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Three-way decision approaches to conflict analysis using decision-theoretic rough set theory

Guangming Lang[a,b,c], Duoqian Miao[a,c,*], Mingjie Cai[d]

[a] Department of Computer Science and Technology, Tongji University, Shanghai 201804, PR China
[b] School of Mathematics and Statistics, Changsha University of Science and Technology, Changsha, Hunan 410114, PR China
[c] The Key Laboratory of Embedded System and Service Computing, Ministry of Education, Tongji University, Shanghai 201804, PR China
[d] College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410004, PR China

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ABSTRACT

Social progress normally occurs through a sequence of struggles and conflicts, and there has been relatively little progress in developing effective methods for conflict analysis. Decision-theoretic rough set theory is a powerful mathematical tool for depicting ambiguous information, and it can provide constructive advice for decision making. In this paper, we first present the concepts of probabilistic conflict, neutral, and allied sets of conflicts and then discuss the mechanism for computing the thresholds α and β for conflict analysis using decision-theoretic rough set theory. Then, we describe incremental algorithms for constructing the probabilistic conflict, neutral, and allied sets in dynamic information systems, and their effectiveness is illustrated by experimental results. In light of the relationship between maximal coalitions and allied sets, we finally provide efficient approaches to help a government adjust various policies according to changes in the present international situation to calculate the maximal coalitions in dynamic information systems.

1. Introduction

Conflict, as an important characteristic of human nature, exists in various real-world situations, and we are primarily interested in finding the essence of the conflict issue to reduce tensions and improve the relationship between the two sides of a conflict. Conflict analysis [9,18,23,24,31,33,35,42], which aims to explore the nature of conflict, has recently attracted increasing attention. For example, Pawlak [23] initially considered auxiliary functions and distance functions based on rough set theory, which offers a deeper insight into the structure of conflicts and enables the analysis of the relationships between parties and the issues being debated. Sun et al. [33] subsequently proposed a conflict analysis decision model and developed a matrix approach for conflict analysis based on rough set theory over two universes. Yang et al. [35] investigated evidence conflict and belief convergence based on the analysis of the degree of coherence between two sources of evidence and illustrated the stochastic interpretation for the basic probability assignment. Yu et al. [42] presented the supporting probability distance to characterize the differences among bodies of evidence and a new combination rule for the combination of conflicting evidence. In essence, because of the newness of this research area, the methods for conflict analysis via rough set theory are artificially constrained. In particular, there has also been a relatively small number of known investigations

* Corresponding author.
E-mail addresses: langguangming1984@tongji.edu.cn (G. Lang), dqmiao@tongji.edu.cn (D. Miao), cmjlong@hnu.edu.cn (M. Cai).

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on the conflict analysis of dynamic information systems. In our view, research on conflict analysis is increasing with the development of rough set theory.

Pawlak's rough set theory is effective in handling imprecise information in conflict analysis, but the conditions of the equivalence relation are so strict that it limits its applications in the real world. Actually, probabilistic rough set theory and decision-theoretic rough set theory are generalizations of classical rough set theory, and they construct probabilistic positive, boundary, and negative regions with two thresholds given by experts and using loss functions, respectively. In contrast, the two types of rough set theories are more effective than Pawlak's model for dealing with uncertain information in practical situations. Furthermore, there are some similarities between the positive, boundary, and negative regions and the conflict, neutral, and allied sets of conflicts. Therefore, we employ probabilistic rough set theory and decision-theoretic rough set theory to cope with conflict problems. Information systems for conflicts vary with time owing to the characteristics of the data collection, and non-incremental approaches for constructing conflict, neutral, and allied sets in dynamic information systems are usually costly or even intractable. Therefore, it is necessary to provide effective approaches for computing the conflict, neutral, and allied sets in dynamic information systems. In what follows, we briefly review probabilistic rough set theory, decision-theoretic rough set theory, three-way decision theory, and knowledge discovery of dynamic information systems.

(1) Probabilistic rough set theory and decision-theoretic rough set theory

Much effort has been applied in probabilistic rough set theory [5,38,43,50], game-theoretic rough sets [40], Bayesian rough set models [32,34] and other decision-making models [13,25,26], and these models provide effective tools for knowledge discovery of information systems in the big data era. In particular, the abundance of research on decision-theoretic rough set theory [3,4,8,15,21,27,29,41,47,48] offers a mathematical way to interpret the two thresholds in probabilistic rough set theory. For instance, Jia et al. [8] defined the minimum cost attribute reduction in decision-theoretic rough set models and designed several approaches for computing the minimum cost attribute reduction of information systems. Liang et al. [15] proposed triangular fuzzy decision-theoretic rough sets to satisfy a fuzzy environment to determine the values of loss functions with the aid of multiple-attribute group decision making. Ma et al. [21] incorporated (α, β)-positive region distribution preservation reduction, (α, β)-boundary region distribution preservation reduction and (α, β)-negative region distribution preservation reduction in the decision-theoretic rough set model; they also provided heuristic reduction algorithms for constructing decision region distribution preservation reducts by considering variants of the conditional information entropy. Qian et al. [27] developed multi-granulation decision-theoretic rough sets and discussed the relationships among multi-granulation decision-theoretic rough sets, decision-theoretic rough sets and single granulation rough sets. Zhao et al. [47] presented decision-theoretic rough set approaches in the frameworks of fuzzy and interval-valued fuzzy probabilistic approximation spaces that have the ability to directly address real-valued and interval-valued data in practice.

(2) Three-way decision theory

Many researchers have combined three-way decision theory [12,45,46] with probabilistic rough set theory [6,38,39], decision-theoretic rough set theory [14,16,19,37,49], and game-theoretic rough set theory[40]. For example, in the theoretical aspects, Hu et al. [6] introduced axiomatic definitions for decision measurement, decision condition and evaluation function and presented a variety of three-way decisions on three-way decision spaces. With the aid of group decision making, Liang et al. [16] studied three-way decisions based on decision-theoretic rough sets under a linguistic assessment. Liu et al. [19] introduced incomplete information to decision-theoretic rough set theory and employed a hybrid information table to address the integrated information system. Yao [39] drew a systematic comparison among probabilistic three-way decisions, probabilistic two-way decisions and qualitative three-way decisions in the standard rough set model. In the applied field, Yao and Azam [40] extended game-theoretic rough set models to analyze the uncertainty involved in medical decision making and enhance the decision-making capabilities of Web-based medical decision support systems. Yu et al. [41] put forward an efficient automatic method by extending the decision-theoretic rough set model to clustering and designed a new clustering validity evaluation function based on the risk calculated using loss functions and probabilities. Zhang et al. [45] provided a framework integrating three-way decisions and random forests and built a random forest to predict the probability that a user likes an item. Zhou et al. [49] designed three email folders instead of two in a three-way spam filtering system that reduces the chances of misclassification.

(3) Knowledge discovery of dynamic information systems

Many investigations [1,2,7,10–12,17,20,28,30,36,44] have focused on knowledge discovery of dynamic information systems with variations of object sets, attribute sets, and attribute values. For instance, Chen et al. [1] provided an incremental method for updating the approximations of variable-precision rough set models when objects in the information system are dynamically altered. Lang et al. [11] proposed incremental approaches for computing the second and sixth lower and upper approximations of sets using characteristic matrices in dynamic covering approximation spaces and investigated knowledge reduction of dynamic covering decision information systems when varying the covering cardinality. Liang et al. [17] discussed incremental mechanisms for three representative information entropies and developed a group incremental rough feature selection algorithm using information entropy. Luo et al. [20] studied the update properties for the dynamic maintenance of approximations and presented fast algorithms for computing rough approximations in set-valued decision systems while updating criteria values. Raza et al. [28] proposed the concept of the incremental dependency class and calculated the attribute dependency without using the positive region for large datasets. Yang et al. [36] put forward algorithms that are both naive and fast for updating multi-granulation rough approximations with the increase in the granular structures. Zhang

Table 1
Information system for the Middle East conflict.

U	a	b	c	d	e
Israel	−1	+1	+1	+1	+1
Egypt	+1	0	−1	−1	−1
Palestine	+1	−1	−1	−1	0
Jordan	0	−1	−1	0	−1
Syria	+1	−1	−1	−1	−1
Saudi Arabia	0	+1	−1	0	+1

et al. [44] designed a new dynamic method for incrementally updating approximations of a concept under neighborhood rough sets.

The contributions of this paper are as follows. First, we present the concepts of probabilistic conflict, neutral, and allied sets of conflicts by employing decision-theoretic rough sets. We also provide an algorithm for computing the probabilistic conflict, neutral, and allied sets in information systems. Second, we focus on conflict analysis and resolutions for dynamic information systems. Specifically, we discuss the properties of the probabilistic conflict, neutral, and allied sets and analyze the mechanism of computing them in dynamic information systems. We also present incremental algorithms for computing the probabilistic conflict, neutral, and allied sets in dynamic information systems. Third, we generate ten dynamic information systems using Pawlak's information system for the Middle East conflict and perform an experiment on them using incremental algorithms. The experimental results are used to verify the effectiveness of the incremental algorithms in computing the probabilistic conflict, neutral, and allied sets in dynamic information systems. Finally, we discuss the relationship between the maximal coalitions and the allied sets, which provide fast methods for constructing the maximal coalitions in conflict analysis. In addition, we put forward incremental algorithms for computing the maximal coalitions in dynamic information systems for conflict. Three comprehensive examples are employed to illustrate how to construct maximal coalitions with the proposed approaches in dynamic information systems for conflicts.

The rest of this paper is organized as follows: Section 2 briefly reviews the concepts of conflict analysis and decision-theoretic rough set theory. Section 3 presents the concepts of probabilistic conflict, neutral, and allied sets and constructs the thresholds for conflict analysis using decision-theoretic rough set theory. In Section 4, we provide incremental approaches to computing the probabilistic conflict, neutral, and allied sets in dynamic information systems. In Section 5, the experimental results demonstrate that incremental algorithms are effective for calculating the probabilistic conflict, neutral, and allied sets. Section 6 discusses the relationship between the maximal coalitions and allied sets and constructs the maximal coalitions in dynamic information systems. We conclude the paper in Section 7.

2. Preliminaries

In this section, we review briefly Pawlak's model for conflict analysis and decision-theoretic rough set theory.

2.1. Pawlak's model for conflict analysis

Definition 2.1 [22]**.** An information system is a 4-tuple $S=(U,A,V,f)$, where $U=\{x_1,x_2,\dots,x_n\}$ is a finite set of objects, A is a finite set of attributes, $V=\{V_a \mid a\in A\}$, where V_a is the set of values of attribute a, and $card(V_a)>1$, f is a function from $U\times A$ into V.

An information system, in which objects are measured using a finite number of attributes, represents all available information and knowledge. Throughout this paper, U is referred to as the object set, and $S=(U,A,V,f)$ is denoted as $S=(U,A)$. Actually, there exist many types of objects such as country, person, and company.

Definition 2.2 [23]**.** Let $S=(U,A)$ be an information system. Then the auxiliary function $\phi_a(x, y)$ for any $a\in A$ is defined as follows:

$$\phi_a(x,y)=\begin{cases}1, & \text{if} \quad a(x)\cdot a(y)=1\vee x=y;\\ 0, & \text{if} \quad a(x)\cdot a(y)=0\wedge x\neq y;\\ -1, & \text{if} \quad a(x)\cdot a(y)=-1.\end{cases}$$

If the auxiliary function $\phi_a(x,y)=1$, then the objects x and y have the same opinion about issue a; if the auxiliary function $\phi_a(x,y)=0$, then it means that at least one object x or y has a neutral opinion about issue a; and if the auxiliary function $\phi_a(x,y)=-1$, then the objects x and y have different opinions about issue a. We provide the following example in [23] to illustrate the information system for the Middle East conflict.

Example 2.3 [23]**.** Table 1 shows the information system for the Middle East conflict as follows.

Remarks. We denote Israel, Egypt, the Palestinians, Jordan, Syria, and Saudi Arabia as x_1, x_2, x_3, x_4, x_5, and x_6, respectively. Moreover, a refers to an Autonomous Palestinian state in the West Bank and Gaza; b denotes an Israeli military outpost

Table 2
Distance matrix for the Middle East conflict.

U	x_1	x_2	x_3	x_4	x_5	x_6
x_1						
x_2	0.9					
x_3	0.9	0.2				
x_4	0.8	0.3	0.3			
x_5	1.0	0.1	0.1	0.2		
x_6	0.4	0.5	0.5	0.4	0.6	

along the Jordan River; c stands for Israel retaining East Jerusalem; d is Israeli military outposts on the Golan Heights; and e denotes Arab countries granting citizenship to Palestinians who choose to remain within their borders.

Subsequently, Pawlak [23] put forward the concept of distance function between two objects for conflict analysis as follows.

Definition 2.4 [23]**.** Let $S=(U,A)$ be an information system. Then the distance function $\rho_A(x, y)$ for $x, y \in U$ is defined as follows:

$$\rho_A(x,y)=\frac{\sum_{a\in A}\phi_a^*(x,y)}{|A|},$$

where

$$\phi_a^*(x,y)=\frac{1-\phi_a(x,y)}{2}=\begin{cases}0, & \text{if} \quad a(x)\cdot a(y)=1\vee x=y;\\ 0.5, & \text{if} \quad a(x)\cdot a(y)=0\wedge x\neq y;\\ 1, & \text{if} \quad a(x)\cdot a(y)=-1.\end{cases}$$

By Definition 2.4, we obtain the conflict space $S=(U,\rho_A)$, where ρ_A denotes the distance function. Pawlak provided the conflict, neutral, and allied relations for conflict analysis using the distance function ρ_A as follows.

Definition 2.5 [23]**.** Let $S=(U,\rho_A)$ be a conflict space, and the distance function $\rho_A(x, y)$ for $x, y \in U$. Then a pair x and y is said to be

(1) conflict if $\rho_A(x, y) > 0.5$;
(2) neutral if $\rho_A(x,y)=0.5$;
(3) allied if $\rho_A(x, y) < 0.5$.

By Definition 2.5, Pawlak presented the allied, conflict, and neutral sets as follows.

Definition 2.6 [23]**.** Let $S=(U,\rho_A)$ be a conflict space. Then the conflict, neutral, and allied sets of $x \in U$ are defined as follows:

(1) $CO(x)=\{y\in U \mid \rho_A(x,y)>0.5\}$;
(2) $NE(x)=\{y\in U \mid \rho_A(x,y)=0.5\}$;
(3) $AL(x)=\{y\in U \mid \rho_A(x,y)<0.5\}$.

By Definition 2.6, we get the conflict, neutral, and allied sets of each object, which reveals the relationship between two objects.

Definition 2.7 [23]**.** Let $S=(U,\rho_A)$ be a conflict space. Then the distance matrix M_A is defined as follows:

$$M_A=[\rho_A(x,y)]_{n\times n}=\left[\frac{\sum_{a\in A}\phi_a^*(x,y)}{|A|}\right]_{n\times n},$$

where

$$\phi_a^*(x,y)=\begin{cases}0, & \text{if} \quad a(x)\cdot a(y)=1\vee x=y;\\ 0.5, & \text{if} \quad a(x)\cdot a(y)=0\wedge x\neq y;\\ 1, & \text{if} \quad a(x)\cdot a(y)=-1.\end{cases}$$

We employ an example in [23] to illustrate the distance matrix, the conflict, neutral, and allied sets as follows.

Example 2.8 [23] (Continued from Example 2.3)**.** By Definition 2.7, we have the distance matrix shown in Table 2 for the Middle East conflict as follows.

By Definition 2.6, we have the conflict, neutral, and allied sets for each object as follows:

In Example 2.8, we find that all objects are classified into the conflict, neutral, and allied sets with a threshold of 0.5. In practical situations, if object x, which actually is in conflict with y, is classified into the allied set of y with a threshold 0.5, then object y suffers a great loss. Therefore, it is important to choose the optimal thresholds for classifying objects.

Table 3
Conflict, Neutral, and Allied sets for the Middle East conflict.

U	$CO(x_i)$	$NE(x_i)$	$AL(x_i)$
x_1	$\{x_2, x_3, x_4, x_5\}$	$\emptyset$	$\{x_1, x_6\}$
x_2	$\{x_1\}$	$\{x_6\}$	$\{x_2, x_3, x_4, x_5\}$
x_3	$\{x_1\}$	$\{x_6\}$	$\{x_2, x_3, x_4, x_5\}$
x_4	$\{x_1\}$	$\emptyset$	$\{x_2, x_3, x_4, x_5, x_6\}$
x_5	$\{x_1, x_6\}$	$\emptyset$	$\{x_2, x_3, x_4, x_5\}$
x_6	$\{x_5\}$	$\{x_2, x_3\}$	$\{x_1, x_4, x_6\}$

Table 4
Loss function.

Action	$X(P)$	$\neg X(N)$
a_P	λ_{PP}	λ_{PN}
a_B	λ_{BP}	λ_{BN}
a_N	λ_{NP}	λ_{NN}

2.2. Decision-theoretic rough set theory

Definition 2.9. [38] Let $S = (U, A)$ be an information system, $P(X|[x] = \frac{|[x]\cap X|}{|[x]|}$ for $X \subseteq U$, and $0 \le \beta \le \alpha \le 1$. Then the probabilistic lower and upper approximations $\underline{apr}_{(\alpha,\beta)}(X)$ and $\overline{apr}_{(\alpha,\beta)}(X)$ of $X \subseteq U$ are defined as follows:

$$\underline{apr}_{(\alpha,\beta)}(X) = \{x \in U \mid P(X|[x]) \ge \alpha\};$$
$$\overline{apr}_{(\alpha,\beta)}(X) = \{x \in U \mid P(X|[x]) \ge \beta\}.$$

The probabilistic lower and upper approximations of sets are generalizations of Pawlak's lower and upper approximations of sets, respectively, the tolerance of which is higher than that of Pawlak's rough sets. Therefore, the probabilistic rough set theory is better than Pawlak's model for handling uncertain and imprecise information.

Definition 2.10. [38] Let $S = (U, A)$ be an information system, and $0 \le \beta \le \alpha \le 1$. Then the probabilistic positive, boundary, and negative regions $POS_{(\alpha,\beta)}(X)$, $BND_{(\alpha,\beta)}(X)$, and $NEG_{(\alpha,\beta)}(X)$ of $X \subseteq U$ are defined as follows:

$$POS_{(\alpha,\beta)}(X) = \{x \in U \mid P(X|[x]) \ge \alpha\};$$
$$BND_{(\alpha,\beta)}(X) = \{x \in U \mid \beta < P(X|[x]) < \alpha\};$$
$$NEG_{(\alpha,\beta)}(X) = \{x \in U \mid P(X|[x]) \le \beta\}.$$

Consequently, the decision-theoretic rough set model was proposed for computing the two thresholds α and β, which contain 2 states ($\Omega = \{X, \neg X\}$) and 3 actions ($\mathscr{A} = \{a_P, a_B, a_N\}$), where X and $\neg X$ indicate that an object is in X and not in X, respectively, and a_P, a_B, and a_N denote three actions in classifying an object x into $POS_{(\alpha,\beta)}(X)$, $BND_{(\alpha,\beta)}(X)$, and $NEG_{(\alpha,\beta)}(X)$, respectively. In Table 4, λ_{PP}, λ_{BP}, and λ_{NP} stand for the losses of taking actions a_P, a_B, and a_N, respectively, when an object belongs to X, and λ_{PN}, λ_{BN}, and λ_{NN} mean the losses of taking actions a_P, a_B, and a_N, respectively, when an object belongs to $\neg X$.

By Table 4, the expected losses $R(a_P|[x])$, $R(a_B|[x])$, and $R(a_N|[x])$ associated with taking the individual actions for an object x are shown as follows:

$$R(a_P|[x]) = \lambda_{PP}P(X|[x]) + \lambda_{PN}P(\neg X|[x]);$$
$$R(a_B|[x]) = \lambda_{BP}P(X|[x]) + \lambda_{BN}P(\neg X|[x]);$$
$$R(a_N|[x]) = \lambda_{NP}P(X|[x]) + \lambda_{NN}P(\neg X|[x]).$$

The Bayesian decision procedure suggests the following minimum-cost decision rules:
(P): If $R(a_P|[x]) \le R(a_B|[x])$ and $R(a_P|[x]) \le R(a_N|[x])$, then $x \in POS_{(\alpha,\beta)}(X)$;
(B): If $R(a_B|[x]) \le R(a_P|[x])$ and $R(a_B|[x]) \le R(a_N|[x])$, then $x \in BND_{(\alpha,\beta)}(X)$;
(N): If $R(a_N|[x]) \le R(a_P|[x])$ and $R(a_N|[x]) \le R(a_B|[x])$, then $x \in NEG_{(\alpha,\beta)}(X)$.
Suppose $\lambda_{PP} \le \lambda_{BP} \le \lambda_{NP}$ and $\lambda_{NN} \le \lambda_{BN} \le \lambda_{PN}$, since $P(X|[x]) + P(\neg X|[x]) = 1$, the rules (P), (B), and (N) are simplified as follows:
(P): If $P(X|[x]) \ge \alpha$ and $P(X|[x]) \ge \gamma$, then $x \in POS_{(\alpha,\beta)}(X)$;
(B): If $P(X|[x]) < \alpha$ and $P(X|[x]) > \beta$, then $x \in BND_{(\alpha,\beta)}(X)$;
(N): If $P(X|[x]) \le \beta$ and $P(X|[x]) \le \gamma$, then $x \in NEG_{(\alpha,\beta)}(X)$, where

$$\alpha = \frac{\lambda_{PN} - \lambda_{BN}}{\lambda_{PN} - \lambda_{BN} + \lambda_{BP} - \lambda_{PP}}, \beta = \frac{\lambda_{BN} - \lambda_{NN}}{\lambda_{BN} - \lambda_{NN} + \lambda_{NP} - \lambda_{BP}}, \gamma = \frac{\lambda_{PN} - \lambda_{NN}}{\lambda_{PN} - \lambda_{NN} + \lambda_{NP} - \lambda_{PP}}.$$

Table 5
Information system for the conflict.

U	a
x_1	−1
x_2	+1
x_3	+1
x_4	0
x_5	+1
x_6	0

Table 6
Distance matrix for the conflict.

U	x_1	x_2	x_3	x_4	x_5	x_6
x_1						
x_2	1					
x_3	1	0				
x_4	0.5	0.5	0.5			
x_5	1.0	0	0	0.5		
x_6	0.5	0.5	0.5	0.5	0.5	

Table 7
Information system for the conflict.

U	a	b	c	d	e
x_1	+1	+1	+1	+1	+1
x_2	+1	+1	+1	−1	−1
x_3	+1	−1	−1	−1	0
x_4	0	−1	−1	0	−1
x_5	+1	−1	−1	−1	−1
x_6	0	+1	−1	0	+1

2.3. Illustration of motivations with two examples

In this section, we consider two examples to illustrate the motivation of conflict analysis based on decision-theoretic rough sets.

Example 2.11. Let Table 5 be an information system for the conflict, where $U = \{x_1, x_2, \ldots, x_6\}$, and $A = \{a\}$. Then the distance matrix M_A is shown in Table 6 as follows:

There is only an attribute a and three values {1, 0.5, 0} for the conflict in Tables 5 and 6. Therefore, the value 0.5 is effective to identify that two objects are in conflict, allied or neutral.

In Example 2.11, Pawlak's model is effective to conduct conflict analysis on the information system whose distance matrix contains three values {1,0.5,0}. In practical situations, there are more than three values in the distance matrix of an information system, which limits the application of Pawlak's model.

Example 2.12. Let Tables 7 and 8 be information systems for the conflict and the loss function related to $x, y \in U$. In Table 8, a_C, a_N, and a_A denote three actions in classifying y into $CO(x)$, $NE(x)$, and $AL(x)$, respectively; $\lambda^*_{CC}, \lambda^*_{NC}$, and λ^*_{AC} mean losses of taking actions of a_C, a_N, and a_A with respect to the attribute *, respectively, when the object belongs to $CO(x)$; $\lambda^*_{CA}, \lambda^*_{NA}$, and λ^*_{AA} stand for losses of taking actions of a_C, a_N, and a_A with respect to the attribute *, respectively, when the object belongs to $AL(x)$.

From Definitions 2.4 and 2.6, we determine that objects x_1 and x_2 are allied since $\rho_A(x_1, x_2) = 0.4 < 0.5$. In fact, we see that objects x_1 and x_2 are allied on attributes a, b and c, but they are in conflict on attributes d and e in Table 7. Therefore, there are some losses if we classify objects x_1 and x_2 into the coalition.

In Table 8, we observe that the loss is $\lambda^d_{AC} + \lambda^e_{AC} = 80 + 80 = 160$ if we put x_1 and x_2 into the allied set; $\lambda^a_{CA} + \lambda^b_{CA} + \lambda^c_{CA} = 10 + 10 + 10 = 30$ if we classify x_1 and x_2 into the conflict set; and $\lambda^a_{NA} + \lambda^b_{NA} + \lambda^c_{NA} + \lambda^d_{NC} + \lambda^e_{NC} = 5 + 5 + 5 + 40 + 40 = 95$ if we add x_1 and x_2 into the neutral set. Thus, the objects x_1 and x_2 are in conflict with respect to the loss since $30 < 95 < 160$, which is different from the results determined using Pawlak's model. Therefore, it is necessary to study conflict analysis based on decision-theoretic rough sets.

Table 8
Loss function related to $x_1, x_2 \in U$.

A	Action	$x_2 \in CO(x_1)$	$x_2 \in AL(x_1)$
a	a_C	$\lambda^a_{CC} = 0$	$\lambda^a_{CA} = 10$
	a_N	$\lambda^a_{NC} = 5$	$\lambda^a_{NA} = 5$
	a_A	$\lambda^a_{AC} = 10$	$\lambda^a_{AA} = 0$
b	a_C	$\lambda^b_{CC} = 0$	$\lambda^b_{CA} = 10$
	a_N	$\lambda^b_{NC} = 5$	$\lambda^b_{NA} = 5$
	a_A	$\lambda^b_{AC} = 10$	$\lambda^b_{AA} = 0$
c	a_C	$\lambda^c_{CC} = 0$	$\lambda^c_{CA} = 10$
	a_N	$\lambda^c_{NC} = 5$	$\lambda^c_{NA} = 5$
	a_A	$\lambda^c_{AC} = 10$	$\lambda^c_{AA} = 0$
d	a_C	$\lambda^d_{CC} = 0$	$\lambda^d_{CA} = 80$
	a_N	$\lambda^d_{NC} = 40$	$\lambda^d_{NA} = 40$
	a_A	$\lambda^d_{AC} = 80$	$\lambda^d_{AA} = 0$
e	a_C	$\lambda^e_{CC} = 0$	$\lambda^e_{CA} = 80$
	a_N	$\lambda^e_{NC} = 40$	$\lambda^e_{NA} = 40$
	a_A	$\lambda^e_{AC} = 80$	$\lambda^e_{AA} = 0$

3. Conflict analysis based on decision-theoretic rough sets

In this section, we provide a conflict analysis model based on decision-theoretic rough sets. First, we propose the concepts of probabilistic conflict, neutral, and allied relations as follows.

Definition 3.1. Let $S = (U, \rho_A)$ be a conflict space, and $0 \le \beta \le \alpha \le 1$. Then the pair x and y is said to be

(1) probabilistic conflict if $\rho_A(x, y) > \alpha$;
(2) probabilistic neutral if $\alpha \ge \rho_A(x, y) \ge \beta$;
(3) probabilistic allied if $\rho_A(x, y) < \beta$.

The probabilistic conflict, neutral, and allied relations are generalizations of Pawlak's conflict, neutral, and allied relations, respectively. In particular, the probabilistic conflict, neutral, and allied relations are the same as those of Pawlak's model when the two thresholds $\alpha = \beta = 0.5$. More pairs of objects will be probabilistically neutral by Definition 3.1 if $\alpha \ge 0.5$ and $\beta \le 0.5$.

We concurrently present the concepts of probabilistic conflict, neutral, and allied sets as follows.

Definition 3.2. Let $S = (U, \rho_A)$ be a conflict space, and $0 \le \beta \le \alpha \le 1$. For any $x \in U$, the probabilistic conflict, neutral, and allied sets $CO^\alpha_\beta(x)$, $NE^\alpha_\beta(x)$, and $AL^\alpha_\beta(x)$ of x are defined as follows:

(1) $CO^\alpha_\beta(x) = \{y \in U \mid \rho_A(x, y) > \alpha\}$;
(2) $NE^\alpha_\beta(x) = \{y \in U \mid \alpha \ge \rho_A(x, y) \ge \beta\}$;
(3) $AL^\alpha_\beta(x) = \{y \in U \mid \rho_A(x, y) < \beta\}$.

Second, we discuss the properties of the probabilistic allied, conflict, and neutral sets as follows.

Theorem 3.3. *Let $S = (U, \rho_A)$ be a conflict space, and $0 \le \beta \le \alpha \le 1$. For any $x, y \in U$, we have*

(1) $y \in CO^\alpha_\beta(x) \Leftrightarrow x \in CO^\alpha_\beta(y)$;
(2) $y \in NE^\alpha_\beta(x) \Leftrightarrow x \in NE^\alpha_\beta(y)$;
(3) $y \in AL^\alpha_\beta(x) \Leftrightarrow x \in AL^\alpha_\beta(y)$.

Proof.

(1) By Definition 3.2(1), we have $CO^\alpha_\beta(x) = \{y \in U \mid \rho_A(x, y) > \alpha\}$ and $CO^\alpha_\beta(y) = \{z \in U \mid \rho_A(y, z) > \alpha\}$. If $y \in CO^\alpha_\beta(x)$, we obtain $\rho_A(x, y) > \alpha$. Thus $\rho_A(y, x) > \alpha$. It follows $x \in CO^\alpha_\beta(y)$, and vice versa. Therefore, $y \in CO^\alpha_\beta(x) \Leftrightarrow x \in CO^\alpha_\beta(y)$.
(2) By Definition 3.2 (2), we get $NE^\alpha_\beta(x) = \{y \in U \mid \beta \le \rho_A(x, y) \le \alpha\}$ and $NE^\alpha_\beta(y) = \{z \in U \mid \beta \le \rho_A(y, z) \le \alpha\}$. If $y \in NE^\alpha_\beta(x)$, we have $\beta \le \rho_A(x, y) \le \alpha$. Thus $\beta \le \rho_A(y, x) \le \alpha$. It follows $x \in NE^\alpha_\beta(y)$, and vice versa. Therefore, $y \in NE^\alpha_\beta(x) \Leftrightarrow x \in NE^\alpha_\beta(y)$.
(3) By Definition 3.2 (3), we obtain $AL^\alpha_\beta(x) = \{y \in U \mid \rho_A(x, y) < \beta\}$ and $AL^\alpha_\beta(y) = \{z \in U \mid \rho_A(y, z) < \beta\}$. If $y \in AL^\alpha_\beta(x)$, we have $\rho_A(x, y) < \beta$. Thus $\rho_A(y, x) < \beta$. It follows $x \in AL^\alpha_\beta(y)$, and vice versa. Therefore, $y \in AL^\alpha_\beta(x) \Leftrightarrow x \in AL^\alpha_\beta(y)$. □

Table 9
Probabilistic Conflict, Neutral, and Allied sets for the Middle East conflict.

U	$CO^{\alpha}_{\beta}(x_i)$	$NE^{\alpha}_{\beta}(x_i)$	$AL^{\alpha}_{\beta}(x_i)$
x_1	$\{x_2, x_3, x_4, x_5\}$	$\{x_6\}$	$\{x_1\}$
x_2	$\{x_1\}$	$\{x_4, x_6\}$	$\{x_2, x_3, x_5\}$
x_3	$\{x_1\}$	$\{x_4, x_6\}$	$\{x_2, x_3, x_5\}$
x_4	$\{x_1\}$	$\{x_2, x_3, x_6\}$	$\{x_4, x_5\}$
x_5	$\{x_1\}$	$\{x_6\}$	$\{x_2, x_3, x_4, x_5\}$
x_6	$\emptyset$	$\{x_1, x_2, x_3, x_4, x_5\}$	$\{x_6\}$

Table 10
Loss function related to $x, y \in U$ with respect to A.

Action	$y \in CO(x)$	$y \in AL(x)$
a_C	λ_{CC}	λ_{CA}
a_N	λ_{NC}	λ_{NA}
a_A	λ_{AC}	λ_{AA}

Theorem 3.3(1) illustrates the relationship between an object and its probabilistic conflict set; Theorem 3.3(2) demonstrates the relationship between an object and its probabilistic neutral set; and Theorem 3.3(3) depicts the relationship between an object and its probabilistic allied set.

Example 3.4 (Continued from Example 2.8)**.** By Definition 3.2, we have the probabilistic conflict, neutral, and allied sets of each object when taking $\alpha = 0.75$ and $\beta = 0.25$ as follows:

There are some differences between the probabilistic conflict, neutral, and allied sets in Tables 3 and 9. In particular, we observe that more objects are classified into the probabilistic neutral sets in Table 9. If we classify the objects of the probabilistic neutral sets into the probabilistic conflict sets and allied sets without enough information, it will generate risks in practical situations. Therefore, the thresholds α and β are important for computing the probabilistic conflict, neutral, and allied sets in conflict analysis.

Third, we calculate the parameters α and β for conflict analysis using decision-theoretic rough set theory.

Theorem 3.5. *Let $S = (U, \rho_A)$ be a conflict space, the distance $\rho_A(x, y)$ for $x, y \in U$, and the losses λ_{CC}, λ_{NC}, λ_{AC}, λ_{AA}, λ_{NA}, and λ_{CA}, where $0 \leq \lambda_{CC} \leq \lambda_{NC} \leq \lambda_{AC}$ and $0 \leq \lambda_{AA} \leq \lambda_{NA} \leq \lambda_{CA}$. Then*

(1) *If $\rho_A(x, y) > \alpha$, then $y \in CO^{\alpha}_{\beta}(x)$;*
(2) *If $\alpha \geq \rho_A(x, y) \geq \beta$, then $y \in NE^{\alpha}_{\beta}(x)$;*
(3) *If $\rho_A(x, y) < \beta$, then $y \in AL^{\alpha}_{\beta}(x)$, where*

$$\alpha = \frac{\lambda_{CA} - \lambda_{NA}}{\lambda_{CA} - \lambda_{NA} + \lambda_{NC} - \lambda_{CC}}, \beta = \frac{\lambda_{NA} - \lambda_{AA}}{\lambda_{NA} - \lambda_{AA} + \lambda_{AC} - \lambda_{NC}}, \gamma = \frac{\lambda_{CA} - \lambda_{AA}}{\lambda_{CA} - \lambda_{AA} + \lambda_{AC} - \lambda_{CC}}.$$

Proof. By Table 10, we have the expected losses $R^x(a_C|y)$, $R^x(a_N|y)$, and $R^x(a_A|y)$ associated with taking the individual actions for the object y as follows:

$$R^x(a_C|y) = \lambda_{CC} * \rho_A(x, y) + \lambda_{CA} * (1 - \rho_A(x, y));$$
$$R^x(a_N|y) = \lambda_{NC} * \rho_A(x, y) + \lambda_{NA} * (1 - \rho_A(x, y));$$
$$R^x(a_A|y) = \lambda_{AC} * \rho_A(x, y) + \lambda_{AA} * (1 - \rho_A(x, y)).$$

The Bayesian decision procedure suggests the following minimum-cost decision rules:
(C): If $R^x(a_C|y) \leq R^x(a_N|y)$ and $R^x(a_C|y) \leq R^x(a_A|y)$, then $y \in CO^{\alpha}_{\beta}(x)$;
(N): If $R^x(a_N|y) \leq R^x(a_C|y)$ and $R^x(a_N|y) \leq R^x(a_A|y)$, then $y \in NE^{\alpha}_{\beta}(x)$;
(A): If $R^x(a_A|y) \leq R^x(a_P|y)$ and $R^x(a_A|y) \leq R^x(a_N|y)$, then $y \in AL^{\alpha}_{\beta}(x)$.
Suppose $\lambda_{CC} \leq \lambda_{NC} \leq \lambda_{AC}$ and $\lambda_{AA} \leq \lambda_{NA} \leq \lambda_{CN}$, we simply the rules ($C$), ($N$), and ($A$) as follows:
(C): If $\rho_A(x, y) > \alpha$ and $\rho_A(x, y) > \gamma$, then $y \in CO^{\alpha}_{\beta}(x)$;
(N): If $\rho_A(x, y) \leq \alpha$ and $\rho_A(x, y) \geq \beta$, then $y \in NE^{\alpha}_{\beta}(x)$;
(A): If $\rho_A(x, y) < \beta$ and $\rho_A(x, y) < \gamma$, then $y \in AL^{\alpha}_{\beta}(x)$, where

$$\alpha = \frac{\lambda_{CA} - \lambda_{NA}}{\lambda_{CA} - \lambda_{NA} + \lambda_{NC} - \lambda_{CC}}, \beta = \frac{\lambda_{NA} - \lambda_{AA}}{\lambda_{NA} - \lambda_{AA} + \lambda_{AC} - \lambda_{NC}}, \gamma = \frac{\lambda_{CA} - \lambda_{AA}}{\lambda_{CA} - \lambda_{AA} + \lambda_{AC} - \lambda_{CC}}.$$

Based on Theorem 3.5, objects are classified into the probabilistic conflict, neutral, and allied sets with the minimum losses, which avoids the misclassification of objects in the conflict analysis. □

Table 11
Loss function related to x, $y \in U$ with respect to A.

Action	$y \in CO(x)$	$y \in AL(x)$
a_C	$\lambda_{CC} = 0$	$\lambda_{CA} = 5$
a_N	$\lambda_{NC} = 2$	$\lambda_{NA} = 2$
a_A	$\lambda_{AC} = 6$	$\lambda_{AA} = 0$

Table 12
Probabilistic Conflict, Neutral, and Allied sets for the Middle East conflict.

U	$CO^{\alpha}_{\beta}(x_i)$	$NE^{\alpha}_{\beta}(x_i)$	$AL^{\alpha}_{\beta}(x_i)$
x_1	$\{x_2, x_3, x_4, x_5\}$	$\{x_6\}$	$\{x_1\}$
x_2	$\{x_1\}$	$\{x_6\}$	$\{x_2, x_3, x_4, x_5\}$
x_3	$\{x_1\}$	$\{x_6\}$	$\{x_2, x_3, x_4, x_5\}$
x_4	$\{x_1\}$	$\{x_6\}$	$\{x_2, x_3, x_4, x_5\}$
x_5	$\{x_1\}$	$\{x_6\}$	$\{x_2, x_3, x_4, x_5\}$
x_6	$\emptyset$	$\{x_1, x_2, x_3, x_4, x_5\}$	$\{x_6\}$

Finally, we propose non-incremental algorithm for computing the probabilistic conflict, neutral, and allied sets(**NIC**) as follows.

Algorithm 3.6. Input: information system $S = (U, A)$;
Output: $CO^{\alpha}_{\beta}(x), NE^{\alpha}_{\beta}(x),$ and $AL^{\alpha}_{\beta}(x)$.
Step 1: Input information system $S = (U, A)$;
Step 2: Compute the distance matrix M_A;
Step 3: Construct the thresholds α, β, and γ;
Step 4: Calculate $CO^{\alpha}_{\beta}(x), NE^{\alpha}_{\beta}(x),$ and $AL^{\alpha}_{\beta}(x)$;
Step 5: Output $CO^{\alpha}_{\beta}(x), NE^{\alpha}_{\beta}(x),$ and $AL^{\alpha}_{\beta}(x)$.

The time complexity of Step 2 is $O(mn^2)$, where $|U| = n$ and $|A| = m$, and the time complexity of Step 4 is $O(n^2)$. Therefore, the time complexity of Algorithm 3.6 is $O(mn^2 + n^2)$.

Example 3.7. (Continued from Example 2.8) By Theorem 3.5, we have the thresholds α, β, and γ using Table 11 as follows:

$$\alpha = \frac{\lambda_{CA} - \lambda_{NA}}{\lambda_{CA} - \lambda_{NA} + \lambda_{NC} - \lambda_{CC}} = \frac{5-2}{5-2+2-0} = \frac{3}{5},$$
$$\beta = \frac{\lambda_{NA} - \lambda_{AA}}{\lambda_{NA} - \lambda_{AA} + \lambda_{AC} - \lambda_{NC}} = \frac{2-0}{2-0+6-2} = \frac{1}{3},$$
$$\gamma = \frac{\lambda_{CA} - \lambda_{AA}}{\lambda_{CA} - \lambda_{AA} + \lambda_{AC} - \lambda_{CC}} = \frac{5-0}{5-0+6-0} = \frac{5}{11}.$$

Subsequently, we get the probabilistic conflict, neutral, and allied sets as shown in Table 12.

4. Conflict analysis in dynamic information systems

In this section, we discuss the mechanism of constructing the probabilistic conflict, neutral, and allied sets in dynamic information systems.

4.1. Conflict analysis when adding objects

We first provide incremental approach for constructing the probabilistic conflict, neutral, and allied sets when adding objects.

Definition 4.1. Let (U, A) and (U^+, A) be information systems, where $U = \{x_1, x_2, \ldots, x_n\}$ and $U^+ = \{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{n+t}\}(t \geq 1)$. Then (U^+, A) is called a dynamic information system of (U, A).

For simplicity, the information system (U, A) is called the original information system of (U^+, A). In practical situations, there are dynamic information systems with variations of the object sets, attribute sets, and attribute values. We only discuss dynamic information systems when adding objects in this section. Furthermore, we employ an example to illustrate the relationship between (U, A) and (U^+, A) as follows.

Table 13
Dynamic information system for the Middle East conflict.

U^+	a	b	c	d	e
x_1	−1	+1	+1	+1	+1
x_2	+1	0	−1	−1	−1
x_3	+1	−1	−1	−1	0
x_4	0	−1	−1	0	−1
x_5	+1	−1	−1	−1	−1
x_6	0	+1	−1	0	+1
x_7	−1	+1	+1	+1	+1

Example 4.2 (Continued from Example 2.3)**.** Let (U, A) and (U^+, A) be shown in Tables 1 and 13, respectively. Then we have $U^+ = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\} = U \cup \{x_7\}$, and (U^+, A) is a dynamic information system of (U, A).

Second, we investigate how to compute the distance matrix and the probabilistic conflict, neutral, and allied sets in dynamic information system.

Theorem 4.3. *Let (U^+, A) and (U, A) be information systems, the distance matrices $M_A^+ = [\rho_A^+(x, y)]_{(n+t)\times(n+t)}$ and $M_A = [\rho_A(x, y)]_{n\times n}$. Then*

$$\rho_A^+(x,y) = \begin{cases} \rho_A(x,y), & \text{if } x, y \in U; \\ \dfrac{\sum_{a\in A} \phi_a^*(x,y)}{|A|}, & otherwise. \end{cases}$$

where

$$\phi_a^*(x,y) = \begin{cases} 0, & \text{if} \quad a(x)\cdot a(y) = 1 \vee x = y; \\ 0.5, & \text{if} \quad a(x)\cdot a(y) = 0 \wedge x \neq y; \\ 1, & \text{if} \quad a(x)\cdot a(y) = -1. \end{cases}$$

Proof. By Definition 2.7, the proof is straightforward. □

Theorem 4.3 illustrates the relationship between M_A^+ and M_A, which reduces the computation time greatly in practical situations. Moreover, we obtain almost all elements of M_A^+ without computation using M_A, which is helpful for calculating the probabilistic allied, conflict, and neutral sets.

Theorem 4.4. *Let (U^+, A) and (U, A) be information systems, and the thresholds $0 \leq \beta \leq \alpha \leq 1$. Then we have the probabilistic conflict, neutral, and allied sets of $x \in U^+$ as follows:*

$$(1)\ CO_\beta^{\alpha+}(x) = \begin{cases} CO_\beta^\alpha(x) \cup \{y \in U^+/U \mid \rho_A^+(x,y) > \alpha\}, & \text{if} \quad x \in U; \\ \{y \in U^+ \mid \rho_A^+(x,y) > \alpha\}, & \text{if} \quad x \in U^+/U. \end{cases}$$

$$(2)\ NE_\beta^{\alpha+}(x) = \begin{cases} NE_\beta^\alpha(x) \cup \{y \in U^+/U \mid \alpha \geq \rho_A^+(x,y) \geq \beta\}, & \text{if} \quad x \in U; \\ \{y \in U^+ \mid \alpha \geq \rho_A^+(x,y) \geq \beta\}, & \text{if} \quad x \in U^+/U. \end{cases}$$

$$(3)\ AL_\beta^{\alpha+}(x) = \begin{cases} AL_\beta^\alpha(x) \cup \{y \in U^+/U \mid \rho_A^+(x,y) < \beta\}, & \text{if} \quad x \in U; \\ \{y \in U^+ \mid \rho_A^+(x,y) < \beta\}, & \text{if} \quad x \in U^+/U. \end{cases}$$

Proof. (1) By Definition 3.2(1), we get $CO_\beta^{\alpha+}(x) = \{y \in U^+ \mid \rho_A^+(x,y) > \alpha\} = \{y \in U \mid \rho_A(x,y) > \alpha\} \cup \{y \in U^+/U \mid \rho_A^+(x,y) > \alpha\}$ and $CO_\beta^\alpha(x) = \{y \in U \mid \rho_A(x,y) > \alpha\}$ for any $x \in U$. Thus, $CO_\beta^{\alpha+}(x) = CO_\beta^\alpha(x) \cup \{y \in U^+/U \mid \rho_A^+(x,y) > \alpha\}$. On the other hand, we have $CO_\beta^{\alpha+}(x) = \{y \in U^+ \mid \rho_A^+(x,y) > \alpha\}$ for any $x \in U^+/U$.

(2) By Definition 3.2(2), we obtain $NE_\beta^{\alpha+}(x) = \{y \in U^+ \mid \alpha \geq \rho_A^+(x,y) \geq \beta\} = \{y \in U \mid \alpha \geq \rho_A(x,y) \geq \beta\} \cup \{y \in U^+/U \mid \alpha \geq \rho_A^+(x,y) \geq \beta\}$ and $NE_\beta^\alpha(x) = \{y \in U \mid \alpha \geq \rho_A(x,y) \geq \beta\}$ for any $x \in U$. Thus, $NE_\beta^{\alpha+}(x) = NE_\beta^\alpha(x) \cup \{y \in U^+/U \mid \alpha \geq \rho_A^+(x,y) \geq \beta\}$. On the other hand, we have $NE_\beta^{\alpha+}(x) = \{y \in U^+ \mid \alpha \geq \rho_A^+(x,y) \geq \beta\}$ for any $x \in U^+/U$.

(3) By Definition 3.2(3), we get $AL_\beta^{\alpha+}(x) = \{y \in U^+ \mid \rho_A^+(x,y) < \beta\} = \{y \in U \mid \rho_A(x,y) < \beta\} \cup \{y \in U^+/U \mid \rho_A^+(x,y) < \beta\}$ and $AL_\beta^\alpha(x) = \{y \in U \mid \rho_A(x,y) < \beta\}$ for any $x \in U$. Thus, $AL_\beta^{\alpha+}(x) = AL_\beta^\alpha(x) \cup \{y \in U^+/U \mid \rho_A^+(x,y) < \beta\}$. On the other hand, we have $AL_\beta^{\alpha+}(x) = \{y \in U^+ \mid \rho_A^+(x,y) < \beta\}$ for any $x \in U^+/U$. □

Theorem 4.4 illustrates the probabilistic sets $CO_\beta^{\alpha+}(x), NE_\beta^{\alpha+}(x)$, and $AL_\beta^{\alpha+}(x)$, which are constructed using the sets $CO_\beta^\alpha(x), NE_\beta^\alpha(x)$, and $AL_\beta^\alpha(x)$, respectively. Moreover, we propose incremental algorithms for computing the probabilistic conflict, neutral, and allied sets in dynamic information systems based on Theorem 4.4 as follows.

Algorithm 4.5 (Incremental algorithm for computing. $CO_\beta^{\alpha+}(x), NE_\beta^{\alpha+}(x)$, and $AL_\beta^{\alpha+}(x)$(**ICA**)**)**

Table 14
Distance matrix for the Middle East conflict.

U^+	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_1							
x_2	0.9						
x_3	0.9	0.2					
x_4	0.8	0.3	0.3				
x_5	1.0	0.1	0.1	0.2			
x_6	0.4	0.5	0.5	0.4	0.6		
x_7	0	0.9	0.9	0.8	1.0	0.4	

Table 15
Probabilistic Conflict, Neutral, and Allied sets for the Middle East conflict.

U^+	$CO_\beta^{\alpha+}(x_i)$	$NE_\beta^{\alpha+}(x_i)$	$AL_\beta^{\alpha+}(x_i)$
x_1	$\{x_2, x_3, x_4, x_5\}$	$\{x_6\}$	$\{x_1, x_7\}$
x_2	$\{x_1, x_7\}$	$\{x_6\}$	$\{x_2, x_3, x_4, x_5\}$
x_3	$\{x_1, x_7\}$	$\{x_6\}$	$\{x_2, x_3, x_4, x_5\}$
x_4	$\{x_1, x_7\}$	$\{x_6\}$	$\{x_2, x_3, x_4, x_5\}$
x_5	$\{x_1, x_7\}$	$\{x_6\}$	$\{x_2, x_3, x_4, x_5\}$
x_6	$\emptyset$	$\{x_1, x_2, x_3, x_4, x_5, x_7\}$	$\{x_6\}$
x_7	$\{x_2, x_3, x_4, x_5\}$	$\{x_6\}$	$\{x_1, x_7\}$

Input: information system $S^+ = (U^+, A)$;
Output: $CO_\beta^{\alpha+}(x), NE_\beta^{\alpha+}(x)$, and $AL_\beta^{\alpha+}(x)$.
Step 1: Input information system $S^+ = (U^+, A)$;
Step 2: Compute the distance matrix M_A^+;
Step 3: Calculate $CO_\beta^{\alpha+}(x), NE_\beta^{\alpha+}(x)$, and $AL_\beta^{\alpha+}(x)$;
Step 4: Output $CO_\beta^{\alpha+}(x), NE_\beta^{\alpha+}(x)$, and $AL_\beta^{\alpha+}(x)$.

The time complexity of Step 2 is $O(2mt + mt^2)$, and the time complexity of Step 3 is $O(2nt + t^2)$. Thus, the time complexity of Algorithm 4.5 is $O(mt^2 + t^2 + 2mt + 2nt)$, which is lower than that of Algorithm 3.6 for computing the probabilistic conflict, neutral, and allied sets in dynamic information systems.

Third, we employ an example to show how to compute the probabilistic conflict, neutral, and allied sets in dynamic information systems.

Example 4.6. (Continued from Example 4.2) By Theorem 4.3, we have the distance matrix shown in Table 14 for the Middle East conflict as follows.

By Theorem 4.4, we compute the probabilistic conflict, neutral, and allied sets as given in Table 15.

4.2. Conflict analysis when deleting objects

In this section, we first put forwards incremental approaches to computing the probabilistic conflict, neutral, and allied sets when deleting objects.

Definition 4.7. Let (*U, A*) and (U^-, A) be information systems, where $U = \{x_1, x_2, \ldots, x_n\}$ and $U^- = \{x_{l_1}, x_{l_2}, \ldots, x_{l_k}\}(k < n)$. Then (U^-, A) is called a dynamic information system of (*U, A*).

The information system (*U, A*) is called the original information system of (U^-, A). We only discuss dynamic information systems when deleting objects in this section. Subsequently, we employ an example to illustrate the relationship between (*U, A*) and (U^-, A) as follows.

Example 4.8. Let (*U, A*) and (U^-, A) be shown in Tables 1 and 16, respectively. Then (U^-, A) is a dynamic information system of (*U, A*).

Second, we investigate how to compute the distance matrix and the probabilistic conflict, neutral, and allied sets in dynamic information systems.

Theorem 4.9. *Let (U^-, A) and (U, A) be information systems, the distance matrices $M_A^- = [\rho_A^-(x, y)]_{l_k \times l_k}$ and $M_A = [\rho_A(x, y)]_{n \times n}$. Then $\rho_A^-(x, y) = \rho_A(x, y)$ for any $x, y \in U^-$.*

Proof. By Definition 2.7, the proof is straightforward. □

Table 16
Dynamic information system for the Middle East conflict.

U^-	a	b	c	d	e
x_1	−1	+1	+1	+1	+1
x_2	+1	0	−1	−1	−1
x_3	+1	−1	−1	−1	0
x_4	0	−1	−1	0	−1
x_5	+1	−1	−1	−1	−1

Table 17
Distance matrix for the Middle East conflict.

	x_1	x_2	x_3	x_4	x_5
x_1					
x_2	0.9				
x_3	0.9	0.2			
x_4	0.8	0.3	0.3		
x_5	1	0.1	0.1	0.2	

Theorem 4.9 shows the relationship between M^-_A and M_A, which reduces the computation time greatly in practical situations. Furthermore, we obtain almost all elements of M^-_A without computation using M_A, which is helpful for calculating the probabilistic allied, conflict, and neutral sets.

Theorem 4.10. *Let* (U^-, A) *and* (U, A) *be information systems, and the thresholds* $0 \leq \beta \leq \alpha \leq 1$. *Then we have the probabilistic conflict, neutral, and allied sets of* $x \in U^-$ *as follows:*

(1) $CO^{\alpha-}_{\beta}(x) = CO^{\alpha}_{\beta}(x)/(U/U^-)$;
(2) $NE^{\alpha-}_{\beta}(x) = NE^{\alpha}_{\beta}(x)/(U/U^-)$;
(3) $AL^{\alpha-}_{\beta}(x) = AL^{\alpha}_{\beta}(x)/(U/U^-)$.

Proof. (1) By Definition 3.2(1), we have $CO^{\alpha}_{\beta}(x) = \{y \in U \mid \rho_A(x, y) > \alpha\} = \{y \in U^- \mid \rho^-_A(x, y) > \alpha\} \cup \{y \in U/U^- \mid \rho_A(x, y) > \alpha\}$ and $CO^{\alpha-}_{\beta}(x) = \{y \in U^- \mid \rho^-_A(x, y) > \alpha\}$ for any $x \in U$. Therefore, $CO^{\alpha-}_{\beta}(x) = CO^{\alpha}_{\beta}(x)/(U/U^-)$.

(2) By Definition 3.2(2), we get $NE^{\alpha}_{\beta}(x) = \{y \in U \mid \alpha \geq \rho_A(x, y) \geq \beta\} = \{y \in U^- \mid \alpha \geq \rho^-_A(x, y) \geq \beta\} \cup \{y \in U/U^- \mid \alpha \geq \rho_A(x, y) \geq \beta\}$ and $NE^{\alpha-}_{\beta}(x) = \{y \in U^- \mid \alpha \geq \rho^-_A(x, y) \geq \beta\}$ for any $x \in U$. Therefore, $NE^{\alpha-}_{\beta}(x) = NE^{\alpha}_{\beta}(x)/(U/U^-)$.

(3) By Definition 3.2(3), we obtain $AL^{\alpha}_{\beta}(x) = \{y \in U \mid \rho_A(x, y) < \beta\} = \{y \in U^- \mid \rho^-_A(x, y) < \beta\} \cup \{y \in U/U^- \mid \rho_A(x, y) < \beta\}$ and $AL^{\alpha-}_{\beta}(x) = \{y \in U^- \mid \rho^-_A(x, y) < \beta\}$ for any $x \in U$. Therefore, $AL^{\alpha-}_{\beta}(x) = AL^{\alpha}_{\beta}(x)/(U/U^-)$. □

Theorem 4.10 indicates that the probabilistic sets $CO^{\alpha-}_{\beta}(x), NE^{\alpha-}_{\beta}(x)$, and $AL^{\alpha-}_{\beta}(x)$ are constructed using the sets $CO^{\alpha}_{\beta}(x), NE^{\alpha}_{\beta}(x)$, and $AL^{\alpha}_{\beta}(x)$, respectively. Moreover, we present an incremental algorithm for computing the probabilistic allied, conflict, and neutral sets in dynamic information systems by Theorem 4.10 as follows.

Algorithm 4.11. (Incremental algorithm for computing $CO^{\alpha-}_{\beta}(x), NE^{\alpha-}_{\beta}(x)$, and $AL^{\alpha-}_{\beta}(x)$(**ICD**))
Input: information system $S^- = (U^-, A)$;
Output: $CO^{\alpha-}_{\beta}(x), NE^{\alpha-}_{\beta}(x)$, and $AL^{\alpha-}_{\beta}(x)$.
Step 1: Input information system $S^- = (U^-, A)$;
Step 2: Compute the distance matrix M^-_A;
Step 3: Calculate $CO^{\alpha-}_{\beta}(x), NE^{\alpha-}_{\beta}(x)$, and $AL^{\alpha-}_{\beta}(x)$;
Step 4: Output $CO^{\alpha-}_{\beta}(x), NE^{\alpha-}_{\beta}(x)$, and $AL^{\alpha-}_{\beta}(x)$.

The time complexity of Step 2 is $O(l_k^2)$; the time complexity of Step 3 is $O(nl_k - l_k^2)$. Thus, the time complexity of Algorithm 4.11 is $O(nl_k)$. Therefore, the time complexity of Algorithm 4.11 is lower than that of Algorithm 3.6 for computing the probabilistic allied, conflict, and neutral sets in dynamic information systems.

Third, we explore an example to show how to compute the distance matrix and the probabilistic conflict, neutral, and allied sets in dynamic information systems.

Example 4.12 (Continued from Examples 2.3, 2.8, and 4.8)**.** By Theorem 4.9, we get the distance matrix Table 17 for the Middle East conflict as follows.

By Theorem 4.10, we have the probabilistic conflict, neutral, and allied sets for the Middle East conflict shown in Table 18 as below.

Table 18
Conflict, Neutral, and Allied sets for the Middle East conflict.

U^-	$CO_\beta^{\alpha-}(x_i)$	$NE_\beta^{\alpha-}(x_i)$	$AL_\beta^{\alpha-}(x_i)$
x_1	$\{x_2, x_3, x_4, x_5\}$	$\emptyset$	$\{x_1\}$
x_2	$\{x_1\}$	$\emptyset$	$\{x_2, x_3, x_4, x_5\}$
x_3	$\{x_1\}$	$\emptyset$	$\{x_2, x_3, x_4, x_5\}$
x_4	$\{x_1\}$	$\emptyset$	$\{x_2, x_3, x_4, x_5\}$
x_5	$\{x_1\}$	$\emptyset$	$\{x_2, x_3, x_4, x_5\}$

Table 19
Information systems for experiments.

No.	Name	$|U_i|$	$|A|$
1	(U_1, A)	600	5
2	(U_2, A)	1200	5
3	(U_3, A)	1800	5
4	(U_4, A)	2400	5
5	(U_5, A)	3000	5
6	(U_6, A)	3600	5
7	(U_7, A)	4200	5
8	(U_8, A)	4800	5
9	(U_9, A)	5400	5
10	(U_{10}, A)	6000	5

Table 20
The experimental environment.

No.	Name	Model	Parameters
1	CPU	Intel(R) Dual-Core CPU (TM)i5-4590	3.30 GHZ
2	Memory	ADAT DDR3	8G
3	Hard disk	SATA	300G
4	System	Windows 7	64 bit
5	Platform	Matlab R2014a	64 bit

5. Experimental analysis

In this section, we discuss the series of experiments carried out to verify the effectiveness of Algorithms 3.6, 4.5, and 4.11 for the maintenance of the probabilistic conflict, neutral, and allied sets in dynamic information systems with the variation of the object sets.

5.1. Information systems and experimental environment

To evaluate the performance of Algorithms 4.5 and 4.11, we generated ten information systems $\{(U_i, A)|1 \leq i \leq 10\}$ using Pawlak's information system for the Middle East conflict in Table 1. Specifically, we derive the information systems $\{(U_i, A)|1 \leq i \leq 10\}$ shown in Table 19 by taking 100, 200, 300, 400, 500, 600, 700, 800, 900, and 1000 times Pawlak's information system.

All computations were performed on a PC with an Intel(R) Dual-Core CPU (TM)i5-4590 @ 3.30 GHz and 8GB memory running 64-bit Windows 7; the software used was 64-bit Matlab R2014a. The details of the hardware and software are given in Table 20.

Remarks. From a practical standpoint, there are not large enough information systems for conflict analysis, so we generate ten information systems using Pawlak's information system for the Middle East conflict. Furthermore, considering the condition of the hardware, we only test Algorithms 4.5 and 4.11 using dynamic information systems with a medium number of objects.

5.2. Experimental results

In this section, we compare the experimental results using the non-incremental algorithm with those of the incremental algorithms in dynamic information systems including variations of the object sets.

5.2.1. Stability of Algorithms 3.6, 4.5, and 4.11 for computing the probabilistic conflict, neutral, and allied sets

To test the stability of Algorithms 3.6, 4.5, and 4.11, we generated the dynamic information systems (U_i^+, A) and (U_i^-, A) by adding an object into the information system (U_i, A) and deleting an object from the information system (U_i, A), respectively, where $i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. For example, we obtained the dynamic information system (U_1^+, A) by adding x_{601}, where $U_1^+ = \{x_1, x_2, \ldots, x_{600}, x_{601}\}$, and we obtained the dynamic information system (U_1^-, A) when deleting x_{600}, where $U_1^- = \{x_1, x_2, \ldots, x_{599}\}$. Additionally, we take the loss function in Table 11 for each pair of objects and obtain the thresholds $\alpha = \frac{3}{5}$ and $\beta = \frac{1}{3}$ for computing the probabilistic conflict, neutral, and allied sets.

To illustrate the performance of Algorithms 3.6, 4.5, and 4.11, we computed the conflict, neutral, and allied sets using each of them in the dynamic information systems (U_i^+, A) and $(U_i^-, A)(1 \leq i \leq 10)$, and the computation times are shown in Tables 21 and 22, respectively. We performed each experiment ten times to ensure the accuracy of the experimental results. Specifically, Table 21 depicts the times for computing the probabilistic conflict, neutral, and allied sets using Algorithms 3.6 and 4.5 in dynamic information systems $\{(U_i^+, A) \mid 1 \leq i \leq 10\}$; Table 22 shows the times for constructing the probabilistic conflict, neutral, and allied sets using Algorithms 3.6 and 4.11 in dynamic information systems $\{(U_i^-, A) \mid 1 \leq i \leq 10\}$. In theory, the computation time using Algorithm 3.6 in dynamic information system $(U_i^+, A)(1 \leq i \leq 10)$ is larger than that when using Algorithm 3.6 in dynamic information system $(U_i^-, A)(1 \leq i \leq 10)$. Actually, the computation time using Algorithm 3.6 in dynamic information system $(U_i^+, A)(1 \leq i \leq 4)$ is larger than that in dynamic information system $(U_i^-, A)(1 \leq i \leq 4)$, and the computation time using Algorithm 3.6 in dynamic information system $(U_i^+, A)(5 \leq i \leq 10)$ is smaller than that in dynamic information system $(U_i^-, A)(5 \leq i \leq 10)$. Simultaneously, the computation time for Algorithm 4.5 in dynamic information system $(U_i^+, A)(1 \leq i \leq 10)$ is larger than that for Algorithm 4.11 in dynamic information system $(U_i^-, A)(1 \leq i \leq 10)$. Therefore, the experimental results in Tables 21 and 22 illustrate that Algorithms 4.5 and 4.11 are more stable than Algorithm 3.6 for computing the probabilistic conflict, neutral, and allied sets in large-scale dynamic information systems.

More details about the efficiency of Algorithms 3.6, 4.5, and 4.11 for computing the probabilistic conflict, neutral, and allied sets in dynamic information systems $\{(U_i^+, A) \mid 1 \leq i \leq 10\}$ and $\{(U_i^-, A) \mid 1 \leq i \leq 10\}$ are shown by Figs. 1 and 3, respectively. Specifically, Fig. 1 depicts the times for computing the probabilistic conflict, neutral, and allied sets using Algorithms 3.6 and 4.5 in dynamic information system $(U_i^+, A)(1 \leq i \leq 10)$. It is clear that the curve of Algorithm 3.6 fluctuates more strongly than that of Algorithm 4.5 in each sub-figure of Fig. 1. Fig. 3 shows the times for calculating the probabilistic conflict, neutral, and allied sets by Algorithms 3.6 and 4.11 in dynamic information system $(U_i^-, A)(1 \leq i \leq 10)$. It is also clear that the curve of Algorithm 3.6 fluctuates more strongly than that of Algorithm 4.11 in each sub-figure of Fig. 3. Therefore, Figs. 1 and 3 illustrate that Algorithms 4.5 and 4.11 are more stable than Algorithm 3.6 in dynamic information systems with variations of the object sets.

In addition, we show the average times and the standard deviations of the computation times using Algorithms 3.6, 4.5, and 4.11 in Tables 21 and 22. The standard deviations demonstrate that Algorithms 3.6, 4.5, and 4.11 are more stable for computing the probabilistic conflict, neutral, and allied sets in conflict analysis. In particular, the standard deviations of the computation times using Algorithms 4.5 and 4.11 are smaller than that for Algorithm 3.6, which illustrates that Algorithms 4.5 and 4.11 are more stable than Algorithm 3.6 for constructing the probabilistic conflict, neutral, and allied sets in conflict analysis.

Figs. 2 and 4 also illustrate the stability of Algorithms 3.6, 4.5 and 4.11 in computing the probabilistic conflict, neutral, and allied sets in dynamic information systems $\{(U_i^+, A) \mid 1 \leq i \leq 10\}$ and $\{(U_i^-, A) \mid 1 \leq i \leq 10\}$, respectively. It is clear that the standard deviations of the computation times using Algorithm 3.6 are larger than those for Algorithms 4.5 and 4.11. Furthermore, the curves of the standard deviations of the computation times using Algorithm 3.6 fluctuate strongly between dynamic information systems with different object sets, but the curves of the standard deviations of the computation times using Algorithms 4.5 and 4.11 are smooth.

Briefly, we draw the following conclusions: (1) Algorithms 3.6, 4.5, and 4.11 are stable when calculating the probabilistic conflict, neutral, and allied sets in dynamic information systems; (2) Algorithms 4.5 and 4.11 are more stable than Algorithm 3.6 for computing the probabilistic conflict, neutral, and allied sets in dynamic information systems.

Remarks. In Tables 21 and 22, $t(s)$ denotes the measurement of the time in seconds; $\bar{t}$ indicates the average time; *SD* is the standard deviation; and NIC, ICA, and ICD stand for Algorithms 3.6, 4.5, and 4.11, respectively. In Figs. 1, 3, 5 and 6, $-\circ-$, $\triangleright$, and $-\Box-$ mean the computation times for Algorithms 3.6, 4.5, and 4.11, respectively. In Figs. 2 and 4, $-\circ-$, $\triangleright$, and $-\Box-$ denote the standard deviations using Algorithms 3.6, 4.5, and 4.11, respectively, the x-coordinate refers to the information system (U_i, A), and the y-coordinate denotes the standard deviation of the computation times. In Figs. 1 and 3, the x-coordinate refers to the i-th experiment, while the y-coordinate denotes the computation time. In Figs. 5 and 6, the x-coordinate refers to the information system (U_i, A), while the y-coordinate denotes the computation time.

5.2.2. Running times compared with those of non-incremental algorithms when adding objects

In this section, we compare the computation times using Algorithm 3.6 with those using Algorithm 4.5 in dynamic information systems with different numbers of objects.

Table 21 depicts the computation times for computing the probabilistic conflict, neutral, and allied sets using Algorithms 3.6 and 4.5 in the dynamic information systems $\{(U_i^+, A) \mid 1 \leq i \leq 10\}$. It is clear that Algorithm 4.5 is more

Table 21
The computing times using Algorithms 3.6 and 4.5.

No\t(s)	Algo.	1	2	3	4	5	6	7	8	9	10	$\bar{t}$	SD
(U_1, A)	NIC	15.6661	15.9191	15.6257	15.5783	15.6294	15.6255	15.5909	15.5295	15.5695	15.5713	15.6305	0.1087
	ICA	0.0660	0.0602	0.0594	0.0588	0.0584	0.0592	0.0589	0.0591	0.0606	0.0603	0.0601	0.0022
(U_2, A)	NIC	62.9139	63.1647	62.4523	62.4584	62.4356	62.5105	62.5668	63.1906	63.1743	62.6676	62.7535	0.3237
	ICA	0.1303	0.1319	0.1325	0.1312	0.1302	0.1306	0.1324	0.1320	0.1316	0.1307	0.1313	0.0008
(U_3, A)	NIC	141.9937	142.3379	141.5171	141.4366	141.8641	141.4537	141.5227	141.4444	141.3873	141.2906	141.6248	0.3315
	ICA	0.2134	0.2139	0.2138	0.2121	0.2153	0.2125	0.2143	0.2143	0.2147	0.2140	0.2138	0.0010
(U_4, A)	NIC	253.0806	254.4429	253.0319	253.3616	254.5597	254.8520	254.2747	255.3814	254.9215	255.3786	254.3285	0.8869
	ICA	0.3205	0.3315	0.3226	0.3312	0.3239	0.3285	0.3233	0.3277	0.3229	0.3318	0.3264	0.0042
(U_5, A)	NIC	405.1555	405.0269	404.3308	405.6184	405.5268	406.3696	406.3895	406.5132	405.7323	405.4138	405.6077	0.6867
	ICA	0.4631	0.4815	0.4820	0.4810	0.4850	0.4817	0.4862	0.4817	0.4847	0.4819	0.4809	0.0065
(U_6, A)	NIC	605.4882	618.9992	621.6452	634.7630	634.3375	635.4958	632.1862	636.9073	632.4613	630.8307	628.3114	9.9670
	ICA	0.6619	0.6816	0.7010	0.6826	0.7041	0.6894	0.7029	0.6896	0.7109	0.6933	0.6917	0.0142
(U_7, A)	NIC	841.8805	909.3873	937.7201	895.7383	900.9209	922.0898	897.4986	920.6675	924.5891	929.9039	908.0396	27.2472
	ICA	0.8030	0.7986	0.8028	0.7994	0.7944	0.8018	0.7973	0.7986	0.8003	0.7944	0.7991	0.0031
(U_8, A)	NIC	1205.8824	1292.4676	1324.5616	1364.9962	1347.3933	1361.1639	1358.0847	1373.5498	1377.5968	1384.1327	1338.9829	54.1768
	ICA	1.3081	1.3404	1.4546	1.3172	1.4693	1.3470	1.4822	1.3229	1.4864	1.3577	1.3886	0.0746
(U_9, A)	NIC	1607.7516	1837.9054	1829.8274	1825.4398	1803.1033	1807.6194	1793.8114	1804.8390	1800.7830	1810.4823	1792.1563	66.2971
	ICA	1.6627	1.6813	1.8746	1.6874	1.9192	1.6630	1.9080	1.6985	1.9497	1.6981	1.7742	0.1212
(U_{10}, A)	NIC	2023.9591	2332.6384	2405.7080	2572.2311	2496.3432	2550.4050	2495.3971	2570.6065	2480.3520	2511.7288	2443.9369	165.1217
	ICA	1.6384	1.7157	1.6466	1.6359	1.6463	1.6357	1.6396	1.6378	1.6332	1.6409	1.6470	0.0245

Table 22
The computing times using Algorithms 3.6 and 4.11.

No\t(s)	Algo.	1	2	3	4	5	6	7	8	9	10	$\bar{t}$	SD
(U_1, A)	NIC	15.4622	15.4489	15.4809	15.4596	15.4458	15.4459	15.4724	15.4555	15.4323	15.4266	15.4530	0.0168
	ICD	0.0364	0.0326	0.0317	0.0319	0.0321	0.0314	0.0313	0.0316	0.0311	0.0313	0.0321	0.0016
(U_2, A)	NIC	62.4870	62.4351	62.4248	62.5108	62.4042	62.4902	62.4547	62.5222	62.5098	62.4887	62.4728	0.0405
	ICD	0.0693	0.0705	0.0673	0.0678	0.0685	0.0674	0.0685	0.0684	0.0679	0.0686	0.0684	0.0010
(U_3, A)	NIC	141.1699	141.9959	141.0741	140.8329	140.9947	140.9273	140.8830	141.1469	140.7790	140.6514	141.0455	0.3718
	ICD	0.1105	0.1147	0.1108	0.1100	0.1092	0.1092	0.1085	0.1097	0.1086	0.1082	0.1099	0.0019
(U_4, A)	NIC	253.9424	255.0751	253.9363	254.7018	253.3578	254.1371	253.9095	253.8701	253.8232	253.8992	254.0653	0.4843
	ICD	0.1623	0.1693	0.1583	0.1620	0.1596	0.1610	0.1602	0.1615	0.1597	0.1631	0.1617	0.0030
(U_5, A)	NIC	405.7237	409.5678	405.9848	409.3968	407.5470	408.3826	407.0939	408.7489	407.0870	408.6365	407.8169	1.3453
	ICD	0.2291	0.2349	0.2324	0.2323	0.2325	0.2336	0.2332	0.2320	0.2338	0.2330	0.2327	0.0015
(U_6, A)	NIC	627.5000	645.7361	642.2223	656.7541	660.3149	657.3012	659.9242	653.2816	660.8158	657.4189	652.1269	10.6786
	ICD	0.3556	0.3595	0.3860	0.3875	0.3927	0.3855	0.3909	0.3870	0.3937	0.3905	0.3829	0.0136
(U_7, A)	NIC	916.0046	951.7778	962.1899	972.8548	963.5182	964.2910	967.9458	970.6460	970.2738	979.6041	961.9106	17.7447
	ICD	0.4846	0.4804	0.5300	0.5487	0.5421	0.5442	0.5404	0.5469	0.5419	0.5463	0.5305	0.0259
(U_8, A)	NIC	1378.3072	1514.6036	1528.9275	1574.1568	1563.1570	1570.7919	1564.8533	1565.5718	1566.0144	1564.7198	1539.1103	59.7353
	ICD	0.7289	0.7382	0.8720	0.8939	0.9154	0.8948	0.8993	0.9001	0.8987	0.9042	0.8646	0.0699
(U_9, A)	NIC	1838.4454	2041.6057	2028.9997	2133.4258	2129.3575	2159.3269	2227.2655	2248.7268	2138.0855	2112.6080	2105.7847	116.4052
	ICD	0.9228	0.9367	1.1448	1.1678	1.1899	1.1746	1.2005	1.1633	1.1785	1.1762	1.1255	0.1043
(U_{10}, A)	NIC	2436.8072	2572.0284	2555.1484	2835.2796	2858.9778	2825.2697	2840.7726	2823.3825	2845.1748	2814.2174	2740.7058	155.8090
	ICD	1.0897	1.1137	1.3725	1.4159	1.3997	1.4397	1.4140	1.4183	1.4192	1.4206	1.3503	0.1323

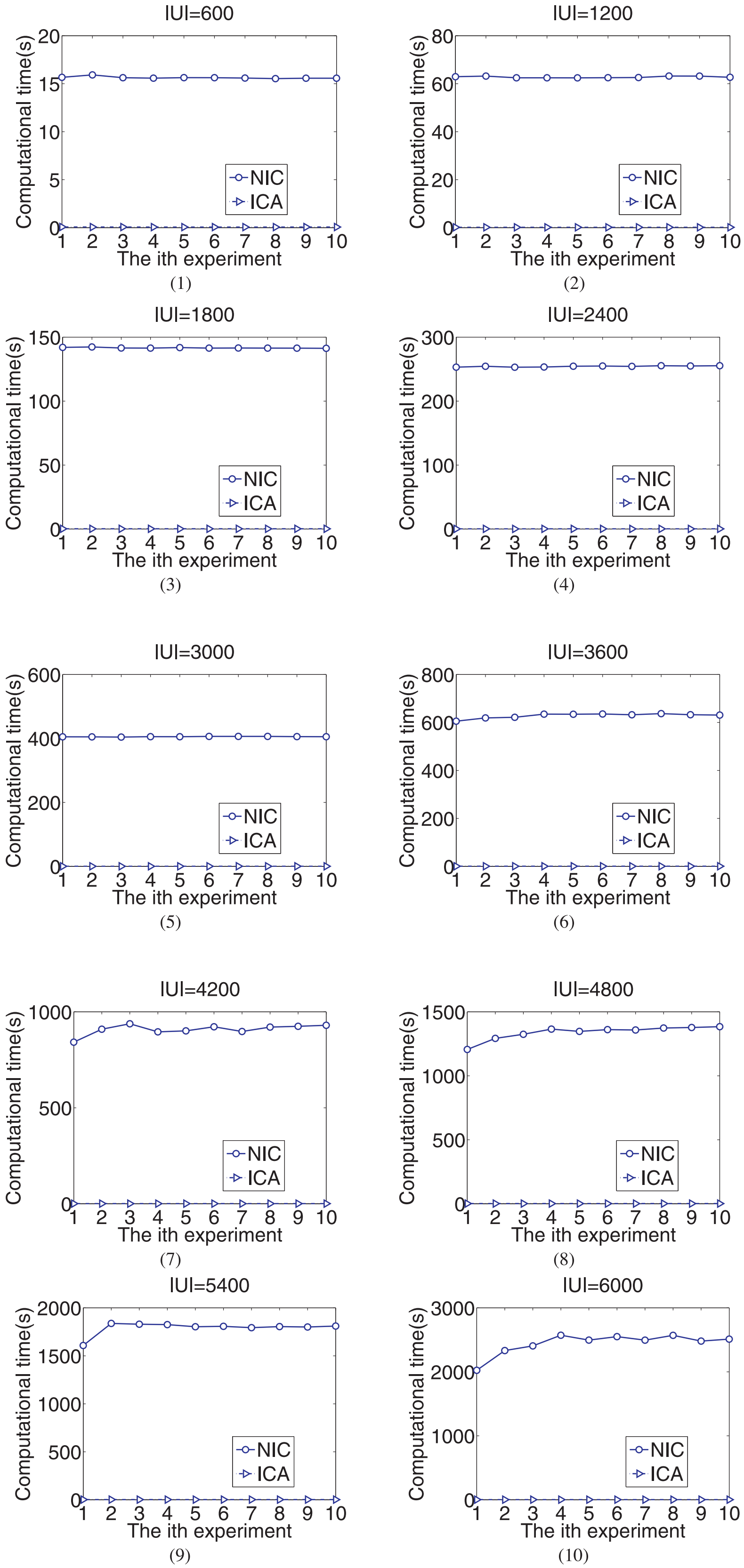

Fig. 1. Computational times using Algorithms 3.6 and 4.5 when adding an object.

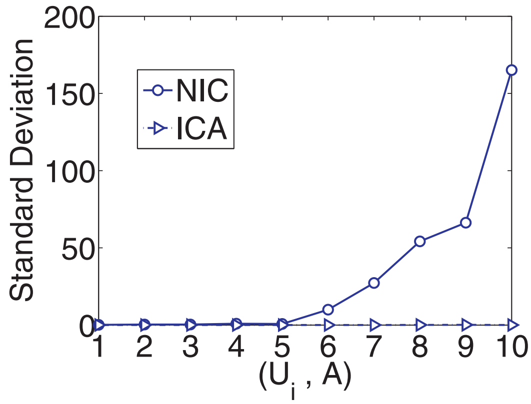

Fig. 2. Standard deviations of computational times using Algorithms 3.6 and 4.5.

effective and efficient than Algorithm 3.6 for computing the probabilistic conflict, neutral, and allied sets in dynamic information systems when adding an object.

Fig. 1 demonstrates that Algorithm 3.6 is more effective and efficient than Algorithm 4.6 for calculating the probabilistic conflict, neutral, and allied sets in the dynamic information systems $\{(U_i^+, A) \mid 1 \leq i \leq 10\}$. In each sub-figure of Fig. 1, we see that the times for computing the probabilistic conflict, neutral, and allied sets using Algorithm 3.6 are larger than those for Algorithm 4.5 in the dynamic information systems $\{(U_i^+, A) \mid 1 \leq i \leq 10\}$, and Algorithm 4.5 executes faster than Algorithm 3.6.

More details on the trendline showing the efficiency of Algorithms 3.6 and 4.5 with an increasing number of objects are shown in Fig. 5, which illustrates that the times for computing the probabilistic conflict, neutral, and allied sets using Algorithms 3.6 and 4.5 increase with the increasing number of objects, but those using Algorithm 3.6 increase faster than those using Algorithm 4.5 with the increasing number of objects.

In summary, we reach the following conclusions: (1) Algorithm 4.5 greatly reduces the running time for computing the probabilistic conflict, neutral, and allied sets in dynamic information systems when adding an object. (2) Algorithm 4.5 executes faster than Algorithm 3.6 for calculating the probabilistic conflict, neutral, and allied sets in dynamic information systems when adding an object.

Remarks. For simplicity, we only construct the probabilistic conflict, neutral, and allied sets when adding a single object in this section. However, we can also compute the probabilistic conflict, neutral, and allied sets using incremental approaches in dynamic information systems when adding more objects.

5.2.3. Running times compared with those of non-incremental algorithms when deleting objects

In this section, we compare the computation times using Algorithm 3.6 with those of Algorithm 4.11 in dynamic information systems with different numbers of objects.

Table 22 depicts the computation times for computing the probabilistic conflict, neutral, and allied sets using Algorithms 3.6 and 4.11 in the dynamic information systems $\{(U_i^-, A) \mid 1 \leq i \leq 10\}$. It is clear that Algorithm 4.11 is more effective and efficient than Algorithm 3.6 for calculating the probabilistic conflict, neutral, and allied sets in the dynamic information systems $\{(U_i^-, A) \mid 1 \leq i \leq 10\}$.

Fig. 3 demonstrates that Algorithm 4.11 executes faster than Algorithm 3.6 for constructing the probabilistic conflict, neutral, and allied sets in the dynamic information systems $\{(U_i^-, A) \mid 1 \leq i \leq 10\}$. In each sub-figure of Fig. 3, we see that the time for computing the probabilistic conflict, neutral, and allied sets using Algorithm 3.6 is larger than that using Algorithm 4.11 in the dynamic information systems $\{(U_i^-, A) \mid 1 \leq i \leq 10\}$.

More details regarding the changes in the trendlines of the efficiency of Algorithms 3.6 and 4.11 with an increasing number of objects are shown in Fig. 6, which illustrates that the times for calculating the probabilistic conflict, neutral, and allied sets using Algorithms 3.6 and 4.11 increase with the increasing number of objects, although the computing times using Algorithm 3.6 increase faster.

To conclude, we draw the following conclusions: (1) Algorithm 4.11 greatly reduces the running time for constructing the probabilistic conflict, neutral, and allied sets in dynamic information systems when deleting an object. (2) Algorithm 4.11 executes faster than Algorithm 3.6 for computing the probabilistic conflict, neutral, and allied sets in dynamic information systems when deleting an object.

Remarks. For simplicity, we only compute the probabilistic conflict, neutral, and allied sets when deleting a single object in this section. However, we can also calculate the probabilistic conflict, neutral, and allied sets using incremental approaches in dynamic information systems when deleting more objects.

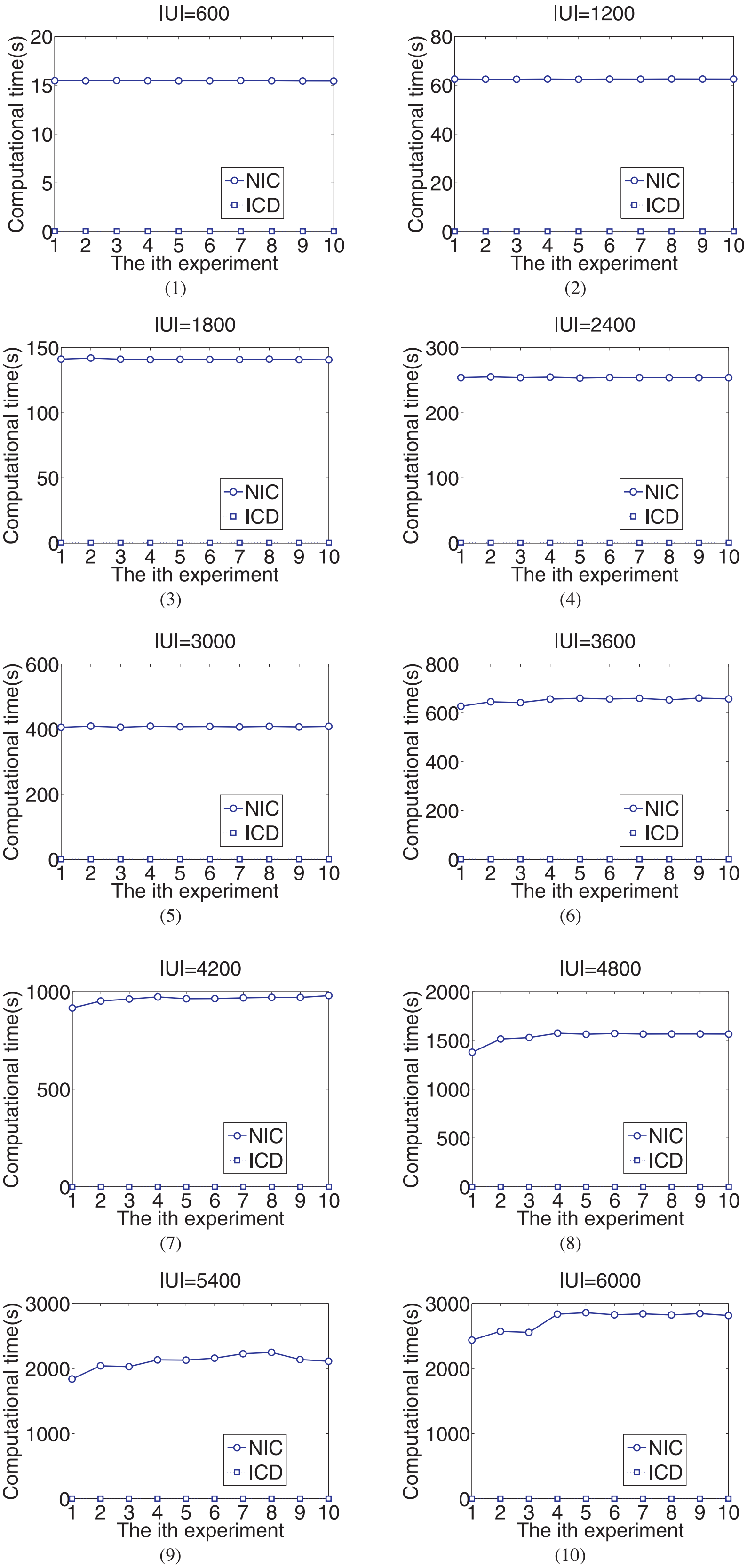

Fig. 3. Computational times using Algorithms 3.6 and 4.11 when deleting an object.

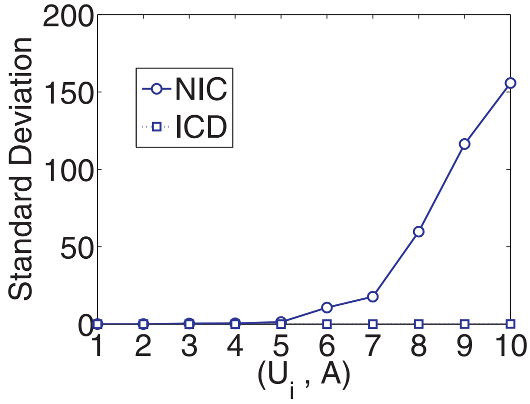

Fig. 4. Standard deviations of computational times using Algorithms 3.6 and 4.11.

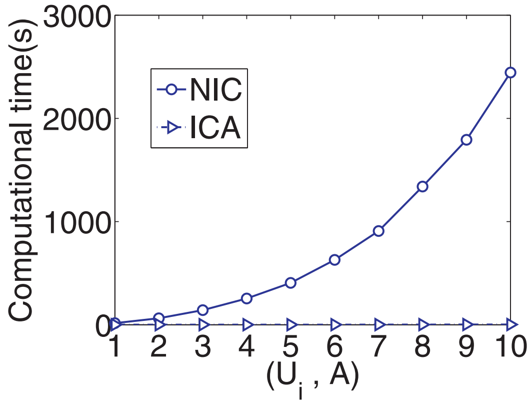

Fig. 5. Computational times using Algorithms 3.6 and 4.5.

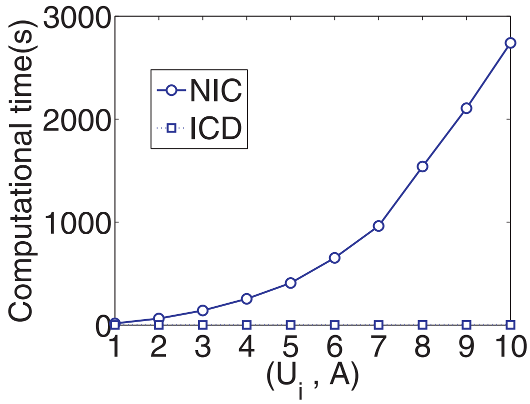

Fig. 6. Computational times using Algorithms 3.6 and 4.11.

6. The maximal coalitions in dynamic information systems

In this section, we discuss the relationship between maximal coalitions and allied sets in dynamic information systems.

Definition 6.1. Let $S = (U, \rho_A)$ be a conflict space, $X \subseteq U$, and $0 \leq \beta \leq \alpha \leq 1$.

(1) If $\rho_A(x, y) < \beta$ for any $x, y \in X$, then X is called a coalition.
(2) If X is a coalition, and there is not a coalition Y satisfying $X \subset Y$, then X is called a maximal coalition.

By Definition 6.1, we observe the concept of the coalition is a generalization of Pawlak's model, and each subset of the maximal coalition is a coalition.

Theorem 6.2. *Let $S=(U,\rho_A)$ be a conflict space, $X \subseteq U$, and $0 \leq \beta \leq \alpha \leq 1$. Then X is a maximal coalition if and only if $\bigcap_{x\in X} AL^{\alpha}_{\beta}(x)=X$.*

Proof. If $X \subseteq U$ is a maximal coalition, by Definition 6.1, we have that $\rho_A(x,y) < \beta$ for any $x, y \in X$. It follows $X \subseteq AL^{\alpha}_{\beta}(x)$, which implies $X \subseteq \bigcap_{x\in X} AL^{\alpha}_{\beta}(x)$. If $X \neq \bigcap_{x\in X} AL^{\alpha}_{\beta}(x)$, then there exists $z \in \bigcap_{x\in X} AL^{\alpha}_{\beta}(x)$ such as $z \notin X$. In other words, there is a coalition $X \cup \{z\}$ such as $X \subset Y = X \cup \{z\}$, which is a contradiction. Therefore, $\bigcap_{x\in X} AL^{\alpha}_{\beta}(x)=X$.

If $\bigcap_{x\in X} AL^{\alpha}_{\beta}(x)=X$ for $X \subseteq U$, we get that $\rho_A(x,y) < \beta$ for any $x, y \in X$. It implies X is a coalition. If there exists $z \in U/X$ such as $\rho_A(x,z) < \beta$ for any $x \in X$, then $z \in \bigcap_{x\in X} AL^{\alpha}_{\beta}(x)$, which is a contradiction. Therefore, X is a maximal coalition. □

Theorem 6.2 provides an effective approach to computing the maximal coalition for the conflict analysis. Specifically, we find that $\{x\}$ is a maximal coalition if $AL^{\alpha}_{\beta}(x)=\{x\}$, which accelerates the construction of the maximal coalition for conflict analysis.

Proposition 6.3. *Let $S=(U,\rho_A)$ be a conflict space, and the maximal coalitions $\{MC_1, MC_2, \ldots, MC_k\}$. Then $\bigcup_{1\leq i\leq k} MC_i = U$.*

Proof: By Definition 6.1(2), there exists a maximal coalition MC_j such as $x \in MC_j$ for any $x \in U$. Therefore, we have $\bigcup_{1\leq i\leq k} MC_i = U$. □

We present a non-incremental algorithm for computing the maximal coalitions as follows.

Algorithm 6.4. (NMC))

Input: the allied set $AL^{\alpha}_{\beta}(x)$ for any $x \in U$;
Output: the maximal coalitions $\{MC_1, MC_2, \ldots, MC_k\}$.
Step 1: Input the allied set $AL^{\alpha}_{\beta}(x)$ for any $x \in U$;
Step 2: Construct X satisfying $\bigcap_{x\in X} AL^{\alpha}_{\beta}(x)=X$;
Step 3: Output the maximal coalitions $\{MC_1, MC_2, \ldots, MC_k\}$.

In addition, we simplify the process of computing the maximal coalitions as follows: (1) if $AL^{\alpha}_{\beta}(x)=\{x\}$, then $\{x\}$ is a maximal coalition, so we only consider $U/\{x|AL^{\alpha}_{\beta}(x)=\{x\}\}$. (2) Supposing $AL^{\alpha}_{\beta}(x)=X$ such as $|X| > 1$, we can take $y \in X/\{x\}$. Then, we compute $AL^{\alpha}_{\beta}(x)\cap AL^{\alpha}_{\beta}(y)$. If $AL^{\alpha}_{\beta}(x)\cap AL^{\alpha}_{\beta}(y)=\{x,y\}$, then $\{x, y\}$ is a maximal coalition. Otherwise, we take $z \in \{AL^{\alpha}_{\beta}(x)\cap AL^{\alpha}_{\beta}(y)\}/\{x,y\}$. If $AL^{\alpha}_{\beta}(x)\cap AL^{\alpha}_{\beta}(y)\cap AL^{\alpha}_{\beta}(z)=\{x,y,z\}$, then $\{x, y, z\}$ is a maximal coalition. Otherwise, we will repeat the above process until $\bigcap_{x\in Y} AL^{\alpha}_{\beta}(x)=Y$ for $Y \subseteq X$.

Example 6.5 (Continued from Example 2.8)**.** By Table 12, we have $AL^{\alpha}_{\beta}(x_1)=\{x_1\}$ and $AL^{\alpha}_{\beta}(x_6)=\{x_6\}$. So we only consider $x \in U/\{x_1, x_6\}$. Since $AL^{\alpha}_{\beta}(x_2)\cap AL^{\alpha}_{\beta}(x_3)\cap AL^{\alpha}_{\beta}(x_4)\cap AL^{\alpha}_{\beta}(x_5)=\{x_2,x_3,x_4,x_5\}$, we get the maximal coalitions $\{MC_1, MC_2, MC_3\}$, where $MC_1=\{x_1\}$, $MC_2=\{x_2,x_3,x_4,x_5\}$, and $MC_3=\{x_6\}$.

In practical situations, there are many dynamic information systems. However, we only consider the dynamic information system when adding an object and the remaining number of maximal coalitions.

Theorem 6.6. *Let $S=(U,\rho_A)$ and $S^+=(U^+,\rho^+_A)$ be conflict spaces, $\{MC_1, MC_2, \ldots, MC_k\}$ and $\{MC^+_1, MC^+_2, \ldots, MC^+_k\}$ the maximal coalitions of S and S^+, respectively, where $U=\{x_1,x_2,\ldots,x_n\}$ and $U^+=\{x_1,x_2,\ldots,x_n,x_{n+1}\}$. Then*

$$MC^+_i=\begin{cases} MC_i, & \text{if } \bigcap_{x\in MC_i\cup\{x_{n+1}\}} AL^{\alpha+}_{\beta}(x) \neq MC_i\cup\{x_{n+1}\};\\ MC_i\cup\{x_{n+1}\}, & otherwise. \end{cases}$$

Proof. The proof is straightforward by Theorem 6.2. □

Theorem 6.6 illustrates how to update the maximal coalitions in dynamic information systems when adding an object. Moreover, we put forward an incremental algorithm for computing the maximal coalitions when adding an object as follows.

Algorithm 6.7. (**IAMC**))

Input: the allied set $AL^{\alpha+}_{\beta}(x)$ for any $x \in U^+$;
Output: the maximal coalitions $\{MC^+_1, MC^+_2, \ldots, MC^+_k\}$.
Step 1: Input the allied set $AL^{\alpha+}_{\beta}(x)$ for any $x \in U^+$;
Step 2: Construct the maximal coalitions $\{MC^+_1, MC^+_2, \ldots, MC^+_k\}$;
Step 3: Output the maximal coalitions $\{MC^+_1, MC^+_2, \ldots, MC^+_k\}$.

Using Algorithm 6.7, we have the following results: (1) if $\bigcap_{x\in MC_i\cup\{x_{n+1}\}} AL^{\alpha+}_{\beta}(x)=MC_i\cup\{x_{n+1}\}$, then we get $MC^+_i=MC_i\cup\{x_{n+1}\}$; (2) if $\bigcap_{x\in MC_i\cup\{x_{n+1}\}} AL^{\alpha+}_{\beta}(x)\neq MC_i\cup\{x_{n+1}\}$, then we obtain $MC^+_i=MC_i$. An example is employed to illustrate the construction of the maximal coalitions in dynamic information systems.

Example 6.8. (Continued from Examples 4.6 and 6.5) By Definition 3.2, we have $AL^{\alpha+}_{\beta}(x_1)\cap AL^{\alpha+}_{\beta}(x_7)=\{x_1,x_7\}$ and $\bigcap_{x\in U/\{x_1,x_6,x_7\}} AL^{\alpha+}_{\beta}(x)=\{x_2,x_3,x_4,x_5\}$. Therefore, we get $\{MC^+_1, MC^+_2, MC^+_3\}$, where $MC^+_1=\{x_1,x_7\}$, $MC^+_2=\{x_2,x_3,x_4,x_5\}$, and $MC^+_3=\{x_6\}$.

Apart from the dynamic information systems when adding objects, there are some dynamic information systems when deleting objects in practical situations. We only consider the dynamic information system when deleting an object and the remaining number of maximal coalitions.

Theorem 6.9. *Let* $S = (U, \rho_A)$ *and* $S^- = (U^-, \rho_A^-)$ *be conflict spaces,* $\{MC_1, MC_2, \ldots, MC_k\}$ *and* $\{MC_1^-, MC_2^-, \ldots, MC_k^-\}$ *the maximal coalitions of S and* S^-*, respectively, where* $U = \{x_1, x_2, \ldots, x_n\}$ *and* $U^- = \{x_1, x_2, \ldots, x_{n-1}\}$*. Then* $MC_i^- = MC_i/\{x_n\}$ *for* $1 \le i \le k$*.*

Proof. The proof is straightforward by Theorem 6.2. □

We provide an incremental algorithm for computing the maximal coalitions when deleting an object as follows.

Algorithm 6.10. (**IDMC**))
Input: the allied set $AL_\beta^{\alpha-}(x)$ for any $x \in U^-$;
Output: the maximal coalitions $\{MC_1^-, MC_2^-, \ldots, MC_k^-\}$.
Step 1: Input the allied set $AL_\beta^{\alpha-}(x)$ for any $x \in U^-$;
Step 2: Compute the maximal coalitions $\{MC_1^-, MC_2^-, \ldots, MC_k^-\}$;
Step 3: Output the maximal coalitions $\{MC_1^-, MC_2^-, \ldots, MC_k^-\}$.

We see that Algorithm 6.10 provides an effective approach to calculating the maximal coalitions for dynamic information systems when deleting an object. In addition, we explore the following example to illustrate updating the maximal coalitions in dynamic information systems when deleting an object.

Example 6.11 (Continued from Examples 4.12 and 6.5)**.** By Theorem 6.9, we have three maximal coalitions $\{MC_1^-, MC_2^-, MC_3^-\}$ when deleting the object x_5, where $MC_1^- = \{x_1\}, MC_2^- = \{x_2, x_3, x_4\}$, and $MC_3^- = \{x_6\}$.

7. Conclusions

In this paper, we first presented the concepts of probabilistic conflict, neutral, and allied sets of conflicts. Based on decision-theoretic rough set theory, we have demonstrated how to calculate the thresholds for computing the probabilistic conflict, neutral, and allied sets in conflict analysis. Second, we have put forward incremental approaches to constructing the probabilistic conflict, neutral, and allied sets in dynamic information systems. Experimental results have illustrated that the incremental algorithms are effective in computing the probabilistic conflict, neutral, and allied sets in dynamic information systems. Third, we provided the concept of the maximal coalitions and discussed the relationship between the maximal coalitions and allied sets. Finally, we introduced incremental approaches to computing the maximal coalitions in dynamic information systems.

In the future, we will study complex information systems for conflicts such as incomplete information systems, ordered information systems, and fuzzy information systems. Furthermore, we will propose effective algorithms for calculating the conflict, neutral, and allied sets in complex information systems to improve the relationship between objects.

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