS}}_{\alpha,\beta} \in \text{RED}^{\text{POS}}_{\alpha,\beta}(\pi_D)$ has only one attribute c, then $R^{\text{POS}}_{\alpha,\beta}-\{c\}=\emptyset$ is not a boundary quantitative reduct. Otherwise, $R^{\text{POS}}_{\alpha,\beta}$ has at least two attributes to exhibit two cases.*

(1) $R^{\text{POS}}_{\alpha,\beta}$ properly includes a boundary quantitative reduct if

$$\exists c \in R^{\text{POS}}_{\alpha,\beta},\ \gamma_{\text{relative}}(R^{\text{POS}}_{\alpha,\beta}-\{c\}) > \beta,\ i.e.,\ \max_{c\in R^{\text{POS}}_{\alpha,\beta}} \gamma_{\text{relative}}(R^{\text{POS}}_{\alpha,\beta}-\{c\}) > \beta. \tag{45}$$

(2) $R^{\text{POS}}_{\alpha,\beta}$ cannot properly include a boundary quantitative reduct if

$$\forall c \in R^{\text{POS}}_{\alpha,\beta},\ \gamma_{\text{relative}}(R^{\text{POS}}_{\alpha,\beta}-\{c\}) \le \beta,\ i.e.,\ \max_{c\in R^{\text{POS}}_{\alpha,\beta}} \gamma_{\text{relative}}(R^{\text{POS}}_{\alpha,\beta}-\{c\}) \le \beta. \tag{46}$$

On the basis of Theorem 8, Theorem 9 can be easily proved by the quantitative reduct definition. It provides a theoretical framework to seek a potential boundary quantitative reduct in a given positive quantitative reduct. A relevant algorithm (i.e., Algorithm 4) is further developed.

Algorithm 4 A constructional algorithm of a potential boundary quantitative reduct from a positive quantitative reduct.

Input: A decision table T with thresholds α,β, and a positive quantitative reduct $R^{\text{POS}}_{\alpha,\beta} \in \text{RED}^{\text{POS}}_{\alpha,\beta}(\pi_D)$;

Output: A boundary quantitative reduct $R^{\text{BND}}_{\alpha,\beta} \in \text{RED}^{\text{BND}}_{\alpha,\beta}(\pi_D)$ satisfying $R^{\text{BND}}_{\alpha,\beta} \subset R^{\text{POS}}_{\alpha,\beta}$, or a judgement that no boundary quantitative reducts in $R^{\text{POS}}_{\alpha,\beta}$.

if $|R^{\text{POS}}_{\alpha,\beta}|=1$ **then**
  return The judgement that no boundary quantitative reducts in $R^{\text{POS}}_{\alpha,\beta}$.
else
  for $c \in R^{\text{POS}}_{\alpha,\beta}$ **do**
    if $\gamma_{\text{relative}}(R^{\text{POS}}_{\alpha,\beta}-\{c\}) > \beta$ **then**
      Let $R^{\text{BND}}_{\alpha,\beta} = R^{\text{POS}}_{\alpha,\beta}-\{c\}$.
      for $c_* \in R^{\text{POS}}_{\alpha,\beta}-\{c\}$ **do**
        if $\gamma_{\text{relative}}(R^{\text{BND}}_{\alpha,\beta}-\{c_*\}) > \beta$ **then**
          $R^{\text{BND}}_{\alpha,\beta} \leftarrow R^{\text{BND}}_{\alpha,\beta}-\{c_*\}$;
        end if
      end for
      return $R^{\text{BND}}_{\alpha,\beta}$.
    end if
  end for
  return The judgement that no boundary quantitative reducts in $R^{\text{POS}}_{\alpha,\beta}$.
end if

In Algorithm 4, $|R^{\text{POS}}_{\alpha,\beta}|=1$ represents only one attribute in the positive quantitative reduct, so there are no internal boundary quantitative reducts. The external "for" loop aims to find out an attribute c to satisfy the basic condition $\gamma_{\text{relative}}(R^{\text{POS}}_{\alpha,\beta}-\{c\}) > \beta$, and only the existence of attribute c determines the existence of a boundary quantitative reduct and the relevant operation; as a result, the second nonexistence judgement after the "for" loop corresponds to that multiple-element $R^{\text{POS}}_{\alpha,\beta}$ includes only the negative quantitative reduct. In the main case with existing attribute c, $R^{\text{POS}}_{\alpha,\beta}-\{c\}$ becomes the inspection basis; by deleting unnecessary attribute c_* in $R^{\text{POS}}_{\alpha,\beta}-\{c\}$, the internal "for" loop aims to achieve the joint sufficiency $(\text{S}^{\text{BND}}_{\alpha,\beta})$ and individual necessity $(\text{N}^{\text{BND}}_{\alpha,\beta})$ to obtain boundary quantitative reduct $R^{\text{BND}}_{\alpha,\beta}$, where

$$R^{\text{BND}}_{\alpha,\beta} \subseteq R^{\text{POS}}_{\alpha,\beta}-\{c\} \subset R^{\text{POS}}_{\alpha,\beta}.$$

The final boundary quantitative reduct depends on the orders of two "for" test sequences. The measurement, comparison, and renewal related to $|R^{\mathrm{POS}}_{\alpha,\beta}| = m''$ are viewed as basic operations to analyze the worst case, where $R^{\mathrm{BND}}_{\alpha,\beta}$ exists. Thus, the outside loop runs m'' times to finally acquire a starting point for compression, and the time complexity concerns $2m'' + 1$; after reducing an attribute, the inside loop runs $m'' - 1$ times to implement the compression, where a single attribute set is finally achieved by deleting $m'' - 2$ attributes, and the time complexity involves $3(m'' - 1) - 1$; therefore, the total time complexity becomes $T(m'') = 5m'' - 3 = o(m'')$ in the worst case. In a word, Algorithm 4 can effectively gain a concrete example or a nonexistence judgement for the internal boundary quantitative reduct, when given an arbitrary positive quantitative reduct.

Finally, the relationship of core attributes is provided for the positive and boundary quantitative reducts. The relevant computational formulation (Eq. (28)) and dual bounds (Eq. (42)) produce following Corollary 5. As a result, $\mathrm{CORE}^{\mathrm{POS}}_{\alpha,\beta}(\pi_D)$ and $\mathrm{CORE}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D)$ can have some overlaps at the C level. However, for useful attributes, $\mathrm{USEFUL}^{\mathrm{POS}}_{\alpha,\beta}(\pi_D)$ and $\mathrm{USEFUL}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D)$ cannot yield some certain relationships.

Corollary 5. *$\mathrm{CORE}^{\mathrm{POS}}_{\alpha,\beta}(\pi_D)$ and $\mathrm{CORE}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D)$ have the following relationships:*

$$\begin{aligned}
&\mathrm{CORE}^{\mathrm{POS}}_{\alpha,\beta}(\pi_D) \cap \mathrm{CORE}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D) = \{c \mid \gamma_{\mathrm{relative}}(C - \{c\}) \le \beta\};\\
&\mathrm{CORE}^{\mathrm{POS}}_{\alpha,\beta}(\pi_D) - \mathrm{CORE}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D) = \{c \mid \gamma_{\mathrm{relative}}(C - \{c\}) \in (\beta, \alpha)\};\\
&\mathrm{CORE}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D) - \mathrm{CORE}^{\mathrm{POS}}_{\alpha,\beta}(\pi_D) \subseteq \{c \mid \gamma_{\mathrm{relative}}(C - \{c\}) \ge \alpha)\}.
\end{aligned} \tag{47}$$

4.2. Three-way qualitative reducts

Three-way $\gamma_{\mathrm{relative}}$–$(\alpha, \beta)$ quantitative reducts have been established. We now turn to relevant three-way qualitative reducts, which can be produced by the associated degeneration technology.

The reduct target of classical qualitative reducts has an equivalent form:

$$\gamma = \gamma(\pi_D|\pi_C) \Longleftrightarrow \gamma_{\mathrm{relative}} = 1. \tag{48}$$

This result actually underlies the relevant development and degeneration of three-way attribute reducts. For the above development, the classical qualitative reducts are first generalized to the $\gamma_{\mathrm{relative}}$–$\alpha$ quantitative reducts with target $\gamma_{\mathrm{relative}} \ge \alpha$, i.e., the $\gamma_{\mathrm{relative}}$–$(\alpha, \beta)$ positive quantitative reduct; furthermore, the positive quantitative reducts are completely extended to three-way quantitative reducts in system $\gamma_{\mathrm{relative}}$–$(\alpha, \beta)$. In contrast, three-way quantitative reducts can utilize $(\alpha, \beta) = (1, 0)$ to degenerate into three-way qualitative reducts, where $\gamma_{\mathrm{relative}} = 0$ deduces $\gamma = 0$ while $\emptyset$ corresponds to the reduction negation; accordingly, the positive quantitative reducts degenerate into the classical qualitative reducts. This degeneration strategy can naturally achieve three-way qualitative reducts and their relevant results via the established platform of three-way quantitative reducts. To formally link up the classical qualitative reducts, the initial γ style (rather than the equivalent $\gamma_{\mathrm{relative}}$ form) is mainly adopted for three-way qualitative reducts, and they are first defined by referring to three-way quantitative reducts (Definition 8).

Definition 9. Three-way qualitative reducts concern the positive, boundary, and negative reducts. (1) First, the positive qualitative reducts are exactly the classical qualitative reducts defined in Definition 3. For systematic consistency, new symbols ($\mathrm{S}^{\mathrm{POS}}$), ($\mathrm{N}^{\mathrm{POS}}$), and ($\mathrm{N}'^{\mathrm{POS}}$) replace previous (S), (N), and (N′), respectively; the sets of attribute reducts and core/useful attributes are updated by:

$$\begin{aligned}
\mathrm{RED}^{\mathrm{POS}}(\pi_D) &= \mathrm{RED}(\pi_D),\\
\mathrm{CORE}^{\mathrm{POS}}(\pi_D) &= \mathrm{CORE}(\pi_D),\\
\mathrm{USEFUL}^{\mathrm{POS}}(\pi_D) &= \mathrm{USEFUL}(\pi_D).
\end{aligned} \tag{49}$$

(2) Then, a subset of condition attributes $R \subseteq C$ is a boundary qualitative reduct of C if it satisfies two conditions:

$$\begin{aligned}
&(\mathrm{S}^{\mathrm{BND}})\ \ \gamma(\pi_D|\pi_R) \in (0, \gamma(\pi_D|\pi_C)),\\
&(\mathrm{N}^{\mathrm{BND}})\ \ \forall c \in R(\gamma(\pi_D|\pi_{R-\{c\}}) = 0).
\end{aligned}$$

The set of boundary qualitative reducts is denoted by $\mathrm{RED}^{\mathrm{BND}}(\pi_D)$. Accordingly, the sets of core and useful attributes are denoted and defined by:

$$\begin{aligned}
\mathrm{CORE}^{\mathrm{BND}}(\pi_D) &= \cap \mathrm{RED}^{\mathrm{BND}}(\pi_D),\\
\mathrm{USEFUL}^{\mathrm{BND}}(\pi_D) &= \cup \mathrm{RED}^{\mathrm{BND}}(\pi_D).
\end{aligned} \tag{50}$$

(3) Finally, a subset of condition attributes $R \subseteq C$ is a negative qualitative reduct of C if it is neither a positive qualitative reduct nor a boundary qualitative reduct. The set of negative qualitative reducts is denoted by $\mathrm{RED}^{\mathrm{NEG}}(\pi_D)$, and

$$\mathrm{RED}^{\mathrm{NEG}}(\pi_D) = 2^C - (\mathrm{RED}^{\mathrm{POS}}(\pi_D) \cup \mathrm{RED}^{\mathrm{BND}}(\pi_D)). \tag{51}$$

In Definition 9, three-way qualitative reducts are established by measure γ and its maximum $\gamma(\pi_D|\pi_C)$ and minimum 0. The positive qualitative reducts have monotonic target $\gamma = \gamma(\pi_D|\pi_C)$ to implement the complete certainty and accuracy for attribute reduction; the boundary qualitative reducts have nonmonotonic target $\gamma \in (0, \gamma(\pi_D|\pi_C))$ to represent the complete uncertainty and large possibility; while the negative qualitative reducts make the surplus collection. The positive qualitative reducts have been studied by the classical qualitative reducts. For the boundary qualitative reducts, the individual necessity ($\mathrm{N}^{\mathrm{BND}}$) has an equivalent description:

$(\mathrm{N}'^{\,\mathrm{BND}})$ $\forall R' \subset R(\gamma(\pi_D|\pi_{R'}) = 0)$.

Moreover, the set of core attributes exhibits a form of lower and upper bounds:

$$\begin{aligned} &\{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) = \gamma(\pi_D|\pi_C)\} \\ \subseteq\ &\mathrm{CORE}^{\mathrm{BND}}(\pi_D) \\ \subseteq\ &\{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) \notin (0, \gamma(\pi_D|\pi_C))\} \\ =\ &\{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) = 0\} \cup \{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) = \gamma(\pi_D|\pi_C)\}. \end{aligned} \tag{52}$$

The negative qualitative reducts mainly consist of three blocks, i.e.,

$$\begin{aligned} &\mathrm{RED}^{\mathrm{NEG}}(\pi_D) = \mathrm{Block}^{\mathrm{i}} \cup \mathrm{Block}^{\mathrm{ii}} \cup \mathrm{Block}^{\mathrm{iii}}, \\ &\begin{cases} \mathrm{Block}^{\mathrm{i}} = \{A \mid \gamma(\pi_D|\pi_A) = \gamma(\pi_D|\pi_C)\} - \mathrm{RED}^{\mathrm{POS}}(\pi_D), \\ \mathrm{Block}^{\mathrm{ii}} = \{A \mid \gamma(\pi_D|\pi_A) \in (0, \gamma(\pi_D|\pi_C))\} - \mathrm{RED}^{\mathrm{BND}}(\pi_D), \\ \mathrm{Block}^{\mathrm{iii}} = \{A \mid \gamma(\pi_D|\pi_A) = 0\}. \end{cases} \end{aligned} \tag{53}$$

$\mathrm{Block}^{\mathrm{i}}$ collects the subset which reaches the joint sufficient ($\mathrm{S}^{\mathrm{POS}}$) rather than individual necessity ($\mathrm{N}^{\mathrm{POS}}$) (with respect to the positive qualitative reducts); $\mathrm{Block}^{\mathrm{ii}}$ is similar for Conditions ($\mathrm{S}^{\mathrm{BND}}$) and ($\mathrm{N}^{\mathrm{BND}}$) (with respect to the boundary qualitative reducts); and $\mathrm{Block}^{\mathrm{iii}}$ collects the surplus subset with minimum $\gamma = 0$. Furthermore, the three-way qualitative reducts completely classify 2^C according to Theorem 7 (or Fig. 2).

Corollary 6. *Three-way qualitative reducts are pairwise of* 2^C*. That is,*

$$\begin{aligned} &\mathrm{RED}^{\mathrm{POS}}(\pi_D) \cup \mathrm{RED}^{\mathrm{BND}}(\pi_D) \cup \mathrm{RED}^{\mathrm{NEG}}(\pi_D) = 2^C; \\ &\begin{cases} \mathrm{RED}^{\mathrm{POS}}(\pi_D) \cap \mathrm{RED}^{\mathrm{BND}}(\pi_D) = \emptyset, \\ \mathrm{RED}^{\mathrm{POS}}(\pi_D) \cap \mathrm{RED}^{\mathrm{NEG}}(\pi_D) = \emptyset, \\ \mathrm{RED}^{\mathrm{BND}}(\pi_D) \cap \mathrm{RED}^{\mathrm{NEG}}(\pi_D) = \emptyset. \end{cases} \end{aligned} \tag{54}$$

Based on the constructional mechanism and partition result, three-way qualitative reducts naturally adopt three-way decisions in the qualitative environment. As the classical qualitative reducts, the positive qualitative reducts in $\mathrm{RED}^{\mathrm{POS}}(\pi_D)$ are related to the reduction affirmation and thus gain the acceptance decision; as a generalized notion, the negative qualitative reducts in $\mathrm{RED}^{\mathrm{NEG}}(\pi_D)$ are related to the complete reduction negation and thus gain the rejection decision; as a new type of qualitative reducts, the qualitative boundary reducts in $\mathrm{RED}^{\mathrm{BND}}(\pi_D)$ are related to the reduction uncertainty and possibility and thus gain the noncommitment decision.

At the C level, Theorems 8, and 9, Corollary 5 of three-way quantitative reducts can derive corresponding conclusions of three-way qualitative reducts. As an example, Theorem 9 produces following Corollary 7. This theoretical result clarifies the potential derivation from the positive to boundary qualitative reducts, so it underlies a relevant structural algorithm, which is similar to previous Algorithm 4 for three-way quantitative reducts.

Corollary 7. *If the positive qualitative reduct* $R^{\mathrm{POS}} \in \mathrm{RED}^{\mathrm{POS}}(\pi_D)$ *has only one attribute* c*, then* $R^{\mathrm{POS}} - \{c\} = \emptyset$ *is not a boundary qualitative reduct. Otherwise,* R^{POS} *has at least two attributes to exhibit two cases.*
(1) R^{POS} *properly includes a boundary qualitative reduct if*

$$\exists c \in R^{\mathrm{POS}},\ \gamma(\pi_D|\pi_{R^{\mathrm{POS}}-\{c\}}) > 0,\ i.e.,\ \max_{c \in R^{\mathrm{POS}}} \gamma(\pi_D|\pi_{R^{\mathrm{POS}}-\{c\}}) > 0. \tag{55}$$

(2) R^{POS} *cannot properly include a boundary qualitative reduct if*

$$\forall c \in R^{\mathrm{POS}},\ \gamma(\pi_D|\pi_{R^{\mathrm{POS}}-\{c\}}) = 0,\ i.e.,\ \max_{c \in R^{\mathrm{POS}}} \gamma(\pi_D|\pi_{R^{\mathrm{POS}}-\{c\}}) = 0. \tag{56}$$

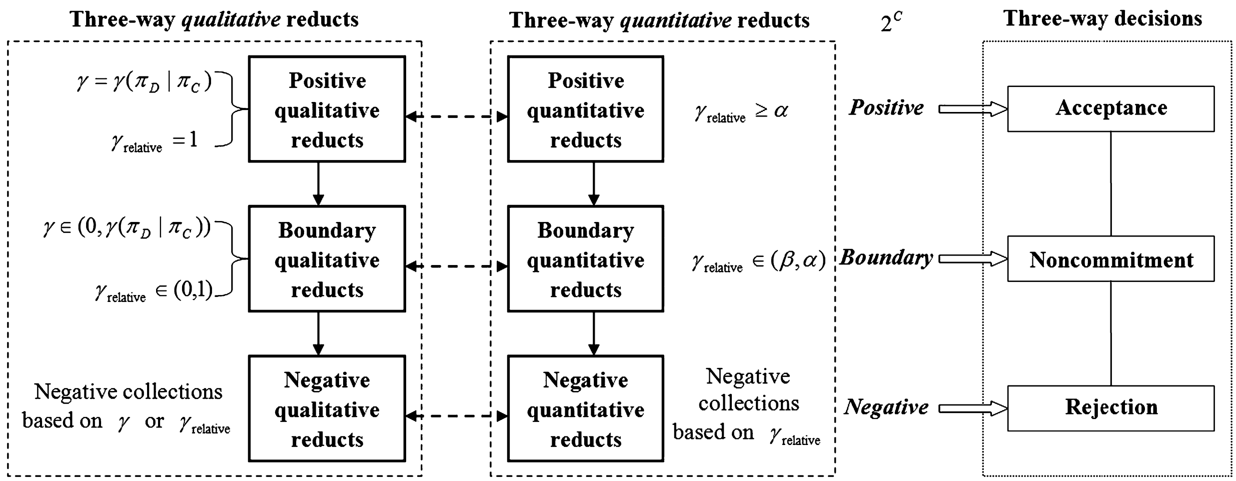

Fig. 3. A summary of three-way quantitative and qualitative reducts.

4.3. A summary of three-way quantitative and qualitative reducts

Thus far, three-way reducts are established in both the quantitative and qualitative patterns. Aiming at three-way quantitative and qualitative reducts, we next make their systematic summary, including their mutual relationship. For this purpose, a summary figure – Fig. 3 – is first provided.

According to Fig. 3, three-way reducts concern three contents.

(1) Three-way qualitative reducts can be equivalently described by the dependency degree γ and relative dependency degree γ_{relative}. The positive and boundary qualitative reducts originate from high target $\gamma = \gamma(\pi_D|\pi_C)$ (or $\gamma_{\text{relative}} = 1$) and moderate target $\gamma \in (0, \gamma(\pi_D|\pi_C))$ (or $\gamma_{\text{relative}} \in (0, 1)$), respectively, while the negative qualitative reducts mainly implement the negative collection based on γ (or γ_{relative}).
(2) Three-way quantitative reducts are described by only the relative dependency degree. The positive and boundary quantitative reducts originate from high target $\gamma_{\text{relative}} \geq \alpha$ and moderate target $\gamma_{\text{relative}} \in (\beta, \alpha)$, respectively, while the negative quantitative reducts mainly implement the negative collection based on γ_{relative}.
(3) The three-way reducts, including both the qualitative and quantitative patterns, completely divide 2^C, which is the existing space of attribute reducts. The positive, boundary, and negative reducts adopt the acceptance, noncommitment, and rejection decisions, respectively, for the attribute reduction action. As a result, three-way reducts closely adhere to attribute reducts and three-way decisions.

For measures, novel γ_{relative} improves initial γ, and the former can fully replace the latter to implement the function for qualitative reducts. In other words, three-way quantitative and qualitative reducts can be uniformly described by γ_{relative} (rather than γ). Next, we want to explain the special case with $\gamma(\pi_D|\pi_C) = 0$. There, any nonempty attribute subset satisfies $\gamma = 0 = \gamma(\pi_D|\pi_C)$ and $\gamma_{\text{relative}} = 1$. With the addition of the individual necessity, the three-way quantitative and qualitative reducts have the same results. The positive reducts include only any single element set, the boundary reducts exhibit emptiness, while the negative reducts contain all the surplus subsets in 2^C (including $\emptyset$). That is, the sets of positive, boundary, and negative reducts exhibit $\{\{c\} \mid c \in C\}$, $\emptyset$, and $2^C - \{\{c\} \mid c \in C\}$, respectively. These results become natural for attribute reducts. In particular, the positive reducts can be realized by $\gamma_{\text{relative}} = 1$ rather than $\gamma_{\text{relative}} = 0$. This correlation of the reduct and measure clarifies the rationality of the γ_{relative} definition (Definition 5) as well as the necessary difference of non-isomorphism (Eq. (23)).

Three-way reducts exhibit clear systematic structures at the 2^C level. They can be further analyzed at the C level. In particular, their reduct derivation (which originates from the measure difference or target strength) becomes an emphasis, for the quantitative and qualitative patterns.

(1) For the pattern interior, three-way reducts exhibit the potential derivation from the higher measure degree to the lower measure degree. The potential derivation relationships are labeled in Fig. 3 by solid arrows. The quantitative derivation is illustrated by Theorems 8, 9, and Algorithm 4, while the qualitative derivation is partly provided by Corollary 7.
(2) For the pattern connection, three-way reducts exhibit the expansion, approximation, and strength between the qualitative and quantitative patterns. These relationships are together labeled in Fig. 3 by dashed double-headed arrows. The relationships between the positive qualitative and quantitative reducts are revealed by Theorems 5, 6, and Algorithm 3. Next, relevant connections of the boundary/negative reducts are provided between the qualitative and quantitative patterns.

The boundary reducts have the nonmonotonic target with respect to the measure interval, thus becoming a new type of attribute reducts. By virtue of measure γ_{relative}, the quantitative target $\gamma_{\text{relative}} \in (\beta, \alpha)$ can produce the qualitative target $\gamma_{\text{relative}} \in (0, 1)$ by setting up $(\alpha, \beta) = (1, 0)$. Moreover, the former is stronger than the latter, i.e.,

$$\gamma_{\text{relative}} \in (\beta, \alpha) \Longrightarrow \gamma_{\text{relative}} \in (0, 1). \tag{57}$$

As a result, the following expansion and strength of boundary reducts become clear.

Proposition 1. *The boundary quantitative reducts expand the boundary qualitative reducts and can degenerate into the latter by setting up* $(\alpha, \beta) = (1, 0)$.

Lemma 3. *For the joint sufficiency of boundary reducts, the quantitative condition strengthens the qualitative condition; for the individual necessity of boundary reducts, the quantitative condition weakens the qualitative condition. Concretely, that R satisfies Condition* $(\mathrm{S}^{\mathrm{BND}}_{\alpha,\beta})$ *deduces that R satisfies Condition* $(\mathrm{S}^{\mathrm{BND}})$*, while that R satisfies Condition* $(\mathrm{N}^{\mathrm{BND}}_{\alpha,\beta})/(\mathrm{N}'^{\mathrm{BND}}_{\alpha,\beta})$ *can be realized by that R satisfies Condition* $(\mathrm{N}^{\mathrm{BND}})/(\mathrm{N}'^{\mathrm{BND}})$.

Proposition 2. *The boundary quantitative reducts are stronger than the boundary qualitative reducts. They have relationships:*

$$\begin{aligned}&\forall R^{\mathrm{BND}}_{\alpha,\beta} \in \mathrm{RED}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D),\ \exists R^{\mathrm{BND}} \in \mathrm{RED}^{\mathrm{BND}}(\pi_D),\ s.t.,\ R^{\mathrm{BND}} \subseteq R^{\mathrm{BND}}_{\alpha,\beta};\\ &\mathrm{CORE}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D) \supseteq \mathrm{CORE}^{\mathrm{BND}}(\pi_D).\end{aligned} \tag{58}$$

In theory, the boundary quantitative reducts exhibit the expansion significance to include the boundary qualitative reducts. In practice, the boundary quantitative reducts depend on concrete thresholds to approximate and strengthen the boundary qualitative reducts. With respect to the boundary qualitative reducts, the boundary quantitative reducts exhibit the expansion, approach, and strengthening, which correspond to $(\alpha, \beta) = (1, 0)$, $(\alpha, \beta) \to (1, 0)$, and $(\alpha, \beta) \prec (1, 0)$ (which denotes $\alpha < 1$ and $\beta > 0$), respectively. According to Proposition 2, a boundary quantitative reduct can internally derive a boundary qualitative reduct; by simulating Algorithm 3, we can similarly develop a constructional algorithm, where incalculable $\mathrm{CORE}^{\mathrm{BND}}(\pi_D)$ needs to be not concerned.

The boundary reducts become a new type of reducts to exhibit some specific features, when compared to the positive reducts. The boundary and positive reducts focus on a nonmonotonic target with the moderate measure interval and a monotonic target with the high measure value, respectively. For the reduct strength, the boundary quantitative reducts are stronger than the boundary qualitative reducts, while the opposite holds for the positive reducts, i.e., the positive quantitative reducts are weaker than the positive qualitative reducts.

Thus far, we uncover the connections of the positive/boundary reducts between the quantitative and qualitative patterns. Furthermore, we can further analyze the connections of the negative reducts. At the 2^C level, the positive/boundary quantitative and qualitative reducts can have the usual set relationship, and this conclusion can be theoretically clarified by the strength relationship between the positive/boundary quantitative and qualitative reducts, i.e., Theorem 6 and Proposition 2. As a result, the negative quantitative and qualitative reducts may have the usual set relationships. At the C level, the positive/boundary quantitative and qualitative reducts mainly concern the expansion, approximation, and strength, so the negative quantitative and qualitative reducts have the corresponding properties; however, the strength and derivation have no significance for the negative collection of reducts.

Corollary 8. *The negative quantitative reducts expand the negative qualitative reducts and can degenerate into the latter by setting up* $(\alpha, \beta) = (1, 0)$.

For the negative quantitative and qualitative reducts, their theoretical expansion is presented in Corollary 8 by $(\alpha, \beta) = (1, 0)$, and their practical approximation can be explained by $(\alpha, \beta) \to (1, 0)$. In particular, the negative reducts consist of six blocks (Eqs. (43) and (53)), i.e.,

$$\mathrm{Block}^{\mathrm{i}}_{\alpha,\beta},\ \mathrm{Block}^{\mathrm{ii}}_{\alpha,\beta},\ \mathrm{Block}^{\mathrm{iii}}_{\alpha,\beta},\ \text{and}\ \mathrm{Block}^{\mathrm{i}},\ \mathrm{Block}^{\mathrm{ii}},\ \mathrm{Block}^{\mathrm{iii}}.$$

These six blocks correspondingly exhibit the expansion and approximation. Moreover, Blocks i and ii are closely related to conditions of the positive and boundary reducts, and their set relationships can be obtained by Lemmas 2 and 3; based on the measure degree, the inclusion relationship of block iii can be achieved, i.e.,

$$\{A \mid \gamma_{\text{relative}}(A) \le \beta\} \supseteq \{A \mid \gamma(\pi_D|\pi_A) = 0\}.$$

Therefore, these blocks exhibit following results at the 2^C level, although their collections – $\mathrm{RED}^{\mathrm{NEG}}_{\alpha,\beta}(\pi_D)$ and $\mathrm{RED}^{\mathrm{NEG}}(\pi_D)$ – usually have no inclusion or extension relationships.

Proposition 3. *For the negative reducts, their three internal blocks exhibit the inclusion or extension between the quantitative and qualitative patterns. That is,*

$$\begin{aligned}&\text{Block}^{\text{i}}_{\alpha,\beta} \supseteq \text{Block}^{\text{i}},\\ &\text{Block}^{\text{ii}}_{\alpha,\beta} \subseteq \text{Block}^{\text{ii}},\\ &\text{Block}^{\text{iii}}_{\alpha,\beta} \supseteq \text{Block}^{\text{iii}}.\end{aligned} \tag{59}$$

From a viewpoint of measure γ_{relative}, the above relationship between the quantitative and qualitative reducts originates from the relationship between quantitative thresholds (α, β) and $(1, 0)$. The relevant conclusion can be generalized for three-way quantitative reducts by relationships $\alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$. As an example, Proposition 4 is next provided to extend previous Corollary 4, where the positive quantitative reducts are described by $\alpha_1 \leq \alpha_2$.

Proposition 4. *If $0 \leq \beta_2 \leq \beta_1 < \alpha_1 \leq \alpha_2 \leq 1$, then the γ_{relative}–(α_1, β_1) positive/boundary quantitative reducts are weaker/stronger than the γ_{relative}–(α_2, β_2) positive/boundary quantitative reducts. Concretely, they exhibit two groups of relationships.*

$$\begin{aligned}&\forall R^{\text{POS}}_{\alpha_2,\beta_2} \in \text{RED}^{\text{POS}}_{\alpha_2,\beta_2}(\pi_D),\ \exists R^{\text{POS}}_{\alpha_1,\beta_1} \in \text{RED}^{\text{POS}}_{\alpha_1,\beta_1}(\pi_D),\ s.t.,\ R^{\text{POS}}_{\alpha_1,\beta_1} \subseteq R^{\text{POS}}_{\alpha_2,\beta_2};\\ &\text{CORE}^{\text{POS}}_{\alpha_2,\beta_2}(\pi_D) \supseteq \text{CORE}^{\text{POS}}_{\alpha_1,\beta_1}(\pi_D).\end{aligned} \tag{60}$$

$$\begin{aligned}&\forall R^{\text{BND}}_{\alpha_1,\beta_1} \in \text{RED}^{\text{BND}}_{\alpha_1,\beta_1}(\pi_D),\ \exists R^{\text{BND}}_{\alpha_2,\beta_2} \in \text{RED}^{\text{BND}}_{\alpha_2,\beta_2}(\pi_D),\ s.t.,\ R^{\text{BND}}_{\alpha_2,\beta_2} \subseteq R^{\text{BND}}_{\alpha_1,\beta_1};\\ &\text{CORE}^{\text{BND}}_{\alpha_1,\beta_1}(\pi_D) \supseteq \text{CORE}^{\text{BND}}_{\alpha_2,\beta_2}(\pi_D).\end{aligned} \tag{61}$$

Note that Chen et al. [2] propose the three-way decision reduction by referring to reduction notions in [63]. There, three-way reducts depend on the quantitative model of decision-theoretic rough sets to preserve the positive, negative, and boundary regions by using the absolute cardinality measure, so they implement the indirect quantification and can exhibit nonempty overlaps in 2^C. In contrast, three-way reducts proposed in this paper, which never depend on quantitative models, mainly maintain the rank of γ_{relative} (or γ) by using the relative ratio measure, so they implement the direct quantification and can exhibit the partition feature in the existing space of reducts. Therefore, the latter three-way reducts are more closely related to quantitative reducts and three-way decisions.

The main use of attribute reducts is to provide rules in decision tables, so three-way reducts are finally interpreted from the rule perspective. As is previously pointed out, $\gamma(\pi_D|\pi_C)$ represents the inherit classification power to generate certain rules, and the classical reduct R mainly preserves the same classification ability to produce and simplify reasoning rules. As a result, initial $[x]_C \longrightarrow [x]_D$ is embodied by simpler $[x]_R \longrightarrow [x]_D$. Next, the three-way reducts with R are considered for the rule generation and simplification (i.e., $[x]_R \longrightarrow [x]_D$ from $[x]_C \longrightarrow [x]_D$), where $\gamma_{\text{relative}}(R)$ effectively locates $\gamma(\pi_D|\pi_R)$ in interval $[0, \gamma(\pi_D|\pi_C)]$.

(QL1) The positive qualitative reduct R is exactly the classical reduct, so R corresponds to the maximal classification ability $\gamma(\pi_D|\pi_R) = \gamma(\pi_D|\pi_C)$ to generate and simplify certainty rules.
(QL2) The boundary qualitative reduct R has a classification ability $\gamma(\pi_D|\pi_R) \in (0, \gamma(\pi_D|\pi_C))$ (but not the maximal one) to generate and simplify certainty rules.
(QL3) The negative qualitative reduct R – the surplus collection – cannot generate certainty rules because $\gamma(\pi_D|\pi_R) = 0$ implies no classification abilities, or cannot effectively simplify certainty rules because of the R redundancy.

The above results adhere to three-way qualitative reducts, and the three-way quantitative reducts are similarly analyzed as follows by adding a quantitative background with threshold (α, β).

(QN1) The positive quantitative reduct R corresponds to the high $\gamma(\pi_D|\pi_R) \in [\alpha\gamma(\pi_D|\pi_C), \gamma(\pi_D|\pi_C)]$ to generate and simplify certainty rules.
(QN2) The boundary quantitative reduct R has the middle classification ability $\gamma(\pi_D|\pi_R) \in (\beta\gamma(\pi_D|\pi_C), \alpha\gamma(\pi_D|\pi_C))$ to generate and simplify certainty rules.
(QN3) The negative quantitative reduct R – the surplus collection – is not suitable to generate certainty rules because of the low classification ability $\gamma(\pi_D|\pi_R) \in [0, \beta\gamma(\pi_D|\pi_C))$, or cannot effectively simplify certainty rules because of the R redundancy.

The classical reduct and three-way reducts mainly focus on the reasoning rules, but both carry different classification powers. The standard reduct is the positive qualitative reduct, and has the strongest classification power and the widest certainty rule coverage. However, the classical reduct could cause the overfitting of rule reasoning, especially in the data noise environment. Thus, the positive quantitative reduct adopts the error tolerance to have the relatively strong classification power and the relatively wide reasoning range. Note that the positive reducts concern mainly necessity rules. In contrast, the

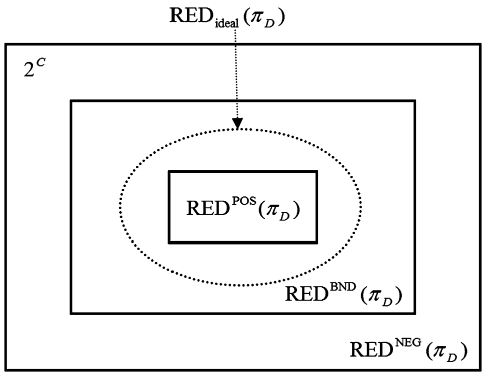

Fig. 4. The division mechanism of three-way qualitative reducts.

boundary reducts further decrease the classification power and necessity rule range, but from the dependency mechanism, they intensify possible rules to accompany necessity rules. Finally, the surplus negative reducts are usually not used for the rule generation and simplification. In short, three-way reducts extend and improve the classical reduct by virtue of the classification power, and they can induce more uncertain rules to be suitable for practical applications.

4.4. The superiority of three-way reducts in contrast to two-way reducts

Yao [46] thoroughly reveals the superiority of three-way decisions in contrast to two-way decisions. Herein, three-way reducts, which have the qualitative and quantitative patterns, also have their relevant superiority. In this subsection, their benefit is analyzed at the 2^C level, via the comparison to the two-way reducts.

Current studies mainly focus on only attribute reducts related to data analyses, not only in the qualitative environment but also in the quantitative environment. From the perspective of 2^C, there are only two classes to include the reduct and nonreduct, and they correspond to only two-way decisions with respect to acceptance and rejection for the reduction action. Thus, three-way reducts make the structure improvement and decision development in both the qualitative and quantitative patterns.

For the classical qualitative reducts, we can achieve the two-way reducts, which are in general, and they constitute two classes of 2^C:

$$\mathrm{RED}(\pi_D) \text{ and } \mathrm{NONRED}(\pi_D) = 2^C - \mathrm{RED}(\pi_D). \tag{62}$$

$\mathrm{RED}(\pi_D)$ is normally accepted and used, while $\mathrm{NONRED}(\pi_D)$ has the implication of strict reduction rejection. In the three-way qualitative reducts,

$$\begin{aligned} \mathrm{RED}(\pi_D) &= \mathrm{RED}^{\mathrm{POS}}(\pi_D), \\ \mathrm{NONRED}(\pi_D) &= \mathrm{RED}^{\mathrm{BND}}(\pi_D) \cup \mathrm{RED}^{\mathrm{NEG}}(\pi_D). \end{aligned} \tag{63}$$

In other words, $\mathrm{RED}(\pi_D)$ still exists to collect the positive qualitative reduct, while $\mathrm{NONRED}(\pi_D)$ is further divided into more detailed subparts: $\mathrm{RED}^{\mathrm{BND}}(\pi_D)$ and $\mathrm{RED}^{\mathrm{NEG}}(\pi_D)$. $\mathrm{RED}^{\mathrm{BND}}(\pi_D)$ consists of the boundary qualitative reduct to represent the reduction possibility and uncertainty; $\mathrm{RED}^{\mathrm{NEG}}(\pi_D)$ implies the reduction impossibility, and its three blocks reveal the concrete causes. The division mechanism of three-way qualitative reducts is presented in Fig. 4. To make the certainty/uncertainty analyses, let $\mathrm{RED}_{\mathrm{ideal}}(\pi_D)$ denote the ideal reduct set for applications. $\mathrm{RED}_{\mathrm{ideal}}(\pi_D)$ is located at the following range with lower and upper bounds:

$$\mathrm{RED}^{\mathrm{POS}}(\pi_D) \subseteq \mathrm{RED}_{\mathrm{ideal}}(\pi_D) \subseteq \mathrm{RED}^{\mathrm{POS}}(\pi_D) \cup \mathrm{RED}^{\mathrm{BND}}(\pi_D). \tag{64}$$

For ideal $\mathrm{RED}_{\mathrm{ideal}}(\pi_D)$, $\mathrm{RED}^{\mathrm{POS}}(\pi_D)$, $\mathrm{RED}^{\mathrm{NEG}}(\pi_D)$, and $\mathrm{RED}^{\mathrm{BND}}(\pi_D)$ represent the positive and negative certainty, and uncertainty, respectively, for the reduction action. For qualitative reducts, $\mathrm{RED}^{\mathrm{POS}}(\pi_D)$ becomes the necessary part to use the acceptance decision, $\mathrm{RED}^{\mathrm{BND}}(\pi_D)$ becomes the possible part to use the noncommitment decision, while $\mathrm{RED}^{\mathrm{NEG}}(\pi_D)$ becomes the negative part to use the rejection decision. Therefore, the three-way qualitative reducts more adhere to the practical methodology of three-way decisions.

To apply to the quantitative environment, three-way quantitative reducts introduce the accuracy approximation (or the fault tolerance) to generalize the three-way qualitative reducts. For the discriminant bar, they actually move specific $(0, 1)$ to usual (α, β) based on $\gamma_{\mathrm{relative}}$. With respect to the three-way qualitative division, the three-way quantitative division usually has no enlargement or lessening relationships in 2^C, but it has the similar structural mechanism. For example,

$$\mathrm{RED}^{\mathrm{POS}}_{\alpha,\beta}(\pi_D) \subseteq \mathrm{RED}^{*}_{\mathrm{ideal}}(\pi_D) \subseteq \mathrm{RED}^{\mathrm{POS}}_{\alpha,\beta}(\pi_D) \cup \mathrm{RED}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D). \tag{65}$$

Table 1
A decision table for illustrating three-way reducts.

OB	c_1	c_2	c_3	c_4	c_5	d
o_1	0	1	1	1	0	1
o_2	1	0	0	0	1	1
o_3	2	2	0	1	0	1
o_4	1	1	1	1	0	2
o_5	0	0	0	0	1	2
o_6	2	2	2	0	1	2
o_7	0	0	1	1	0	3
o_8	1	1	0	1	0	3
o_9	2	2	2	2	2	3

For ideal $\text{RED}^*_{\text{ideal}}(\pi_D)$, three-way quantitative reducts are related to the positive and negative certainty, and uncertainty in the quantitative pattern. Therefore, they have the quantitative advance and also adhere to the three-way decisions. In particular, the positive quantitative reducts are directly constructed by the dependency degree γ_{relative} and the approximate requirement $\gamma_{\text{relative}} \geq \alpha$, and they generalize the classical qualitative reducts to exhibit the practical advantage in quantitative applications. If only considering the positive quantitative reducts in the usual way, we can yield the two-way quantitative reducts, which are in general, and they constitute:

$$\text{RED}_\alpha(\pi_D) \text{ and } \text{NONRED}_\alpha(\pi_D) = 2^C - \text{RED}_\alpha(\pi_D). \tag{66}$$

By introducing bar β,

$$\begin{aligned} \text{RED}_\alpha(\pi_D) &= \text{RED}^{\text{POS}}_{\alpha,\beta}(\pi_D), \\ \text{NONRED}_\alpha(\pi_D) &= \text{RED}^{\text{BND}}_{\alpha,\beta}(\pi_D) \cup \text{RED}^{\text{NEG}}_{\alpha,\beta}(\pi_D). \end{aligned} \tag{67}$$

That is, the γ_{relative}–α quantitative reducts become the γ_{relative}–(α, β) positive quantitative reducts, and $\text{NONRED}_\alpha(\pi_D)$ is reasonably divided into two detailed descriptions with respect to the γ_{relative}–(α, β) boundary and negative quantitative reducts. Therefore, three-way quantitative reducts have the structure improvement and decision development for the two-way reducts.

In the above entire studies, usual assumption $0 \leq \beta < \alpha \leq 1$ (Eq. (36)) theoretically ensures the three-way number of attribute reducts, so three-way qualitative and quantitative reducts generally exist. In fact, the assumption can be enlarged to a reasonable range:

$$0 \leq \beta \leq \alpha \leq 1, \tag{68}$$

by adding a specific case $\alpha = \beta$. As a result, three-way quantitative reducts can exhibit the expansion and degeneration with respect to two-way reducts, where only the positive reducts are mainly utilized in the current style. The relevant result is described by the following theorem.

Theorem 10. *(1) Three-way quantitative reducts expand two-way quantitative reducts and can degenerate into the latter by setting up $\alpha = \beta$. (2) Three-way quantitative reducts expand two-way qualitative reducts and can degenerate into the latter by setting up $\alpha = \beta = 1$.*

Proof. (1) When $\alpha = \beta$, the positive and negative quantitative reducts correspond to $\text{RED}_\alpha(\pi_D)$ and $\text{NONRED}_\alpha(\pi_D)$, respectively, while the boundary quantitative reducts never emerge.

(2) When $\alpha = \beta = 1$, the positive and negative quantitative reducts correspond to $\text{RED}(\pi_D)$ and $\text{NONRED}(\pi_D)$, respectively, while the boundary quantitative reducts never emerge. □

In summary, three-way reducts have four aspects of superiority when compared to the two-way reducts, i.e., the reduct structure, decision semantics, quantitative application, and improved expansion.

5. Three-way reducts illustrations based on an example of decision tables

Aiming at three-way reducts, this section finally provides relevant illustrations by observing a decision table example. Moreover, the practical example will be used to extract several general conclusions regarding the attribute set redundancy and the dependency measure distribution. The concrete decision table is given in Table 1. There, $OB = \{o_1, ..., o_9\}$, $C = \{c_1, c_2, c_3, c_4, c_5\}$, $D = \{d\}$. According to the basic calculation,

$$\begin{aligned} \pi_C &= \{\{o_1\}, \{o_2\}, ..., \{o_9\}\}, \\ \pi_D &= \{\{o_1, o_2, o_3\}, \{o_4, o_5, o_6\}, \{o_7, o_8, o_9\}\}, \end{aligned}$$

i.e., π_C consists of nine granules with the single element style while π_D consists of three specific decision classes.

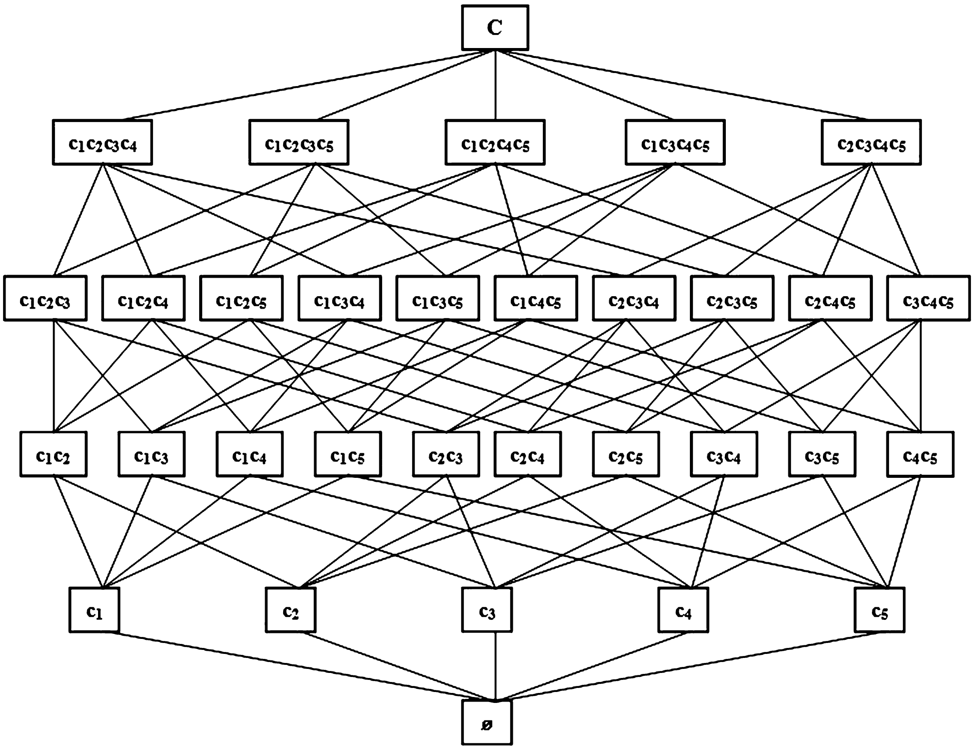

Fig. 5. Lattice $(2^C, \subseteq)$.

5.1. Lattice observations based on measures

As a basis, Table 1 is first used to make some observations for relevant lattices, which are related to the dependency degree γ and relative dependency degree γ_{relative}.

C consists of five condition attributes, so 2^C has $2^5 = 32$ attribute subsets of C to constitute lattice $(2^C, \subseteq)$. The relevant lattice structure is presented in Fig. 5.

The thirty-two attribute subsets produce sixteen classifications to constitute lattice $(2^{\pi_C}, \Leftarrow)$. The transformation between lattices $(2^C, \subseteq)$ and $(2^{\pi_C}, \Leftarrow)$ is completed by homomorphism mapping π. The relevant lattice structure of $(2^{\pi_C}, \Leftarrow)$ is presented in Fig. 6.

In Fig. 6, each node has three lines of data. The first line marks the classification and its serial number, the second line marks the subset origination, and the third line denotes the result of the positive region and measures γ, γ_{relative}. As an example, we explain the data of node 8.

(1) As shown in the line 1, the 8th classification refers to

$$o_1o_7, o_4o_8, \ldots = \{\{o_1, o_7\}, \{o_4, o_8\}, \{o_2\}, \{o_3\}, \{o_5\}, \{o_6\}, \{o_9\}\},$$

which consists of two granules with dual elements and five surplus granules with a single element.

(2) As shown in the line 2, the 8th classification comes from three subsets: $\{c_1, c_4\}$, $\{c_1, c_5\}$, and $\{c_1, c_4, c_5\}$, which are simply denoted by c_1c_4, c_1c_5, and $c_1c_4c_5$, respectively. In particular, $\{c_1, c_4, c_5\}$, which is collected in the bracketed form $(c_1c_4c_5)$, becomes redundant, because its proper subset $\{c_1, c_4\}/\{c_1, c_5\}$ can express the same classification. As a result, $\{c_1, c_4, c_5\}$ never becomes any reduct in the narrow sense, such as the positive and boundary reducts.

(3) As shown in the line 3, the 8th classification produces the positive region $\{o_2, o_3, o_5, o_6, o_9\}$ (which is denoted by $o_2o_3o_5o_6o_9$), dependency degree 5/9, and relative dependency degree 5/9. In this example, the two types of measures become equal, i.e., $\gamma_{\text{relative}} = \gamma$, because of $\gamma(\pi_D|\pi_C) = 1$.

According to the five condition attributes, Fig. 5 has six levels with thirty-two attribute subsets, while Fig. 6 has five levels with sixteen classifications. In general, if $|C| = m$ then lattice $(2^C, \subseteq)$ necessarily has $m+1$ levels with 2^m subsets, while $(2^{\pi_C}, \Leftarrow)$ has at most $m+1$ levels with at most 2^m classifications. The final levels of subset classifications mainly depend on the concrete granulation structure of attributes. For a given attribute number, the lattice structure of $(2^C, \subseteq)$ is stable, but mapping π can produce multiple structures of $(2^{\pi_C}, \Leftarrow)$ according to the formal structure of the concrete decision table. For attribute reducts, $(2^{\pi_C}, \Leftarrow)$ becomes the analysis basis while $(2^C, \subseteq)$ provides the accessible implement. By gathering

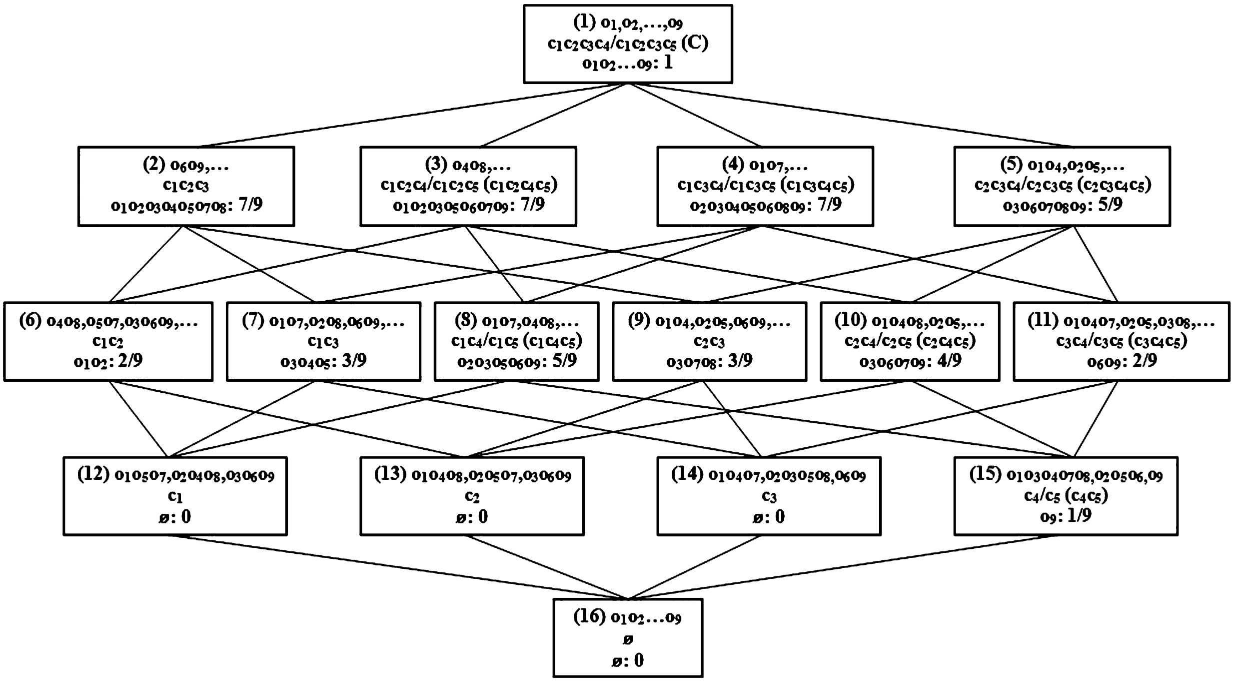

Fig. 6. Lattice $(2^{\pi_C}, \Leftarrow)$ and relevant classifications, subsets, regions, and measures.

information of subsets and classifications, Fig. 6 also reflects the homomorphic feature of mapping π. Moreover, all eight redundant subsets in the brackets of Fig. 6 can be collected and saved in the negative reducts, i.e.,

$$C, \{c_1, c_2, c_4, c_5\}, \{c_1, c_3, c_4, c_5\}, \{c_2, c_3, c_4, c_5\}, \{c_1, c_4, c_5\}, \{c_2, c_4, c_5\}, \{c_3, c_4, c_5\}, \{c_4, c_5\}$$
$$\in \mathrm{RED}^{\mathrm{NEG}}(\pi_D), \mathrm{RED}^{\mathrm{NEG}}_{\alpha,\beta}(\pi_D). \tag{69}$$

The redundant subsets can deduce a theoretical conclusion.

Proposition 5. *If an attribute subset and its proper subset have the same classification, then it is absolutely redundant and never belongs to any reduct in the narrow sense. For the generalized notion of three-way reducts, the absolutely redundant subset belongs to only the negative reducts.*

According to Proposition 5, the absolutely redundant subset, which originates from the formal structure of decision tables, never satisfies arbitrary individual necessity to be denied for usual attribute reducts. It can be first determined to implement the optimal calculation for three-way reducts. Furthermore, the opposite of absolute redundance is the relative redundance. According to concrete individual necessity, the relatively redundant subset becomes redundant in terms of its proper subset and relevant subclassification.

The sixteen classifications produce thirteen positive regions and eight measure values. In view of $\gamma_{\text{relative}} = \gamma$,

$$0 \leq 1/9 \leq 2/9 \leq 3/9 \leq 4/9 \leq 5/9 \leq 7/9 \leq 1 \tag{70}$$

constitutes equivalent lattices $(\{\gamma(A) \mid A \subseteq C\}, \leq)$ and $(\{\gamma_{\text{relative}}(A) \mid A \subseteq C\}, \leq)$, which are a sublattice of lattice $([0, 1], \leq)$. The relevant transformations from lattice $(2^C, \subseteq)$ are realized by homomorphic mappings γ and γ_{relative}, respectively. Moreover, the lattice equality reflects the isomorphism of lattices which originates from $\gamma(\pi_D|\pi_C) = 1 \neq 0$. The final measure results are mainly related to the positive region, where classification π_D is added for consideration. The information of regions and measures are also marked in Fig. 6 by line 3 of data notes. As a result, the order-preservation of homomorphic mappings γ, γ_{relative}, and isomorphic mapping f, i.e., the measure monotonicity in Eqs. (6), (16), and (22), are illustrated by Fig. 6. In particular, Fig. 6 contains enough information to underlie the follow-up illustrations for three-way reducts.

Herein, eight measure values are related to number nine, which is related to the cardinality of universe OB or region $\mathrm{POS}(\pi_D|\pi_C)$. In general, we can give a following distribution conclusion for γ, i.e., Proposition 6. Furthermore, the γ_{relative} distribution is similarly deduced in Corollary 9, mainly by the γ_{relative} connotation with respect to the positive region (Theorem 3). The two conclusions of measure distribution have some comparability. For example, the measure change depends on the positive region, whose nonempty supplementarity necessarily involves at least two condition attributes.

Proposition 6. *For the dependency degree γ, if $|OB| = n$ then it exhibits an integer multiple with respect to factor $1/n$, i.e., k/n ($k \in \{0, 1, ..., n\}$). Its minimum 0 and maximum can be obtained at least by subsets $\emptyset$ and C, respectively. Its maximum is not greater than 1, and its second maximum is not greater than $(n-2)/n$.*

Corollary 9. *For the relative dependency degree γ_{relative}, if $|\text{POS}(\pi_D|\pi_C)| = n'$ then it exhibits an integer multiple with respect to factor $1/n'$, i.e., k/n' ($k \in \{0, 1, ..., n'\}$). Its minimum 0 and maximum 1 can be obtained at least by subsets $\emptyset$ and C, respectively, and its second maximum is not greater than $(n'-2)/n'$.*

5.2. Illustrations for three-way quantitative reducts

On the basis of Table 1 and Fig. 6, three-way quantitative reducts are calculated in this subsection to clarify their internal relationships.

Except for the application requirement, the measure distribution can be considered for the threshold determination. For γ_{relative}, eight ordered values $0, 1/9, 2/9, 3/9, 4/9, 5/9, 7/9, 1$ correspond to 4, 1, 2, 2, 1, 2, 3, 1 classifications, respectively. Thus, we mainly take $(\alpha, \beta) = (7/9, 2/9)$ to illustrate three-way quantitative reducts, and relevant results based on $(\alpha, \beta) = (5/9, 3/9)$ are simply and finally provided.

For the reduct target $\gamma_{\text{relative}} \in [7/9, 1]$, only classifications (1)–(4) reach the joint sufficiency ($\text{S}^{\text{POS}}_{\alpha,\beta}$) but classification (1) never satisfies the individual necessity ($\text{N}^{\text{POS}}_{\alpha,\beta}$). For classification (1), subclassification (2)/(3)/(4) can also reach (S^{POS}) with respect to $[7/9, 1]$. As a result, the subsets related to classification (1) correspond to not the positive qualitative reducts but the negative qualitative reducts; in fact, C and $\{c_1, c_2, c_3, c_4\}/\{c_1, c_2, c_3, c_5\}$ become the absolutely and relatively redundant, respectively. By extracting the positive quantitative reducts from classifications (2)–(4), the sets of attribute reducts and core/useful attributes are obtained:

$$\begin{aligned} \text{RED}^{\text{POS}}_{7/9,2/9}(\pi_D) &= \{\{c_1, c_2, c_3\}, \{c_1, c_2, c_4\}, \{c_1, c_2, c_5\}, \{c_1, c_3, c_4\}, \{c_1, c_3, c_5\}\}, \\ \text{CORE}^{\text{POS}}_{7/9,2/9}(\pi_D) &= \{c_1\}, \\ \text{USEFUL}^{\text{POS}}_{7/9,2/9}(\pi_D) &= C. \end{aligned}$$

For the reduct target $\gamma_{\text{relative}} \in (2/9, 7/9)$, only classifications (5) and (7)–(10) reach the joint sufficiency ($\text{S}^{\text{BND}}_{\alpha,\beta}$) but classification (5) never satisfies the individual necessity ($\text{N}^{\text{BND}}_{\alpha,\beta}$). By extracting the boundary quantitative reducts from classifications (7)–(10), the sets of attribute reducts and core/useful attributes are obtained:

$$\begin{aligned} \text{RED}^{\text{BND}}_{7/9,2/9}(\pi_D) &= \{\{c_1, c_3\}, \{c_1, c_4\}, \{c_1, c_5\}, \{c_2, c_3\}, \{c_2, c_4\}, \{c_2, c_5\}\}, \\ \text{CORE}^{\text{BND}}_{7/9,2/9}(\pi_D) &= \emptyset, \\ \text{USEFUL}^{\text{BND}}_{7/9,2/9}(\pi_D) &= C. \end{aligned}$$

The negative reducts with respect to threshold $(7/9, 2/9)$ are collected as follows:

$$\begin{aligned} \text{RED}^{\text{NEG}}_{7/9,2/9}(\pi_D) = 2^C &- \{\{c_1, c_2, c_3\}, \{c_1, c_2, c_4\}, \{c_1, c_2, c_5\}, \{c_1, c_3, c_4\}, \{c_1, c_3, c_5\}\} \\ &- \{\{c_1, c_3\}, \{c_1, c_4\}, \{c_1, c_5\}, \{c_2, c_3\}, \{c_2, c_4\}, \{c_2, c_5\}\}. \end{aligned}$$

The negative reducts include $32 - 5 - 6 = 21$ subsets, and they can be divided into three blocks according to Eq. (43). (1) $\text{Block}^{\text{i}}_{7/9,2/9}$ mainly collects two groups of subsets:

$$C, \{c_1, c_2, c_4, c_5\}, \{c_1, c_3, c_4, c_5\};$$

$$\{c_1, c_2, c_3, c_4\}, \{c_1, c_2, c_3, c_5\}.$$

These five subsets satisfy ($\text{S}^{\text{POS}}_{\alpha,\beta}$) rather than ($\text{N}^{\text{POS}}_{\alpha,\beta}$). The first group originates from the absolutely redundant subsets with high measure $\gamma_{\text{relative}} \geq \alpha$, which are collected in Eq. (69), while the second group originates from the relatively redundant subsets with respect to ($\text{N}^{\text{POS}}_{\alpha,\beta}$). (2) $\text{Block}^{\text{ii}}_{7/9,2/9}$ mainly collects two groups of subsets:

$$\{c_2, c_3, c_4, c_5\}, \{c_1, c_4, c_5\}, \{c_2, c_4, c_5\};$$

$$\{c_2, c_3, c_4\}, \{c_2, c_3, c_5\}.$$

These five subsets satisfy ($\text{S}^{\text{BND}}_{\alpha,\beta}$) rather than ($\text{N}^{\text{BND}}_{\alpha,\beta}$). The first group originates from redundant subsets with a moderate measure $\gamma_{\text{relative}} \in (2/9, 7/9)$, while the second group corresponds to the relatively redundant subsets with respect to ($\text{N}^{\text{BND}}_{\alpha,\beta}$). (3) $\text{Block}^{\text{iii}}_{7/9,2/9}$ collects the surplus eleven subsets with low level $\gamma_{\text{relative}}(A) \leq 2/9$:

$$\{c_3, c_4, c_5\}, \{c_4, c_5\};$$

$$\{c_1, c_2\}, \{c_3, c_4\}, \{c_3, c_5\}, \{c_1\}, \{c_2\}, \{c_3\}, \{c_4\}, \{c_5\}, \emptyset,$$

where the first group concerns the absolutely redundant subsets with the low measure.

The above calculation process mainly utilizes the lattices $(2^C, \subseteq)$, $(2^{\pi_C}, \Leftarrow)$, and $(\{\gamma_{\text{relative}}(A) \mid A \subseteq C\}, \leq)$ to obtain the three-way quantitative reducts, where the positive and boundary quantitative reducts are first calculated and Fig. 6 is fully utilized. Moreover, all classifications play an important role in relevant connections and make a three-way division. As a result, three-way qualitative reducts completely divide 2^C, and they can implement corresponding three-way decisions in the qualitative environment. According to the quantitative threshold $(\alpha, \beta) = (7/9, 2/9)$, the five positive quantitative reducts $\{c_1, c_2, c_3\}, \{c_1, c_2, c_4\}, \{c_1, c_2, c_5\}, \{c_1, c_3, c_4\}, \{c_1, c_3, c_5\}$ can be accepted and used, the six boundary quantitative reducts $\{c_1, c_3\}, \{c_1, c_4\}, \{c_1, c_5\}, \{c_2, c_3\}, \{c_2, c_4\}, \{c_2, c_5\}$ correspond to the noncommitment decision, while the surplus twenty-one negative quantitative reducts are completely rejected.

To verify the results of core attributes,

$$\begin{aligned} \gamma_{\text{relative}}(C - \{c_1\}) &= 5/9, \\ \gamma_{\text{relative}}(C - \{c_2\}) = \gamma_{\text{relative}}(C - \{c_3\}) &= 7/9, \\ \gamma_{\text{relative}}(C - \{c_4\}) = \gamma_{\text{relative}}(C - \{c_5\}) &= 1 \end{aligned}$$

are calculated to produce a three-way partition for C:

$$\begin{aligned} \{c \mid \gamma_{\text{relative}}(C - \{c\}) \leq 2/9\} &= \emptyset, \\ \{c \mid \gamma_{\text{relative}}(C - \{c\}) \in (2/9, 7/9)\} &= \{c_1\}, \\ \{c \mid \gamma_{\text{relative}}(C - \{c\}) \geq 7/9\} &= \{c_2, c_3, c_4, c_5\}. \end{aligned}$$

On this basis, we have:

$$\begin{aligned} \text{CORE}^{\text{POS}}_{7/9,2/9}(\pi_D) = \{c_1\} &= \{c \mid \gamma_{\text{relative}}(C - \{c\}) \leq 7/9\}, \\ \{c \mid \gamma_{\text{relative}}(C - \{c\}) \leq 2/9\} = \emptyset \subseteq \text{CORE}^{\text{BND}}_{7/9,2/9}(\pi_D) = \emptyset &\subseteq \{c \mid \gamma_{\text{relative}}(C - \{c\}) \notin (2/9, 7/9)\} = \{c_2, c_3, c_4, c_5\}. \end{aligned}$$

For the core attributes, these results practically verify the computational formulation and dual bounds in the previous theoretical studies, i.e., Eqs. (28) and (42) as well as Algorithm 1. For Algorithm 2, on the basis of core attribute c_1, $Sig(\{c_1\}, c_4) = Sig(\{c_1\}, c_5)$ reaches the maximum of heuristic information for adding $c \in \{c_2, c_3, c_4, c_5\}$, so c_4 or c_5 is first added. Similarly, c_2/c_3 is next and finally added. According to the different addition order, Algorithm 2 outputs one positive quantitative reduct of

$$\{c_1, c_2, c_4\},\ \{c_1, c_2, c_5\},\ \{c_1, c_3, c_4\},\ \{c_1, c_3, c_5\},$$

but $\{c_1, c_2, c_3\}$ cannot be obtained. Moreover, we can verify the mutual relationship between $\text{CORE}^{\text{POS}}_{7/9,2/9}(\pi_D)$ and $\text{CORE}^{\text{BND}}_{7/9,2/9}(\pi_D)$, i.e.,

$$\begin{aligned} &\text{CORE}^{\text{POS}}_{7/9,2/9}(\pi_D) \cap \text{CORE}^{\text{BND}}_{7/9,2/9}(\pi_D) = \emptyset = \{c \mid \gamma_{\text{relative}}(C - \{c\}) \leq 2/9\}; \\ &\text{CORE}^{\text{POS}}_{7/9,2/9}(\pi_D) - \text{CORE}^{\text{BND}}_{7/9,2/9}(\pi_D) = \{c_1\} = \{c \mid \gamma_{\text{relative}}(C - \{c\}) \in (2/9, 7/9)\}; \\ &\text{CORE}^{\text{BND}}_{7/9,2/9}(\pi_D) - \text{CORE}^{\text{POS}}_{7/9,2/9}(\pi_D) = \emptyset \subseteq \{c_2, c_3, c_4, c_5\} = \{c \mid \gamma_{\text{relative}}(C - \{c\}) \geq 7/9)\}. \end{aligned}$$

Herein, the sets of useful attributes exhibit the equality, i.e.,

$$\text{USEFUL}^{\text{POS}}_{7/9,2/9}(\pi_D) = C = \text{USEFUL}^{\text{BND}}_{7/9,2/9}(\pi_D).$$

For the relationships of the three-way quantitative reducts, Theorems 8, 9, and Algorithm 4 provide the main results and thus are next illustrated. For all the positive quantitative reducts, $\{c_1, c_2, c_3\}/\{c_1, c_3, c_4\}/\{c_1, c_3, c_5\}$, $\{c_1, c_2, c_4\}$, and $\{c_1, c_2, c_5\}$ properly include boundary quantitative reducts $\{c_1, c_3\}$, $\{c_1, c_4\}$, $\{c_1, c_5\}$, respectively, and they actually satisfy the existence condition with respect to $\gamma_{\text{relative}}(R - \{c\}) > 2/9$. Suppose positive quantitative reduct $\{c_1, c_2, c_3\}$ is put into Algorithm 4. $|\{c_1, c_2, c_3\}| = 3 > 1$, and by the attribute deletion we have:

$$\begin{aligned} \gamma_{\text{relative}}(\{c_1, c_2, c_3\} - \{c_1\}) = \gamma_{\text{relative}}(\{c_1, c_2, c_3\} - \{c_2\}) &= 3/9 > 2/9, \\ \gamma_{\text{relative}}(\{c_1, c_2, c_3\} - \{c_3\}) &= 2/9. \end{aligned}$$

When c_1/c_2 is deleted, Algorithm 4 returns a boundary quantitative reduct $\{c_2, c_3\}/\{c_1, c_3\}$, which is properly included in $\{c_1, c_2, c_3\}$. Moreover, we can verify that an arbitrary boundary quantitative reduct has two attributes and properly includes at least a nonempty negative quantitative reduct.

Finally, three-way quantitative reducts are achieved for the other threshold $(\alpha, \beta) = (5/9, 3/9)$. The positive quantitative reducts exhibit:

$$\begin{aligned}\mathrm{RED}^{\mathrm{POS}}_{5/9,3/9}(\pi_D) &= \{\{c_1,c_2,c_3\},\{c_2,c_3,c_4\},\{c_2,c_3,c_5\},\{c_1,c_4\},\{c_1,c_5\}\},\\ \mathrm{CORE}^{\mathrm{POS}}_{5/9,3/9}(\pi_D) &= \emptyset,\\ \mathrm{USEFUL}^{\mathrm{POS}}_{5/9,3/9}(\pi_D) &= C.\end{aligned}$$

The boundary quantitative reducts yield:

$$\begin{aligned}\mathrm{RED}^{\mathrm{BND}}_{5/9,3/9}(\pi_D) &= \{\{c_2,c_4\},\{c_2,c_5\}\},\\ \mathrm{CORE}^{\mathrm{BND}}_{5/9,3/9}(\pi_D) &= \{c_2\},\\ \mathrm{USEFUL}^{\mathrm{BND}}_{5/9,3/9}(\pi_D) &= \{c_2,c_4,c_5\}.\end{aligned}$$

The negative quantitative reducts produce:

$$\mathrm{RED}^{\mathrm{NEG}}_{5/9,3/9}(\pi_D) = 2^C - \{\{c_1,c_2,c_3\},\{c_2,c_3,c_4\},\{c_2,c_3,c_5\},\{c_1,c_4\},\{c_1,c_5\}\} - \{\{c_2,c_4\},\{c_2,c_5\}\}.$$

These results can be utilized to make similar analyses for the three-way quantitative reducts and their relationships. For example, $\mathrm{CORE}^{\mathrm{BND}}_{5/9,3/9}(\pi_D)$ has the following form with respect to lower and upper bounds (Eq. (42)):

$$\{c \mid \gamma_{\mathrm{relative}}(C-\{c\}) \le 3/9\} = \emptyset \subseteq \mathrm{CORE}^{\mathrm{BND}}_{5/9,3/9}(\pi_D) = \{c_2\} \subseteq \{c \mid \gamma_{\mathrm{relative}}(C-\{c\}) \notin (3/9,5/9)\} = C.$$

5.3. Illustrations for three-way qualitative reducts

In this subsection, Table 1 and Fig. 6 are further used to illustrate the three-way qualitative reducts, which can be described based on γ and $\gamma(\pi_D|\pi_C)$ (or $\gamma_{\mathrm{relative}}$ and $(\alpha,\beta)=(1,0)$).

For the reduct target $\gamma=\gamma(\pi_D|\pi_C)$ (or $\gamma_{\mathrm{relative}}=1$), only classification (1) reaches the joint sufficiency ($\mathrm{S}^{\mathrm{POS}}$) and originates from three attribute subsets: $\{c_1,c_2,c_3,c_4\}$, $\{c_1,c_2,c_3,c_5\}$, and C. Furthermore, only $c_1c_2c_3c_4$ and $c_1c_2c_3c_5$ satisfy the individual necessity ($\mathrm{N}^{\mathrm{POS}}$), while C is absolutely redundant. For the positive qualitative reduct, the sets of attribute reducts and core/useful attributes are obtained:

$$\begin{aligned}\mathrm{RED}^{\mathrm{POS}}(\pi_D) &= \{\{c_1,c_2,c_3,c_4\},\{c_1,c_2,c_3,c_5\}\},\\ \mathrm{CORE}^{\mathrm{POS}}(\pi_D) &= \{c_1,c_2,c_3\},\\ \mathrm{USEFUL}^{\mathrm{POS}}(\pi_D) &= C.\end{aligned}$$

For the reduct target $\gamma\in(0,\gamma(\pi_D|\pi_C))$ (or $\gamma_{\mathrm{relative}}\in(0,1)$), eleven classifications with labels (2)–(11) and (15) reach the joint sufficiency ($\mathrm{S}^{\mathrm{BND}}$), but classifications (2)–(5), (8), (10) and (11) never satisfy the individual necessity ($\mathrm{N}^{\mathrm{BND}}$). As far as classification (8) is concerned, its subclassification (15) can also reach the interval target with respect to ($\mathrm{S}^{\mathrm{BND}}$), so the subset with respect to classification (8) corresponds to not the boundary qualitative reduct but the negative qualitative reduct. Only surplus classifications (6), (7), (9) and (15) and their nonredundant subsets are related to the individual necessity ($\mathrm{N}^{\mathrm{BND}}$) and the boundary reduct. For the boundary qualitative reduct, the sets of attribute reducts and core/useful attributes are obtained:

$$\begin{aligned}\mathrm{RED}^{\mathrm{BND}}(\pi_D) &= \{\{c_1,c_2\},\{c_1,c_3\},\{c_2,c_3\},\{c_4\},\{c_5\}\},\\ \mathrm{CORE}^{\mathrm{BND}}(\pi_D) &= \emptyset,\\ \mathrm{USEFUL}^{\mathrm{BND}}(\pi_D) &= C.\end{aligned}$$

The negative qualitative reducts are collected as follows:

$$\mathrm{RED}^{\mathrm{NEG}}(\pi_D) = 2^C - \{\{c_1,c_2,c_3,c_4\},\{c_1,c_2,c_3,c_5\}\} - \{\{c_1,c_2\},\{c_1,c_3\},\{c_2,c_3\},\{c_4\},\{c_5\}\}.$$

The $32-2-5=25$ subsets can be divided into three blocks $\mathrm{Block}^{\mathrm{i}}$, $\mathrm{Block}^{\mathrm{ii}}$, and $\mathrm{Block}^{\mathrm{iii}}$ according to Eq. (53). (1) For classification (1), C satisfies ($\mathrm{S}^{\mathrm{POS}}$) rather than ($\mathrm{N}^{\mathrm{POS}}$) (with respect the positive qualitative reduct); (2) For classifications (2)–(11) and (15), the following subsets satisfy ($\mathrm{S}^{\mathrm{BND}}$) rather than ($\mathrm{N}^{\mathrm{BND}}$) (with respect to the boundary qualitative reduct), i.e.,

$$\begin{aligned}&\{c_1,c_2,c_4,c_5\},\{c_1,c_3,c_4,c_5\},\{c_2,c_3,c_4,c_5\},\{c_1,c_4,c_5\},\{c_2,c_4,c_5\},\{c_3,c_4,c_5\},\{c_4,c_5\};\\ &\{c_1,c_2,c_3\},\{c_1,c_2,c_4\},\{c_1,c_2,c_5\},\{c_1,c_3,c_4\},\{c_1,c_3,c_5\},\{c_2,c_3,c_4\},\{c_2,c_3,c_5\},\{c_1,c_4\},\{c_1,c_5\},\{c_2,c_4\},\\ &\{c_2,c_5\},\{c_3,c_4\},\{c_3,c_5\}\end{aligned}$$

The twenty subsets exhibit two groups. The first group originates from the absolutely redundant subsets with $\gamma\in(0,\gamma(\pi_D|\pi_C))$ (or $\gamma_{\mathrm{relative}}\in(0,1)$), while the second group corresponds to the relatively redundant subsets with respect to ($\mathrm{N}^{\mathrm{BND}}$). (3) For classifications (12)–(14) and (16), the following four subsets

$\{c_1\}, \{c_2\}, \{c_3\}, \emptyset,$

exhibit the low measure γ (or γ_{relative}) and thus never satisfy (S^{POS}) and (S^{BND}) (with respect to the positive and boundary qualitative reducts).

Three-way qualitative reducts completely divide 2^C into three classes, and they can gain corresponding three-way decisions in the qualitative environment. Two positive qualitative reducts $\{c_1, c_2, c_3, c_4\}$ and $\{c_1, c_2, c_3, c_5\}$ can be accepted and used; five boundary qualitative reducts $\{c_1, c_2\}$, $\{c_1, c_3\}$, $\{c_2, c_3\}$, $\{c_4\}$, and $\{c_5\}$ correspond to the noncommitment decision; and the surplus twenty-five negative qualitative reducts are rejected.

To verify the results of core attributes,

$$\gamma(\pi_D|\pi_{C-\{c_1\}}) = 5/9,$$
$$\gamma(\pi_D|\pi_{C-\{c_2\}}) = \gamma(\pi_D|\pi_{C-\{c_3\}}) = 7/9,$$
$$\gamma(\pi_D|\pi_{C-\{c_4\}}) = \gamma(\pi_D|\pi_{C-\{c_5\}}) = 1;$$

are calculated to produce a three-way partition for C:

$$\{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) = 0\} = \emptyset,$$
$$\{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) \in (0, \gamma(\pi_D|\pi_C))\} = \{c_1, c_2, c_3\},$$
$$\{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) = \gamma(\pi_D|\pi_C)\} = \{c_4, c_5\}.$$

On this basis, we have the computational formulation and dual bounds:

$$\text{CORE}^{\text{POS}}(\pi_D) = \{c_1, c_2, c_3\} = \{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) < \gamma(\pi_D|\pi_C)\},$$
$$\{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) = 0\} = \emptyset \subseteq \text{CORE}^{\text{BND}}(\pi_D) = \emptyset \subseteq \{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) \notin (0, \gamma(\pi_D|\pi_C))\} = \{c_4, c_5\}.$$

Moreover, we can verify the mutual relationship between $\text{CORE}^{\text{POS}}(\pi_D)$ and $\text{CORE}^{\text{BND}}(\pi_D)$, i.e.,

$$\text{CORE}^{\text{POS}}(\pi_D) \cap \text{CORE}^{\text{BND}}(\pi_D) = \emptyset = \{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) = 0\};$$
$$\text{CORE}^{\text{POS}}(\pi_D) - \text{CORE}^{\text{BND}}(\pi_D) = \{c_1, c_2, c_3\} = \{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) \in (0, \gamma(\pi_D|\pi_C))\};$$
$$\text{CORE}^{\text{BND}}(\pi_D) - \text{CORE}^{\text{POS}}(\pi_D) = \emptyset \subseteq \{c_4, c_5\} = \{c \mid \gamma(\pi_D|\pi_{C-\{c\}}) = \gamma(\pi_D|\pi_C)\}.$$

Herein, the sets of useful attributes exhibit the equality, i.e.,

$$\text{USEFUL}^{\text{POS}}(\pi_D) = C = \text{USEFUL}^{\text{BND}}(\pi_D).$$

Finally, we can illustrate the relationships of the three-way qualitative reducts, including the potential derivation in Corollary 7. Based on the practical verification, each positive qualitative reduct properly includes at least a boundary qualitative reduct, while each boundary qualitative reduct with at least two attributes also properly includes at least a nonempty negative qualitative reduct.

5.4. Relationship illustrations between three-way quantitative and qualitative reducts

According to Table 1 and Fig. 6, three-way quantitative/qualitative reducts and their internal relationships have been illustrated. In this subsection, the strength relationships are further illustrated between three-way quantitative and qualitative reducts.

As is previously pointed out, three-way quantitative reducts theoretically expand and practically approximate three-way qualitative reducts. With respect to threshold (α, β), the qualitative reducts are stable on $(1, 0)$, while the quantitative reducts possibly have multiple choices. For the (α, β) determination, the concrete decision table is worth considering, and some special cases may lead to the qualitative degeneration of the quantitative reducts. For this decision table example with Table 1, γ_{relative} has the discrete distribution:

$$0 \le 1/9 \le 2/9 \le 3/9 \le 4/9 \le 5/9 \le 7/9 \le 1.$$

If α is so great to be greater than the second maximum 7/9, then the positive quantitative reducts degenerate into the positive qualitative reducts; furthermore, if β is so small to be smaller than the second minimum 1/9, then the boundary and negative quantitative reducts degenerate into the boundary and negative qualitative reducts, respectively. In contrast, if $1/9 \le \beta < \alpha \le 7/9$, then three-way quantitative reducts necessarily differ from three-way qualitative reducts. These analyses can produce a general conclusion based on the second maximum/minimum.

Proposition 7. *In lattice $(\{\gamma_{\text{relative}}(A) \mid A \subseteq C\}, \le)$, the second maximum and minimum of γ_{relative} are denoted by $\gamma_{\text{relative}}^{max_2}$ and $\gamma_{\text{relative}}^{min_2}$, respectively. If $\alpha > \gamma_{\text{relative}}^{max_2}$, then the positive quantitative reducts degenerate into the positive qualitative reducts; furthermore, if $\beta <$*

$\gamma^{min_2}_{\text{relative}}$, then the three-way quantitative reducts completely degenerate into three-way qualitative reducts. In contrast, if $\alpha \le \gamma^{max_2}_{\text{relative}}$ and $\beta \ge \gamma^{min_2}_{\text{relative}}$, then three-way quantitative reducts differ from three-way qualitative reducts.

According to Proposition 7, three-way quantitative reducts with $\alpha > \gamma^{max_2}_{\text{relative}}$ and $\beta < \gamma^{min_2}_{\text{relative}}$ also exhibit the qualitative degeneration, which involves the previous case $(\alpha, \beta) = (1, 0)$. Moreover, $\alpha \le \gamma^{max_2}_{\text{relative}}$ and $\beta \ge \gamma^{min_2}_{\text{relative}}$ can theoretically ensure the difference and non-degeneration between three-way quantitative and qualitative reducts. In the theoretical sense, the usual case $\gamma^{min_2}_{\text{relative}} \le \beta < \alpha \le \gamma^{max_2}_{\text{relative}}$ avoids the latent qualitative degeneration and implements the practical quantitative applications, and the above quantitative results with thresholds $(7/9, 2/9)$ and $(5/9, 3/9)$ provide some powerful illustrations. As a result, the γ_{relative} distribution is worth considering for the threshold determination.

Next, we focus on the reduct strength, which is described in Theorem 6, Proposition 2, and Algorithm 3. First consider case $(\alpha, \beta) = (7/9, 2/9)$. (1) The positive quantitative reducts are weaker than the positive qualitative reducts.

$$\text{CORE}^{\text{POS}}_{7/9,2/9}(\pi_D) = \{c_1\} \subseteq \{c_1, c_2, c_3\} = \text{CORE}^{\text{POS}}(\pi_D)$$

shows the extension of core attributes between the positive quantitative and qualitative reducts.

$$\text{RED}^{\text{POS}}_{7/9,2/9}(\pi_D) = \{\{c_1, c_2, c_3\}, \{c_1, c_2, c_4\}, \{c_1, c_2, c_5\}, \{c_1, c_3, c_4\}, \{c_1, c_3, c_5\}\},$$
$$\text{RED}^{\text{POS}}(\pi_D) = \{\{c_1, c_2, c_3, c_4\}, \{c_1, c_2, c_3, c_5\}\}.$$

Based on the results of positive reducts, each positive qualitative reduct includes at least one positive quantitative reduct. Suppose qualitative reduct $\{c_1, c_2, c_3, c_4\}$ is put into Algorithm 3, and then positive quantitative reduct $R^{\text{POS}}_{7/9,2/9}$ has the bound form:

$$\{c_1\} \subseteq R^{\text{POS}}_{7/9,2/9} \subseteq \{c_1, c_2, c_3, c_4\}.$$

Only core attribute c_1 cannot be deleted; thus, if first deleting $c_2/c_3/c_4$, then $\{c_1, c_3, c_4\}/\{c_1, c_2, c_4\}/\{c_1, c_2, c_3\}$ is output to become the final quantitative reduct $R^{\text{POS}}_{7/9,2/9}$ in the given qualitative reduct $\{c_1, c_2, c_3, c_4\}$. (2) The boundary quantitative reducts are stronger than the boundary qualitative reducts.

$$\text{CORE}^{\text{BND}}_{7/9,2/9}(\pi_D) = \emptyset = \text{CORE}^{\text{BND}}(\pi_D)$$

accords with the inclusion of core attributes between the positive quantitative and qualitative reducts.

$$\text{RED}^{\text{BND}}_{7/9,2/9}(\pi_D) = \{\{c_1, c_3\}, \{c_1, c_4\}, \{c_1, c_5\}, \{c_2, c_3\}, \{c_2, c_4\}, \{c_2, c_5\}\},$$
$$\text{RED}^{\text{BND}}(\pi_D) = \{\{c_1, c_2\}, \{c_1, c_3\}, \{c_2, c_3\}, \{c_4\}, \{c_5\}\}.$$

Based on the results of boundary reducts, each boundary quantitative reduct includes at least one boundary qualitative reduct. In particular, there are no boundary quantitative reducts to include boundary qualitative reduct $\{c_1, c_2\}$. This fact verifies the basic opposite for the reduct strength. In other words, the weaker reduct can also exist by breaking away from the inclusion relationship from its stronger reduct, although each strong reduct necessarily includes at least one weak reduct [58]. (3) Based on the above results, the positive/boundary quantitative and qualitative reducts have the usual set relationship. For example, we can achieve the following nonempty relationships:

$$\text{RED}^{\text{BND}}_{7/9,2/9}(\pi_D) \cap \text{RED}^{\text{BND}}(\pi_D) = \{\{c_1, c_3\}, \{c_2, c_3\}\},$$
$$\text{RED}^{\text{BND}}_{7/9,2/9}(\pi_D) - \text{RED}^{\text{BND}}(\pi_D) = \{\{c_1, c_4\}, \{c_1, c_5\}, \{c_2, c_4\}, \{c_2, c_5\}\},$$
$$\text{RED}^{\text{BND}}(\pi_D) - \text{RED}^{\text{BND}}_{7/9,2/9}(\pi_D) = \{\{c_1, c_2\}, \{c_4\}, \{c_5\}\}.$$

As a result, the negative quantitative and qualitative reducts also have the usual set relationships. The internal blocks of the negative quantitative and qualitative reducts are calculated in Sections 5.2 and 5.3, respectively. By the simple verification, we have their mutual inclusion/extension relationships as follows:

$$\text{Block}^{\text{i}}_{7/9,2/9} \supseteq \text{Block}^{\text{i}},$$
$$\text{Block}^{\text{ii}}_{7/9,2/9} \subseteq \text{Block}^{\text{ii}},$$
$$\text{Block}^{\text{iii}}_{7/9,2/9} \supseteq \text{Block}^{\text{iii}}.$$

From the quantitative viewpoint, the quantitative and qualitative reducts focus on $(\alpha, \beta) = (7/9, 2/9)$ and $(\alpha, \beta) = (1, 0)$, respectively. Based on the direct verification, the other case $(\alpha, \beta) = (5/9, 3/9)$ is used to similarly illustrate relationships between the three-way quantitative and qualitative reducts. Furthermore, the relevant strength relationships are exhibited by the two types of quantitative reducts, i.e., the positive/boundary reducts based on $(5/9, 3/9)$ are weaker/stronger than the positive/boundary reducts based on $(7/9, 2/9)$.

$$\mathrm{CORE}^{\mathrm{POS}}_{7/9,2/9}(\pi_D)=\{c_1\}\supseteq\emptyset=\mathrm{CORE}^{\mathrm{POS}}_{5/9,3/9}(\pi_D)$$

shows the inclusion of core attributes of the positive quantitative reduct between (7/9, 2/9) and (5/9, 3/9).

$$\mathrm{RED}^{\mathrm{POS}}_{7/9,2/9}(\pi_D)=\{\{c_1,c_2,c_3\},\{c_1,c_2,c_4\},\{c_1,c_2,c_5\},\{c_1,c_3,c_4\},\{c_1,c_3,c_5\}\},$$
$$\mathrm{RED}^{\mathrm{POS}}_{5/9,3/9}(\pi_D)=\{\{c_1,c_2,c_3\},\{c_2,c_3,c_4\},\{c_2,c_3,c_5\},\{c_1,c_4\},\{c_1,c_5\}\}.$$

Each positive quantitative reduct with (7/9, 2/9) includes only one positive quantitative reduct with (5/9, 3/9), but reducts $\{c_2,c_3,c_4\}$ and $\{c_2,c_3,c_5\}$ with (5/9, 3/9) cannot be included.

$$\mathrm{CORE}^{\mathrm{BND}}_{7/9,2/9}(\pi_D)=\emptyset\subseteq\{c_2\}=\mathrm{CORE}^{\mathrm{BND}}_{5/9,3/9}(\pi_D)$$

shows the extension of core attributes of the boundary quantitative reducts between (7/9, 2/9) and (5/9, 3/9).

$$\mathrm{RED}^{\mathrm{BND}}_{7/9,2/9}(\pi_D)=\{\{c_1,c_3\},\{c_1,c_4\},\{c_1,c_5\},\{c_2,c_3\},\{c_2,c_4\},\{c_2,c_5\}\},$$
$$\mathrm{RED}^{\mathrm{BND}}_{5/9,3/9}(\pi_D)=\{\{c_2,c_4\},\{c_2,c_5\}\}.$$

Each boundary quantitative reduct with (5/9, 3/9) includes only one boundary quantitative reduct with (7/9, 2/9), but reducts $\{c_1,c_3\}$, $\{c_1,c_4\}$, $\{c_1,c_5\}$, and $\{c_2,c_3\}$ with (7/9, 2/9) cannot be included.

5.5. Superiority illustrations of three-way reducts in contrast to two-way reducts

In this subsection, Table 1 and Fig. 6 are finally used to illustrate the superiority of three-way reducts in contrast to two-way reducts.

The classical qualitative reducts imply the two-way reducts:

$$\mathrm{RED}(\pi_D)=\{\{c_1,c_2,c_3,c_4\},\{c_1,c_2,c_3,c_5\}\},$$
$$\mathrm{NONRED}(\pi_D)=2^C-\{\{c_1,c_2,c_3,c_4\},\{c_1,c_2,c_3,c_5\}\},$$

and they adopt the two-way decisions with respect to acceptance and rejection, respectively. They develop to three-way reducts, and we can verify the relevant structure improvement:

$$\mathrm{RED}(\pi_D)=\mathrm{RED}^{\mathrm{POS}}(\pi_D),$$
$$\mathrm{NONRED}(\pi_D)=\mathrm{RED}^{\mathrm{BND}}(\pi_D)\cup\mathrm{RED}^{\mathrm{NEG}}(\pi_D).$$

Three-way qualitative reducts are calculated in Section 5.2, and they gain three-way decisions with respect to acceptance, noncommitment, and rejection, respectively. Moreover, the ideal reduct set has the form with lower and upper bounds:

$$\begin{aligned}&\{\{c_1,c_2,c_3,c_4\},\{c_1,c_2,c_3,c_5\}\}\\ &\subseteq\mathrm{RED}_{\mathrm{ideal}}(\pi_D)\\ &\subseteq\{\{c_1,c_2,c_3,c_4\},\{c_1,c_2,c_3,c_5\},\{c_1,c_2\},\{c_1,c_3\},\{c_2,c_3\},\{c_4\},\{c_5\}\}.\end{aligned}$$

When $\alpha=7/9$, the two-way quantitative reducts become:

$$\mathrm{RED}_{7/9}(\pi_D)=\{\{c_1,c_2,c_3\},\{c_1,c_2,c_4\},\{c_1,c_2,c_5\},\{c_1,c_3,c_4\},\{c_1,c_3,c_5\}\},$$
$$\mathrm{NONRED}_{7/9}(\pi_D)=2^C-\{\{c_1,c_2,c_3\},\{c_1,c_2,c_4\},\{c_1,c_2,c_5\},\{c_1,c_3,c_4\},\{c_1,c_3,c_5\}\}.$$

They adopt the two-way decisions with respect to acceptance and rejection, respectively. By introducing the other bar $\beta=2/9$, they are improved to the three-way quantitative reducts. The results of the three-way quantitative reducts are computed in Section 5.3. As a result,

$$\mathrm{RED}_{7/9}(\pi_D)=\mathrm{RED}^{\mathrm{POS}}_{7/9,2/9}(\pi_D),$$
$$\mathrm{NONRED}_{7/9}(\pi_D)=\mathrm{RED}^{\mathrm{BND}}_{7/9,2/9}(\pi_D)\cup\mathrm{RED}^{\mathrm{NEG}}_{7/9,2/9}(\pi_D).$$

Three-way quantitative reducts correspond to the three-way decisions. In the quantitative environment with (7/9, 2/9), the ideal reduct set has the range:

$$\begin{aligned}&\{\{c_1,c_2,c_3\},\{c_1,c_2,c_4\},\{c_1,c_2,c_5\},\{c_1,c_3,c_4\},\{c_1,c_3,c_5\}\}\\ &\subseteq\mathrm{RED}^{*}_{\mathrm{ideal}}(\pi_D)\\ &\subseteq\{\{c_1,c_2,c_3\},\{c_1,c_2,c_4\},\{c_1,c_2,c_5\},\{c_1,c_3,c_4\},\{c_1,c_3,c_5\},\{c_1,c_3\},\{c_1,c_4\},\{c_1,c_5\},\{c_2,c_3\},\{c_2,c_4\},\{c_2,c_5\}\}.\end{aligned}$$

The quantitative reducts based on $\alpha = 5/9$ and $(\alpha, \beta) = (5/9, 3/9)$ can be similarly analyzed.

Herein, case $\alpha = \beta$ implies $\mathrm{RED}^{\mathrm{NEG}}_{\alpha,\beta}(\pi_D) = \emptyset$ and thus is utilized to produce the degenerate pattern of two-way reducts. Suppose $\alpha = \beta = 5/9$, and then the two-way quantitative reducts exhibit:

$$\mathrm{RED}^{\mathrm{POS}}_{5/9,5/9}(\pi_D) = \{\{c_1, c_2, c_3\}, \{c_2, c_3, c_4\}, \{c_2, c_3, c_5\}, \{c_1, c_4\}, \{c_1, c_5\}\} = \mathrm{RED}_{5/9}(\pi_D),$$
$$\mathrm{RED}^{\mathrm{NEG}}_{5/9,5/9}(\pi_D) = 2^C - \{\{c_1, c_2, c_3\}, \{c_2, c_3, c_4\}, \{c_2, c_3, c_5\}, \{c_1, c_4\}, \{c_1, c_5\}\} = \mathrm{NONRED}_{5/9}(\pi_D).$$

By using $\alpha = \beta = 1$, three-way quantitative reducts degenerate into the two-way qualitative reducts, i.e.,

$$\mathrm{RED}^{\mathrm{POS}}_{1,1}(\pi_D) = \{\{c_1, c_2, c_3, c_4\}, \{c_1, c_2, c_3, c_5\}\} = \mathrm{RED}(\pi_D),$$
$$\mathrm{RED}^{\mathrm{NEG}}_{1,1}(\pi_D) = 2^C - \{\{c_1, c_2, c_3, c_4\}, \{c_1, c_2, c_3, c_5\}\} = \mathrm{NONRED}(\pi_D).$$

6. Conclusions

In this paper, three-way decisions are introduced into attribute reducts and thus three-way attribute reducts are systematically investigated at the 2^C and C levels. The analyses of lattices and measures establish the solid mathematical foundation, so the dependency degree γ with inherent dependency semantics is improved to the relative dependency degree $\gamma_{\mathrm{relative}}$. In particular, novel measure $\gamma_{\mathrm{relative}}$ becomes monotonic to relatively describe the attribute dependency and becomes controllable to quantitatively measure attribute reducts. According to advanced measure $\gamma_{\mathrm{relative}}$, we adopt the following research line of reduct construction:

$$\begin{aligned} &\text{the positive qualitative reducts}\\ \longrightarrow\ &\text{the positive quantitative reducts}\\ \longrightarrow\ &\text{three-way quantitative reducts}\\ \longrightarrow\ &\text{three-way qualitative reducts.}\end{aligned}$$

The positive qualitative reducts are the classical qualitative reducts based on γ. For quantitative applications, the positive qualitative reducts utilize the quantification approximation to be generalized to the positive quantitative reducts, i.e., the $\gamma_{\mathrm{relative}}$–$\alpha$ quantitative reducts. By adding bar β, the positive quantitative reducts are completely extended to three-way quantitative reducts. By setting up $(\alpha, \beta) = (1, 0)$, three-way quantitative reducts degenerate into three-way qualitative reducts, which can be described by γ or $\gamma_{\mathrm{relative}}$ and include the classical qualitative reducts. As a result, the positive, boundary, and negative reducts constitute three-way reducts to divide 2^C: the existing space of attribute reducts, and they gain acceptance, noncommitment, and rejection of three-way decisions, respectively. In particular, the positive qualitative reducts are highly related to the approximate reducts [34,35,40,59], while the boundary reducts become a new reduct type to describe the theoretical uncertainty and practical possibility of attribute reduction. For three-way reducts, they have potential derivation from the higher level to the lower level, and they have approximation, expansion, and strength between the quantitative and qualitative patterns. Three-way reducts improve the latent two-way reducts with only acceptance and rejection of two-way decisions. These relevant results are finally illustrated by observing an example of decision tables, where several general conclusions are also extracted. By developing the relative dependency degree with controllability, three-way reducts implement a quantitative generalization for qualitative reducts and a structural completion for attribute reducts. The relevant study provides a new insight into both three-way decisions and attribute reducts.

As an original notion, three-way attribute reducts are worth deeply researching and extensively applying, so several issues are retained for the follow-up studies.

(1) As a monotonic uncertainty measure, the relative dependency degree relatively transforms and quantitatively improves the fundamental dependency degree. For $\gamma_{\mathrm{relative}}$, its dependency semantics and controllability superiority produce the division and development of three-way reducts, and it can effectively replace γ to implement qualitative and quantitative reducts. $\gamma_{\mathrm{relative}}$ needs the in-depth use, by the comparison and replacement for γ. Other monotonic measures, such as information measures, can be relatively considered by relative development or metrical fusion.
(2) The positive quantitative reducts have more generation and optimization to practically apply to the quantitative environment, even with data noise. Thresholds α and β are related to the measure distribution, and their determination can be realized by the expert experience or user requirement. Their threshold modeling is worth considering, such as by the decision theory with cost and risk.
(3) The quantitative reducts originate from the direct and relative quantification of reduct targets and attribute reducts. Within the framework of double quantification [43,57,59], they can be further considered by using the direct and absolute quantification or by combing some indirect quantification.
(4) Three-way reducts are related to the subset classification with large number $2^{|C|}$, so their concrete calculation needs some simplified strategies, such as the mutual relationships and the attribute structures. The systematicness of three-way reducts, including the relationships among the internal parts, the relationships between the quantitative and

qualitative patterns, are worthy of the real-life verification and application, especially in noise scenarios. For three-way reducts, the relevant three-way classification of attributes, i.e., the core, marginal, nonuseful attributes at the C level [53], can be deeply researched to extend the results of core attributes.

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Appendix A. Equivalence proof between Conditions (N_α) and $(\mathrm{N'}_\alpha)$

Proof. (1) $\forall R' \subset R$, then $\exists c \in R - R' \subset R$, s.t., $R' \subseteq R - \{c\} \subset R$. According to the γ_{relative} monotonicity (Theorem 1), $\gamma_{\text{relative}}(R') \le \gamma_{\text{relative}}(R - \{c\}) \le \gamma_{\text{relative}}(R)$; according to Condition (N_α), $\gamma_{\text{relative}}(R - \{c\}) < \alpha$. Hence, $\gamma_{\text{relative}}(R') < \alpha$, i.e., $(\mathrm{N'}_\alpha)$ holds.

(2) $\forall c \in R$, let $R' = R - \{c\} \subset R$. According to Condition $(\mathrm{N'}_\alpha)$, $\gamma_{\text{relative}}(R - \{c\}) < \alpha$, i.e., (N_α) holds.

Based on the above two items, (N_α) and (N'_α) become equivalent. □

Appendix B. Proof of Eq. (28): $\mathrm{CORE}_\alpha(\pi_D) = \{c \mid \gamma_{\text{relative}}(C - \{c\}) < \alpha\}$

Proof. (1) If $c \notin \mathrm{CORE}_\alpha(\pi_D)$, then $\exists R \in \mathrm{RED}_\alpha(\pi_D)$ but $c \notin R$. Thus, $\gamma_{\text{relative}}(R) \ge \alpha$, and $R \subseteq C - \{c\} \subset C$. According to the γ_{relative} monotonicity (Theorem 1), $\gamma_{\text{relative}}(C - \{c\}) \ge \alpha$, so $c \notin \{c \mid \gamma_{\text{relative}}(C - \{c\}) < \alpha\}$. Hence, $\{c \mid \gamma_{\text{relative}}(C - \{c\}) < \alpha\} \subseteq \mathrm{CORE}_\alpha(\pi_D)$.

(2) If $c \notin \{c \mid \gamma_{\text{relative}}(C - \{c\}) < \alpha\}$, then $\gamma_{\text{relative}}(C - \{c\}) \ge \alpha$. Thus, $\exists R \subseteq C - \{c\}$ to satisfy $\gamma_{\text{relative}}(R) \ge \alpha$ and Condition (N_α), so $R \in \mathrm{RED}_\alpha(\pi_D)$. However, $c \notin R$ because $R \subseteq C - \{c\}$. Hence, $c \notin \mathrm{CORE}_\alpha(\pi_D)$, $\mathrm{CORE}_\alpha(\pi_D) \subseteq \{c \mid \gamma_{\text{relative}}(C - \{c\}) < \alpha\}$.

Based on the above two items, $\mathrm{CORE}_\alpha(\pi_D) = \{c \mid \gamma_{\text{relative}}(C - \{c\}) < \alpha\}$. □

Appendix C. Proof of Eq. (42) to provide lower and upper bounds of $\mathrm{CORE}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D)$

Proof. (1) If $c \notin \mathrm{CORE}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D)$, then $\exists R \in \mathrm{RED}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D)$ but $c \notin R$. Thus, $\gamma_{\text{relative}}(R) \in (\beta, \alpha)$, and $R \subseteq C - \{c\} \subset C$. According to the γ_{relative} monotonicity (Theorem 1), $\gamma_{\text{relative}}(C - \{c\}) > \beta$, so $c \notin \{c \mid \gamma_{\text{relative}}(C - \{c\}) \le \beta\}$. Hence, $\{c \mid \gamma_{\text{relative}}(C - \{c\}) \le \beta\} \subseteq \mathrm{CORE}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D)$.

(2) If $c \notin \{c \mid \gamma_{\text{relative}}(C - \{c\}) \notin (\beta, \alpha)\}$, then $\gamma_{\text{relative}}(C - \{c\}) \in (\beta, \alpha)$. Thus, $\exists R \subseteq C - \{c\}$ to satisfy $\gamma_{\text{relative}}(R) \in (\beta, \alpha)$ and Condition $(\mathrm{N}^{\mathrm{BND}}_{\alpha,\beta})$, so $R \in \mathrm{RED}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D)$. However, $c \notin R$ because $R \subseteq C - \{c\}$. Hence, $c \notin \mathrm{CORE}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D)$, $\mathrm{CORE}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D) \subseteq \{c \mid \gamma_{\text{relative}}(C - \{c\}) \notin (\beta, \alpha)\}$.

Based on the above two items, the formula of lower and upper bounds holds, i.e.,

$$\{c \mid \gamma_{\text{relative}}(C - \{c\}) \le \beta\} \subseteq \mathrm{CORE}^{\mathrm{BND}}_{\alpha,\beta}(\pi_D) \subseteq \{c \mid \gamma_{\text{relative}}(C - \{c\}) \notin (\beta, \alpha)\}. \quad \square \tag{C.1}$$

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