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ORIGINAL ARTICLE

Relation granulation and algebraic structure based on concept lattice in complex information systems

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Abstract Normally, there may exist some kind of relationship among different attribute values such as order relationship, similarity relationship or other more complicated relationship hidden in complex information systems. In the case, the binary relation on the universe is probably a kind of more general binary relation rather than equivalence relation, tolerance relation, order relation, etc. For the case, the paper tries to take concept lattice as theoretical foundation, which is appropriate very well for analyzing and processing binary relations, and finally proposes a new rough set model from the perspective of sub-relations. In the model, one general binary relation can be decomposed into several sub-relations, which can be viewed as granules to study algebraic structure and offer solutions to problems such as reduction, core. The algebraic structure mentioned above can organized all of relation granulation results in the form of lattice structure. In addition, the computing process based on concept lattice is often accompanied by high time complexity, aiming at the problem, the paper attempts to overcome it by introducing granular computing, and further converts complex information systems into relatively simple ones. In general, the paper is a new attempt and exploring to the fusion of rough set and concept lattice, and also offers a new idea for the expansion of rough set from the perspective of relation granulation.

Keywords Rough set · Concept lattice · General binary relations · Granular computing · Relation granulation

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1 Introduction

In the age of information globalization, how to process massive data for knowledge acquisition has already become one of the research hotspots. In 1982, Pawlak proposed rough set [31], which is a mathematical theory developed on the basis of predecessors' research theories like Zadeh's fuzzy set theory [54] and Shafer's evidence theory [40] for processing uncertainty information. Although rough set has similarities with theories mentioned above for processing uncertain problems, there are still some differences. The most remarkable distinction is that rough set does not require offering any priori knowledge, such as fuzzy membership function in fuzzy set, belief function in evidence theory, and the assumption that data should comply with some kind of statistical distribution in probability statistics, so rough set is more objective to describe problems. It is known that rough set also has its limitation, namely, it does not contain the mechanism for processing inaccurate or uncertain original data, which means that it should be complemented by other theories such as fuzzy set, evidence theory, and probability theory. After 30 years of development, rough set has become a complete and independent scientific field, and found wide applications in knowledge discovery [15, 16, 58–60].

In the process of exploring the world, people usually understand things first perceptually and then conceptually, meanwhile, they find out common characteristics of things and form final concepts after conclusion and summarization. Originating from people's philosophical understanding that "A concept is an idea unit composed of intent and extent", German scholar Wille initiated formal concept analysis (FCA) [47] in 1982 on the basis of Brikhoff's lattice theory [2]. As the core structure of FCA, concept lattice is a kind of concept hierarchical structure established on the basis of the binary relation between object set and attribute set. Although the hierarchy structure of concept lattice is similar to flat structure, there are still some differences between them, the most remarkable distinction is that concept lattice must be a complete lattice. In recent years, the related research has made many achievements [1, 4, 57], especially, many scholars have combined concept lattice with theories such as fuzzy set, rough set, neural network, and probability theory, thus greatly expanding the theoretical foundation and application scope of concept lattice.

Although rough set and concept lattice are two different type of theories, they still share some similarities in terms of goal and methodology [14]. Studying their relationship and combining them together will definitely help us to analyze and understand data more efficiently. Research results in recent years about the comparison and combination of two theories have made great achievements. For instance, Oosthuizen discussed the connection between rough set and concept lattice [30]; Kent [14] and Yao [51] introduced upper and lower approximate ideas into concept lattice, expanded the definition of concepts; reduction is one of core problems in both concept lattice and rough set [19, 28, 29, 33, 35, 36, 41], many scholars have paid more attention to the fusion of the two theories in terms of reduction, relevant achievements included the relation between concept lattice reduction and rough set reduction [3, 45], and reduction methods for concept lattice based on rough set [25, 27]. Kang and Miao [12] discussed relation granularity, and further proposed a variable precision rough set model based on concept lattice, which is a direct expansion of Pawlak's rough set; Qu and Zhai [37] introduced inclusion degree into concept lattice to express some basic notions in concept lattice. In addition, they also exposed limitations of rough set in data analysis, and pointed out that rough set could be expanded through the technology in concept lattice; Kang et al. [10] proposed a rough set model based on concept lattice, which solved the problem of algebraic structure in information systems. Tan et al. [42] studied connections between covering-based rough sets and concept lattices from the perspective of reduction, structures and approximation operators, and got lots of valuable conclusions. Wang and Liu [44] proposed AFS concept lattices, and offered an approach to find AFS formal concepts by virtue of rough set. By adopting ideas from FCA, Yao [53] presented the notion of RS-definable concepts, and found that the family of RS-definable concepts can produce an atomic boolean algebra. Li et al. [20] from the view of "AND" decision rules, "OR" decision rules, granular rules and disjunctive rules to investigate the relationship between multigranulation rough sets and concept lattices, and finally drew some important conclusions.

Normally, attributes are complicated and varied in complex information systems. In a narrow sense, they can be categorized into nominal attributes, order symbolic attributes, real-valued attributes, etc. The so called "nominal attribute" actually means an attribute value domain composed of several discrete values, and different values are independent of each other, such as attributes describing gender and color; order symbolic attributes refer to an attribute value domain composed of several discrete values, but different values are not independent of each other, and they possess the order characteristic, like some attribute describing grade; real-valued attributes refer to an attribute value domain that is a real number field or a sub-set of real number field, such as some attribute describing temperature.

It is known that the classical rough set, with equivalence relations as the classification basis, forms partitions in the universe, and further acquires knowledge from data. Since equivalence relation is required to be reflexive, symmetrical and transitive strictly, the model can just process information systems only containing nominal attributes. Therefore, it is not practical due to the limitations in many applications. In the case, lots of scholars weakened equivalence relation to some more general binary relations, it greatly widened and extended the intent and extent of classical rough set to some extent, and it is of great theoretical and practical significance for actual applications of rough set theory. With regard to information systems containing real-valued attributes or order symbolic attributes, scholars have further expanded the classical rough set in recent years, and following two types of expanded models are the most common and widely used.

1. Equivalence relation is further expanded into tolerance relation (also called similarity relation sometimes) [18, 39], which meets reflexive and symmetrical properties. This kind of rough set models called tolerance rough set models (TRSM) have strong robustness and fault tolerance. TRSM is effective for processing real-valued attributes, that is, it can discover the similarities among

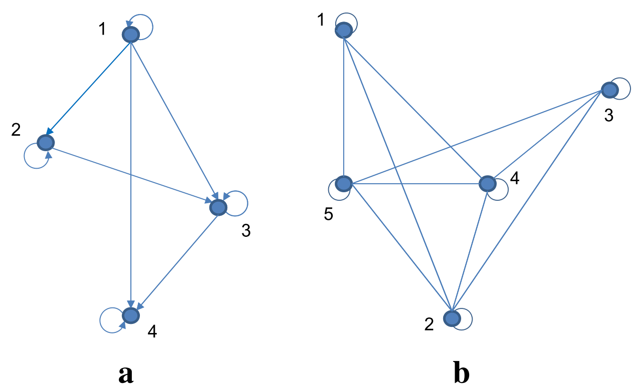

Fig. 1 An order relation graph and a tolerance relation graph

attribute values, and can eliminate their minor differences. For instance, in Fig. 1, the relation graph shown in Fig. 1b is a tolerance relation graph, which only needs to meet reflexivity and symmetry.

2. Equivalence relation is expanded into dominance relation [7, 50], which belongs to the scope of order relation, and meets transitive and reflexive properties. This kind of rough set models called dominance-based rough sets (DRSA). DRSA can clearly express useful information hidden in an information system containing order symbolic attributes. For instance, in Fig. 1, the relation graph shown in Fig. 1a is an order relation graph, which only needs to meet reflexivity and transitivity.

Apart from weakening equivalence relations to dominance relations (order relations) and tolerance relations, some scholars have also generalized the equivalence relation on the universe to more general binary relations in recent years [5, 8, 43, 52, 61]. Although lots of scholars have made great effort in the expansion of rough set, related research results on complex information systems are rare, namely, rough set still needs to be further enriched and improved.

It is known that information granulation is the cornerstone of rough set, and a good granulation mechanism will be the key to success of rough set modeling. In almost all existing granulation models based on relations, the universe U is usually decomposed into several granules such as $X_1, X_2, \ldots, X_n$. If $X_i \cap X_j = \emptyset, 1 \leqslant i, j \leqslant n$, $X_1 \cup X_2 \cup \cdots \cup X_n = U$, then $\mathscr{X} = \{X_1, X_2, \ldots, X_n\}$ is a partition. If $\exists X_i \cap X_j \neq \emptyset$, and $X_1 \cup \cdots \cup X_n = U$, then we say $\mathscr{X}$ is a cover. This kind of information granulation mentioned above are always called universe granulation. However, in some complex information system, the relation on U usually does not meet common mathematical properties such as reflexivity, symmetry and transitivity, therefore, establishing corresponding granulation mechanism on the universe will become very difficult and complicated, and even granulation results of the universe is always actually meaningless.

In view of the above problem, the paper attempts to offer another type of information granulation mechanism, namely relation granulation rather than universe granulation. The so-called relation granulation means that one general binary relation can be decomposed into several sub-relations called block-relations, and further, block-relations can be viewed as granules to get knowledge in complex information systems. In comparison with other granulation mechanisms, the most remarkable distinction is that the one proposed in the paper mainly focuses on the granulation of the general binary relation rather than the granulation of the universe itself. The advantage of relation granulation may be not obvious relative to universe granulation, but it provides a new way of thinking for the related research. In addition, the paper, in virtue of the idea of relation granulation, can not only conduce to better analyzing and solving problems, but also essentially conform to people's features in solving complicated problems.

In fact, we have discussed the similar topic in [12], but classical information systems in [12] have been expended to complex information systems, meanwhile, the tolerance relation in [12] has also been expended to the more general relation. In the case, for the granulation of binary relation, our focus is on the more general relation rather than tolerance relation. From the point of this angle, the relation granulation in [12] is only a special case of the one in the paper, and the method proposed in the paper can be viewed as a necessary complement to the existing method in [12].

Let $\mathscr{E}$ be a set, and $\mathscr{R}$ be a binary relation on $\mathscr{E}$. If $X \times Y \subseteq \mathscr{R}$, and there does not exist $X_1 \times Y_1 \subseteq \mathscr{R}$ satisfying $X \times Y \subset X_1 \times Y_1$, then $X \times Y$ is called a block-relation. The set of all block-relations in $\mathscr{R}$ is denoted as $\kappa(\mathscr{R})$.

For instance, Fig. 2 is a general binary relation graph, which can be decomposed into several block-relations shown in Fig. 3.

Referring to human's granular cognitive mechanism, granular computing (GrC) emerged as a new concept and

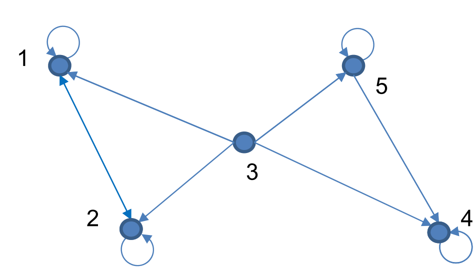

Fig. 2 A general relation graph

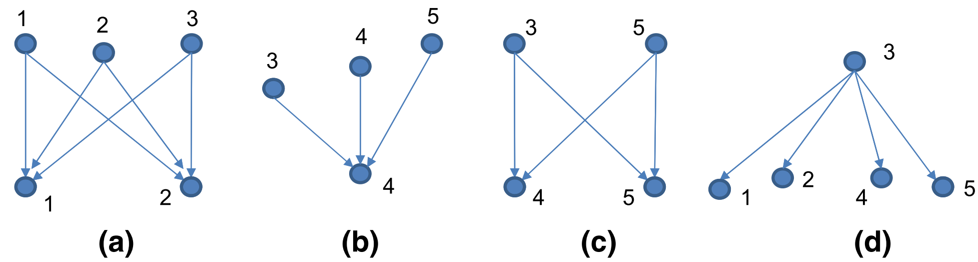

Fig. 3 All block-relations graphs in Fig. 2

method in the artificial intelligence field in the 1970s, which tried to generalize and formalize human's cognitive way so as to distill a set of systematic methods and technologies to simulate their mechanism of analyzing and solving problems. At present, there are three typical GrC theories, namely fuzzy set [55], rough set [32] and quotient space theory [56], which greatly facilitate the research and development of GrC. Recently, research on GrC and the combination theory of GrC , rough set and concept lattice has obtained many findings [9, 11, 13, 22, 23, 26, 34, 46, 48, 49]. For instance, Wei and Wan [46] built relationships between equivalence classes and some kinds of concepts, and gave some new judgment theorems for join (meet)-irreducible elements of a concept lattice; Li et al. [23] gave an axiomatic approach to describe three-way concepts by introducing multi-granularity, and built a computing system to find composite three-way cognitive concepts. In addition, for learning three-way cognitive concepts, they also provided a set approximation method to simulate the cognitive processes; Xu and Li [49] studied information granules based on FCA, and further constructed a two-way learning system in fuzzy datasets; Li et al. [22], aiming at concept learning, gave a new way from the standpoint of cognitive computing by introducing GrC. In modeling, they gave a detailed description of the cognitive processes, and built a cognitive computing system.

Normally, in the information system (U, AT, V, f), for any attribute $m \in AT$, (V_m, R_m) can be understood as a directed graph, where R_m is the general binary relation on V_m. If R_m is large-scale, and does not meet common mathematical properties, then (V_m, R_m) will be a very complicated graph. In the case, it will not only bring high time complexity for solving follow-up problems based on concept lattice, but also is not conducive to understand the sketch structure of (V_m, R_m). In addition, there exists one other problem, that is, the computing process based on concept lattice is often accompanied by high time complexity and space complexity. In the case, the paper tries to introduce GrC, which can convert the complicated graph into a relatively simple one under some high granularity. For instance, on the basis of GrC, we can convert the complicated relation shown in Fig. 4 into the simple one shown in Fig. 5, the corresponding granulation process, please see Sect. 3. In fact, vertexes in Fig. 4 are granulated into $A = \{v_1, v_2, v_5, v_6\}$, $B = \{v_3, v_4\}$, $C = \{v_7, v_8\}$, and directed edges in Fig. 4 are granulated into $A \times A$, $B \times B$, $C \times C$ and $B \times C$.

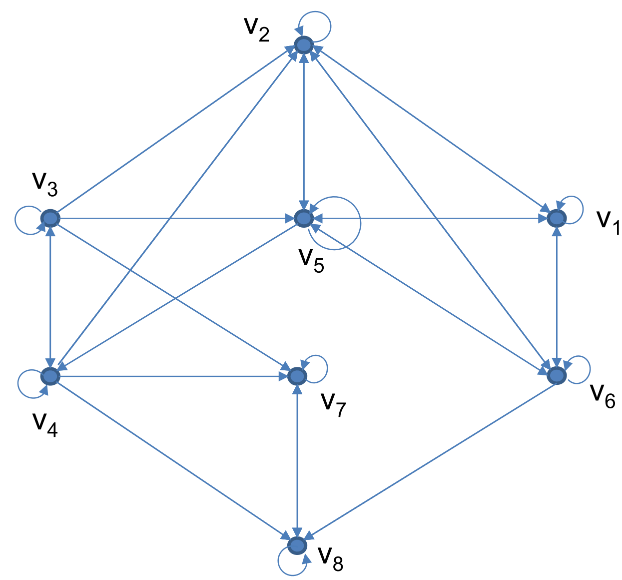

Fig. 4 A complicated binary relation graph

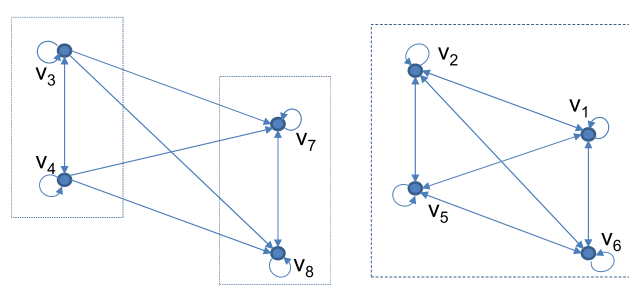

Fig. 5 A relatively simple binary relation graph deducted from Fig. 4 based on GrC

Based on the discussion above, the paper brings concept lattice and GrC into complex information systems, and proposes a new knowledge acquisition model based on concept lattice and GrC. In general, the paper is not only a new attempt and exploration to the knowledge acquisition in complex information systems, but also offers a feasible idea for the expansion of rough set from the view of concept lattice.

Subsequent chapters of the paper are arranged as follows: Sect. 2 simply recalls concept lattice theory; Sect. 3 discusses the granulation of attribute value domain; Sect. 4 decomposes one general binary relation on universe into several block-relations in virtue of granulation idea, and on this basis, builds algebraic structure in complex information systems based on concept lattice; Sect. 5 discusses reduction, core and dependency in complex information systems; Sect. 6 is the summary of the paper.

2 Concept lattice theory

The section briefly introduces concept lattice, for details please refer to [6].

Normally, a formal context is characterized as a triple, that is, (G, M, I), where $I \subseteq G \times M$, and elements in G and M are called objects and attributes separately. If $(g, m) \in I$, we say the attribute m belongs to the object g.

Definition 1 Let $K = (G, M, I)$ be a formal context. For $A \subseteq G, B \subseteq M$ we define

$$A' = \{m \in M | (g, m) \in I, \forall g \in A\}$$
$$B' = \{g \in G | (g, m) \in I, \forall m \in B\}$$

(A, B) is called a formal concept, if $A' = B$ and $B' = A$. The order relation "$\leq$" between concepts (A_1, B_1) and (A_2, B_2) is defined as

$$(A_1, B_1) \leq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 \Leftrightarrow B_1 \supseteq B_2$$

$(\mathscr{B}(K), \leq)$ is a lattice, where $\mathscr{B}(K)$ is the set of all concepts in K.

Proposition 1 *In* (G, M, I), let $A, A_1, A_2 \subseteq G, B, B_1, B_2 \subseteq M$, *then there always exist following simple conclusions:*

(1) $A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$ (2) $B_1 \subseteq B_2 \Rightarrow B_2' \subseteq B_1'$
(3) $A \subseteq A''; B \subseteq B''$ (4) $A' = A'''; B' = B'''$

3 Complex information systems and the granulation of attribute value domain

Normally, an information system is defined as a quadruple (U, AT, V, f), where U called universe is a finite nonempty set, and the elements in U are called objects; AT is also a finite nonempty set, and the elements in AT are called attributes; $V = \bigcup_{m \in AT} V_m$ and V_m is the domain of attribute m; $f: U \times AT \to V$ is a function, that is, $f(x, m) \in V_m$ for any $x \in U$, $m \in AT$.

It is known that the classical rough set does not consider the relationship between different attribute values such as similarity relationship (values describing attributes are similar or dissimilar), order relationship (values describing attributes have merits and demerits in reality) or other more complicated relationship, so it can not clearly express some useful information hidden in information systems. In this case, we presents complex information systems.

$S = (U, AT, V, f, \tau)$ is a complex information system, where $\tau = \{\tau_m | m \in AT\}$. As a binary relation on V_m, $\tau_m \subseteq V_m \times V_m$ can objectively reflect the certain relationship(such as, similarity relationship, order relationship or other more complicated relationship) existed between values in V_m. That is, $(v, w) \in \tau_m$ implies there exists the certain relationship τ_m from $v \in V_m$ to $w \in V_m$. Different attributes may correspond to different relationships. For convenience, $S = (U, AT, V, f, \tau)$ is denoted as $S = (U, AT, \tau)$.

For example, if $m \in AT$ is an order symbolic attribute, then there may exist order relationship between values in V_m; if $m \in AT$ is a real-valued attribute, then there may exist similarity relationship between values in V_m. Table 1 is a typical complex information system, where τ_b and τ_c are shown in sub-table(a) and (b) separately (if the crossing of $v \in V_m$ row and $w \in V_m$ column is denoted as "$\times$", then it means $(v, w) \in \tau_m$). In addition, for the normal attribute a(or d, or e), the corresponding τ_a(or τ_d, or τ_e) is shown in Table 2.

Essentially, for any attribute $m \in AT$, we can perceive $G_m = (V_m, \tau_m)$ as a digraph, where V_m denotes the set of all vertexes, and τ_m is the set of all directed edges. Obviously, a bigger n indicates massive directed edges, meanwhile, graph G_m is definitely a huge and complicated digraph. This not only brings about the high time complexity and space complexity for further computing and solving problems, but also makes no contribution to further understanding the brief structure of digraph generally. To better understand and solve problems, we will convert the complicated graph G_m into relatively a simple one under the high granularity.

As a matter of fact, granulation and granularity are important parts of GrC. Granulation usually refers to decomposing the large-scale complicated information into several simple blocks, and each block is regarded as a granule; granularity can measure the average size of information granules objectively. A familiar example is that the time is granulated according to scales such as year, month, day, hour, minute, and second. GrC can find out a proper granularity to solve problems through

Table 1 A typical complex information system

	a	b	c	d	e
1	Yes	s_1	z_1	Yes	No
2	Yes	s_2	z_2	Yes	No
3	Yes	s_3	z_3	No	No
4	Yes	s_4	z_4	No	No
5	No	s_5	z_5	No	No
6	No	s_6	z_6	No	No
7	No	s_7	z_7	No	Yes
8	No	s_8	z_8	No	Yes

(a) The binary relation τ_b on V_b

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	×	×			×	×		
s_2	×	×			×	×		
s_3		×	×	×	×		×	
s_4		×	×	×			×	×
s_5	×	×		×	×	×		
s_6	×	×			×	×		×
s_7							×	×
s_8							×	×

(b) The binary relation τ_c on V_c

	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
z_1	×	×			×	×		
z_2	×	×	×		×	×		×
z_3	×	×	×	×			×	×
z_4		×	×	×	×		×	×
z_5					×	×		
z_6					×	×	×	
z_7			×	×			×	×
z_8			×	×			×	×

Table 2 The binary relation τ_a on V_a

	No	Yes
No	×	
Yes		×

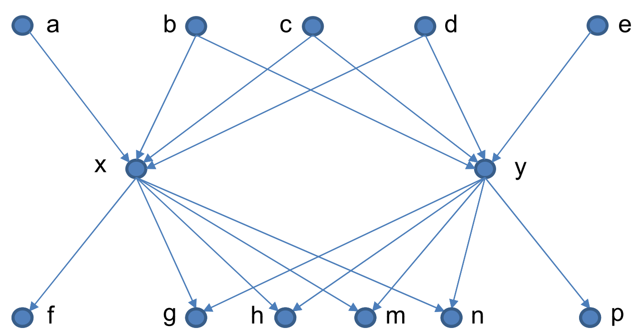

Fig. 6 A binary relation graph

granularity analysis, rather than analyze and solve problems at deeper levels, so it can significantly simplify the solving process of complicated problems. Obviously, carrying out modeling research on complicated data according to human's granular cognitive mechanism can generate new data modeling theories and approaches expectedly, which are of the significant theoretical significance for the development of data mining and knowledge discovery, and also of the application value for enhancing the processing efficiency of massive information.

Definition 2 Let $S = (U, AT, \tau)$ be a complex information system. For $v, w \in V_m$, if $(v, w) \in \tau_m$, we say v is a predecessor of w, and w is a successor of v. In the case, for any $z \in V_m$ we define

$$suc(z) = \{x \in V_m | (z, x) \in \tau_m\}$$
$$pre(z) = \{x \in V_m | (x, z) \in \tau_m\}$$

we say $pre(z)$ is the predecessor neighborhood of z, and $suc(z)$ is the successor neighborhood of z.

Definition 3 Let $S = (U, AT, \tau)$ be a complex information system, $v, w \in V_m$. On the basis of above Definition, the similarity model between v and w is defined as

$$sim(v, w) = min\{pre\_sim(v, w), suc\_sim(v, w)\}$$

where

$$pre\_sim(v, w) = \frac{1}{|pre(v) \cup pre(w)|} \times |pre(v) \cap pre(w)|$$

$$suc\_sim(v, w) = \frac{1}{|suc(v) \cup suc(w)|} \times |suc(v) \cap suc(w)|$$

For instance, in Fig. 6, we can get pre(x) = $\{a, b, c, d\}$, pre(y) = {b,c,d,e}, suc(x) = {f,g,h,m,n}, suc(y) = {g,h,m,n,p}, and further, pre_sim(x,y)=0.6 and suc_sim(x,y)=0.667 can be calculated. Obviously, sim(x,y)=0.6.

In Definition 3, when $suc(v) \cup suc(w) = \emptyset$, we assume $suc\_sim(v, w) = 0$ holds. Similarly, when $pre(v) \cup pre(w) = \emptyset$, we assume $pre\_sim(v, w) = 0$ holds.

In Table 1, on the basis of Definition 3, similarities between values in V_b are shown in Table 3, and the similarities of values in V_c are shown in Table 4.

In fact, based on the similarity model mentioned above, a fuzzy similarity relation matrix can be defined as

$$F_m = (\tilde{r}_{ij})_{|n|\times|n|},\ \tilde{r}_{ij} = sim(v_i, v_j)$$

where $v_i, v_j \in V_m, |V_m| = n$. We also know that the granulation analysis is always on the basis of fuzzy equivalence relation matrix. In the case, for generating fuzzy equivalence relation matrix, we always need to calculate the fuzzy transitive closure

$$F_m^+ = F_m^1 \cup F_m^2 \cup \cdots \cup F_m^{n-1}$$

There are many ways to calculate the fuzzy transitive closure, this paper chooses a simple algorithm with low complexity $O(n^2)$ [17]. For instance, by using above algorithm, F_b^+ shown in Table 5 can be calculated from Table 3, in the same way, F_c^+ shown in Table 6 can be calculated from Table 4.

And further, by introducing the parameter $\theta \in [0, 1]$, we can calculate the equivalence relation matrix θ−cut. Essentially, $F_m^+(\theta)$ is an equivalence relation on V_m, and $V_m^\theta = V_m/F_m^+(\theta)$ is a partition. Obviously, the bigger θ is, the smaller $|V_m^\theta|$ becomes, and vice versa. In addition, if $v \in Q_i$, then the granule $Q_i \in V_m^\theta$ can also be

Table 3 Similarity of values in V_b based on Definition 3

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	1.000	0.667	0.000	0.125	0.800	0.800	0.000	0.000
s_2	0.667	1.000	0.286	0.125	0.800	0.667	0.000	0.000
s_3	0.000	0.286	1.000	0.667	0.167	0.000	0.167	0.167
s_4	0.125	0.125	0.667	1.000	0.250	0.167	0.400	0.167
s_5	0.800	0.800	0.167	0.250	1.000	0.667	0.000	0.000
s_6	0.800	0.667	0.000	0.167	0.667	1.000	0.000	0.143
s_7	0.000	0.000	0.167	0.400	0.000	0.000	1.000	0.600
s_8	0.000	0.000	0.167	0.167	0.000	0.143	0.600	1.000

Table 4 Similarity of values in V_c based on Definition 3

	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
z_1	1.000	0.667	0.250	0.167	0.333	0.400	0.000	0.000
z_2	0.667	1.000	0.500	0.333	0.333	0.286	0.250	0.250
z_3	0.250	0.500	1.000	0.714	0.000	0.125	0.667	0.667
z_4	0.167	0.333	0.714	1.000	0.125	0.000	0.667	0.667
z_5	0.333	0.333	0.000	0.125	1.000	0.667	0.000	0.000
z_6	0.400	0.286	0.125	0.000	0.667	1.000	0.125	0.125
z_7	0.000	0.250	0.667	0.667	0.000	0.125	1.000	0.667
z_8	0.000	0.250	0.667	0.667	0.000	0.125	0.667	1.000

Table 5 The fuzzy transitive closure F_b^+ calculated from Table 3

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8
s_1	1.000	0.800	0.286	0.286	0.800	0.800	0.286	0.286
s_2	0.800	1.000	0.286	0.286	0.800	0.800	0.286	0.286
s_3	0.286	0.286	1.000	0.667	0.286	0.286	0.400	0.400
s_4	0.286	0.286	0.667	1.000	0.286	0.286	0.400	0.400
s_5	0.800	0.800	0.286	0.286	1.000	0.800	0.286	0.286
s_6	0.800	0.800	0.286	0.286	0.800	1.000	0.286	0.286
s_7	0.286	0.286	0.400	0.400	0.286	0.286	1.000	0.600
s_8	0.286	0.286	0.400	0.400	0.286	0.286	0.600	1.000

Table 6 The fuzzy transitive closure F_c^+ calculated from Table 4

	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
z_1	1.000	0.667	0.500	0.500	0.400	0.400	0.500	0.500
z_2	0.667	1.000	0.500	0.500	0.400	0.400	0.500	0.500
z_3	0.500	0.500	1.000	0.714	0.400	0.400	0.667	0.667
z_4	0.500	0.500	0.714	1.000	0.400	0.400	0.667	0.667
z_5	0.400	0.400	0.400	0.400	1.000	0.667	0.400	0.400
z_6	0.400	0.400	0.400	0.400	0.667	1.000	0.400	0.400
z_7	0.500	0.500	0.667	0.667	0.400	0.400	1.000	0.667
z_8	0.500	0.500	0.667	0.667	0.400	0.400	0.667	1.000

denoted as $[v]_\theta$. That is, if $v \in Q_i$ and Q_i is a granule, then $Q_i = [v]_\theta = \{w \in V_m | (v,w) \in F_m^+(\theta)\}$.

Definition 4 In $S = (U, AT, \tau)$, let $\pi \in [0,1]$, then we say τ_m^ϖ is a granulation relation of τ_m, which is defined as

$$(v,w) \in \tau_m^\varpi \iff \alpha_\theta(v,w) \geqslant \pi, \text{ for any } v,w \in V_m$$

where $\varpi = (\theta, \pi)$ and

$$\alpha_\theta(v,w) = \frac{1}{|[v] \times [w]|} \times |\tau_m \cap ([v] \times [w])|$$

For example, $V_b^\theta = \{[s_1],[s_3],[s_7]\}$ with $\theta = 0.6$ can be calculated from Table 5, where $[s_1] = \{s_1,s_2,s_5,s_6\}$, $[s_3] = \{s_3,s_4\}$, $[s_7] = \{s_7,s_8\}$; Similarly, $V_c^\theta = \{[z_1],[z_3],[z_5]\}$ with $\theta = 0.6$ can be calculated from Table 6, where $[z_1] = \{z_1,z_2\}$, $[z_3] = \{z_3,z_4,z_7,z_8\}$, $[z_5] = \{z_5,z_6\}$. In the case, the corresponding granulation relations τ_b^ϖ and τ_c^ϖ with $\pi = 0.6$ are shown in Table 7.

It is obvious that the parameter π can affect $|\tau_m^\varpi|$ to a given degree. That is, with θ remaining unchanged, the bigger π is, the smaller $|\tau_m^\varpi|$ is, and vice versa. Essentially speaking, when increase of both θ and π, we can simplify a complicated and huge digraph (V_m, τ_m) into a simple one $(V_m^\theta, \tau_m^\varpi)$ under the high granularity.

4 The algebraic structure in complex information systems

Concept lattice and rough set are similar to each other in data analysis, so studying their relationship and combining them together will definitely help us analyze and understand data more efficiently.

Concept lattice is a kind of concept hierarchical structure established on the basis of binary relation between sets G and M. As the core data structure of FCA, it is considered a powerful data analysis tool for knowledge discovery. The section will attempt to bring the outstanding mathematical properties of concept lattice into complex information systems, and emphatically probes into the algebraic structure in complex information systems.

In recent years, a good many scholars have discussed equivalence relation, dominance relation, similarity relation, and other complicated relations on the universe. Similarly, for any $B \subseteq AT$, the paper defines the binary relation on the universe U as follows

$$R_B^\varpi = \left\{ \begin{array}{l} (x,y) \in U \times U | \forall m \in B, (v,w) \in \tau_m^\varpi, \\ f(x,m) = v, f(y,m) = w \end{array} \right\}$$

Definition 5 Let $S = (U, AT, \tau)$ be a complex information system. By the following rule

$$((x,y),m) \in I_\varpi \iff (v,w) \in \tau_m^\varpi,\ v = f(x,m), w = f(y,m)$$

Table 7 Granulation relations τ_b^ϖ and τ_c^ϖ with $\theta = 0.6, \pi = 0.6$

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8		z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8
s_1	×	×			×	×			z_1	×	×			×	×		
s_2	×	×			×	×			z_2	×	×			×	×		
s_3			×	×			×	×	z_3			×	×			×	×
s_4			×	×			×	×	z_4			×	×			×	×
s_5	×	×			×	×			z_5					×	×		
s_6	×	×			×	×			z_6					×	×		
s_7							×	×	z_7			×	×			×	×
s_8							×	×	z_8			×	×			×	×

S can be transformed to the following one-valued formal context

$$K_S^{\varpi} = (U \times U, AT, I_{\varpi})$$

In actual applications, we can indirectly change K_S^{ϖ} by changing parameters θ and π. For instance, Table 8 shows the one-valued formal context derived from Table 1.

Theorem 1 *Let K_S^{ϖ} be the derived context of $S = (U, AT, \tau)$, $B \subseteq AT$. By means of operators in* Definition 1*, we can easily obtain the conclusion*

$$B' = R_B^{\varpi}$$

Proof The conclusion can be obtained from Definitions 1 and 5 immediately.

In fact, the relation R_B^{ϖ} is a kind of extremely complicated binary relation. In the case, the paper, in virtue of granulation idea in GrC, decomposes R_B^{ϖ} into several block-relations, which can not only generally conduce to better analyzing and solving problems, but also essentially conform to people's features in solving complicated problems.

Following the definition of block-relations(see the Introduction Section), let $\mathscr{R}_1$ and $\mathscr{R}_2$ be binary relations on $\mathscr{E}$, for any $X_1 \times Y_1 \in \kappa(\mathscr{R}_1)$, if there exists $X_2 \times Y_2 \in \kappa(\mathscr{R}_2)$ satisfying $X_1 \times Y_1 \subseteq X_2 \times Y_2$, then we say $\kappa(\mathscr{R}_2)$ is coarser than $\kappa(\mathscr{R}_1)$, which is denoted by $\kappa(\mathscr{R}_1) \preccurlyeq \kappa(\mathscr{R}_2)$.

Proposition 2 *From above discussions, there are following conclusions*

- *$\kappa(\mathscr{R})$ is a cover of $\mathscr{R}$, that is, $\mathscr{R} = \cup\{X \times Y | X \times Y \in \kappa(\mathscr{R})\}$;*
- *$\mathscr{R}_1 \subseteq \mathscr{R}_2$, if and only if $\kappa(\mathscr{R}_1) \preccurlyeq \kappa(\mathscr{R}_2)$.*

Essentially, in the formal context K_S^{ϖ}, operators in Definition 1 can be described as: for the set $R \subseteq U \times U$,

$$R' = \{m \in AT | ((x, y), m) \in I_{\varpi}, \forall (x, y) \in R\}$$

Table 8 An one-valued formal context derived from Table 1 with $\pi = 0.6, \theta = 0.6$

	a	b	c	d	e
(1, 1)	×	×	×	×	×
(1, 2)	×	×	×	×	×
(1, 3)	×				×
(1, 4)	×				×
(1, 5)		×	×		×
⋮	⋮	⋮	⋮	⋮	⋮
(8, 6)	×			×	
(8, 7)	×	×	×	×	×
(8, 8)	×	×	×	×	×

Correspondingly, for the set $B \subseteq AT$,

$$B' = \{(x, y) \in U \times U | ((x, y), m) \in I_{\varpi}, \forall m \in B\}$$

If $R' = B$ and $B' = R$, then we say $(\kappa(R), B)$ is a block-relation concept. The set of all block-relation concept in K_S^{ϖ} is denoted as $\mathscr{B}(S_{\varpi})$. Let $(\kappa(R_1), B_1)$ and $(\kappa(R_2), B_2)$ be block-relation concepts, we define

$$(\kappa(R_1), B_1) \le (\kappa(R_2), B_2) \Leftrightarrow \kappa(R_1) \preccurlyeq \kappa(R_2) \Leftrightarrow B_2 \subseteq B_1$$

It is not hard to see that $(\mathscr{B}(S_{\varpi}), \le)$ is a lattice structure, which is called block-relation lattice. Meanwhile, it is not hard to see the following conclusion can be inferred immediately

$$(R, B) \in \mathscr{B}(K_S^{\varpi}) \Longleftrightarrow (\kappa(R), B) \in \mathscr{B}(S_{\varpi})$$

Obviously, $(\mathscr{B}(S_{\varpi}), \le)$ can organize all granulation results $\{\kappa(R_B^{\varpi}) | B \subseteq AT\}$ in the form of lattice structure. Therefore, $(\mathscr{B}(S_{\varpi}), \le)$ can be viewed as the core data structure of the whole information system S. In addition, it is known that concept lattice is very suitable for discovering IF-THEN rules, so $(\mathscr{B}(S_{\varpi}), \le)$ also can be used to acquire rules in complex information systems.

Theorem 2 *If $(\kappa(R), B) \in \mathscr{B}(S_{\varpi})$, then $R = R_B^{\varpi}$.*

Proof If $(\kappa(R), B) \in \mathscr{B}(S_{\varpi})$, then $R = B'$. In addition, $B' = R_B^{\varpi}$ can be inferred from Theorem 1. Hence, the conclusion $R = R_B^{\varpi}$ is true.

Theorem 3 *Let $K = (\mathscr{E}, \mathscr{E}, \mathscr{R})$ be a formal context. If $\mathscr{R}$ is a binary relation on the set $\mathscr{E}$, then*

$$X \times Y \in \kappa(\mathscr{R}) \Leftrightarrow (X, Y) \in \mathscr{B}(K)$$

Obviously, in $S = (U, AT, \tau)$, for any $B \subseteq AT$, the corresponding $\kappa(R_B^{\varpi})$ can be calculated in virtue of the following conclusion

- $\kappa(R_B^{\varpi}) = \{X \times Y | (X, Y) \in \mathscr{B}(K)\}$, where $K = (U, U, R_B^{\varpi})$

For instance, let $B = \{b, c, d\}$, $\theta = 0.6$ and $\pi = 0.6$, then we can get the formal context shown in Table 9, it is obvious that (12, 12), (56, 56), (34, 3456) and (3478, 78) are corresponding concepts. And further, we cam obtain all block-relations of R_B^{ϖ}, that is, 12×12, 56×56, 34×3456 and 3478×78.

Based on above discussions, block-relation lattices shown in Figs. 7 and 8 can be deduced from Table 1. In Figs. 7 and 8, only intents of concepts are given. The extents of concepts in Fig. 7 are shown in Table 10. The extents of concepts in Fig. 8 will not be detailed here again. In addition, for the block-relations $U \times \varnothing$ and $\varnothing \times U$, there is not any practical significance, so $U \times \varnothing$ and $\varnothing \times U$ will not be taken into account.

Table 9 A formal context

	1	2	3	4	5	6	7	8
1	×	×						
2	×	×						
3			×	×			×	×
4			×	×			×	×
5					×	×		
6					×	×		
7							×	×
8							×	×

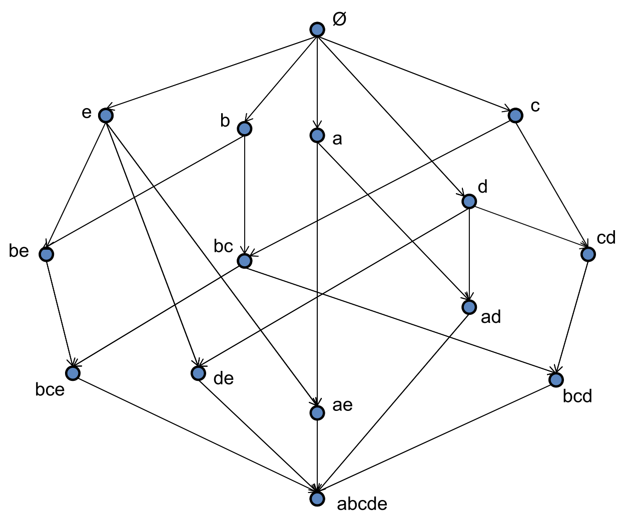

Fig. 7 A block-relation lattice with $\theta = 0.6$ and $\pi = 0.6$

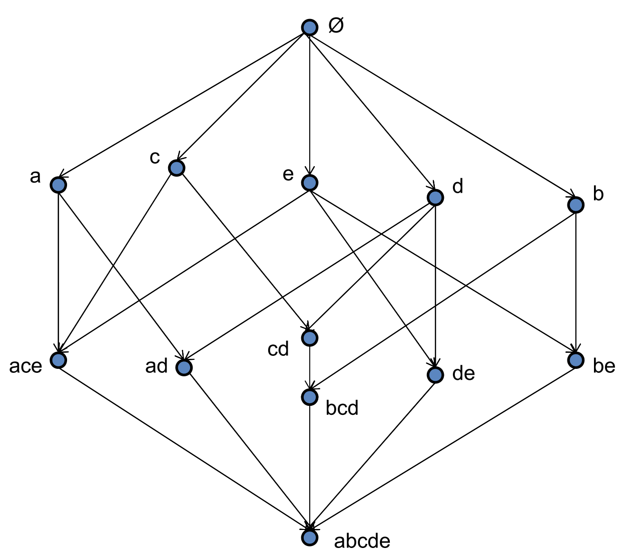

Fig. 8 A block-relation lattice with $\theta = 0.5$ and $\pi = 0.6$

5 Reduction, core and dependency in complicated information systems

In $S = (U, AT, \tau)$, let $B \subseteq AT$, $m \in B$, if $R_D \neq R^{\varpi}_{D-m}$, then we say that m is indispensable in B; If $B \subseteq D$ is the minimal subset satisfying $R^{\varpi}_B = R^{\varpi}_D$, then we say that B is a reduction of D; $core(B)$ consisted of all indispensable attributes in $B \subseteq AT$ is called the core of B.

Theorem 4 *Let* $B \subseteq AT$, $m \in B$. *If* $\kappa(R^{\varpi}_{B-m}) \neq \kappa(R^{\varpi}_B)$ or $(B - m)' \neq B'$, then $m \in core(B)$.

Theorem 5 *Let* $C \subseteq B$. *If C is a minimal subset satisfying* $\kappa(R^{\varpi}_B) = \kappa(R^{\varpi}_C)$ or $B' = C'$, *then C is a reduction of B.*

As a kind of important rule-based knowledge, dependency is close to human's natural thinking, and can be understood easily. Normally, function dependency and order dependency are the most common types, which are described as follows: for any $(x, y) \in U \times U$, if

$$(\forall m \in B, f(x,m) = f(y,m)) \Rightarrow (\forall n \in D, f(x,n) = f(y,n))$$

then $B \rightarrow D$ is called a function dependency; for any $(x, y) \in U \times U$, if

$$(\forall m \in B, f(x,m) \preccurlyeq f(y,m)) \Rightarrow (\forall n \in D, f(x,n) \preccurlyeq f(y,n))$$

$B \rightarrow D$ is called an order dependency. In the paper, function dependency and order dependency are further expanded to the more general dependency defined as follows: for any $(x, y) \in U \times U$, if

$$(\forall m \in B, (v_1, w_1) \in R^{\varpi}_m) \Rightarrow (\forall n \in D, (v_2, w_2) \in R^{\varpi}_n)$$

where $v_1 = f(x,m), w_1 = f(y,m), v_2 = f(x,n), w_2 = f(y,n)$, then we say D depends on B completely, which is denoted as $B \rightarrow D$. In the case, we also say $B \rightarrow D$ is a complex dependency. In essence, a complex dependency is an implication of K^{ϖ}_S, as defined by Ganter and Wille [6]. For more researches on implications, refer to [21, 24, 38].

From above the definition of complex dependency, we can easily proof the following conclusions

Table 10 Extents of block-relation concepts in Fig. 7

Intents	Extents
a	$1234 \times 1234,\ 5678 \times 5678$
b	$1256 \times 1256,\ 34 \times 3478,\ 3478 \times 78$
c	$12 \times 1256,\ 3478 \times 3478,\ 1256 \times 56$
d	$12 \times 12,\ 345678 \times 345678$
e	$123456 \times 123456,\ 78 \times 78$
ad	$12 \times 12,\ 34 \times 34,\ 5678 \times 5678$
ae	$1234 \times 1234,\ 56 \times 56,\ 78 \times 78$
bc	$12 \times 1256,\ 34 \times 3478\ 1256 \times 56,\ 3478 \times 78$
be	$1256 \times 1256,\ 34 \times 34,\ 78 \times 78$
cd	$12 \times 12,\ 56 \times 56,\ 3478 \times 3478$
de	$12 \times 12,\ 78 \times 78,\ 3456 \times 3456$
bcd	$12 \times 12,\ 56 \times 56,\ 34 \times 3478,\ 3478 \times 78$
bce	$34 \times 34,\ 78 \times 78,\ 12 \times 1256,\ 1256 \times 56$
∅	12345678×12345678
abcde	$12 \times 12,\ 34 \times 34,\ 56 \times 56,\ 78 \times 78$

- $R_B^{\varpi} \subseteq R_D^{\varpi}$ or $B' \subseteq D'$, if and only if $B \rightarrow D$ is a complex dependency.
- $\kappa(R_B^{\varpi}) \subseteq \kappa(R_D^{\varpi})$, if and only if $B \rightarrow D$ is a complex dependency.

However, D does not completely depend on B in many cases. In the case, we say D partially depends on B, and the corresponding dependency degree is defined as:

$$\xi_{\varpi}(B/D) = \frac{|\mathrm{pos}_{\varpi}(B/D)|}{|R_B^{\varpi}|},$$

where

$$\mathrm{pos}_{\varpi}(B/D) = \bigcup_{X\times Y \in \kappa(R_D^{\varpi})} \bigcup_{H\times N \in \kappa(R_B^{\varpi})} \{H \times N | H \times N \subseteq X \times Y\}.$$

For example, in Table 1, let $\theta = 0.6, \pi = 0.6, B = \{\text{d,e}\}$ and $D = \{\text{b,c}\}$, then

$\kappa(R_B^{\varpi}) = \{12 \times 12,\ 78 \times 78,\ 3456 \times 3456\}$
$\kappa(R_D^{\varpi}) = \{12 \times 1256,\ 34 \times 3478\ \ 1256 \times 56,\ 3478 \times 78\}$
And then we can obtain

$$\xi_{\varpi}(B/D) = \frac{1}{24} \times |(12 \times 12) \cup (78 \times 78)| = \frac{1}{24} \times 8 = 0.33.$$

Obviously, D partially depends on B, the corresponding dependency degree is 0.33.

An information system usually contains massive dependencies, some of which are valuable, and some are valueless and redundant. In the paper, we define redundant dependencies as follows: Let $B \rightarrow D, B_1 \rightarrow D_1$, if $B \subseteq B_1$ and $D_1 \subseteq D$, then we say $B_1 \rightarrow D_1$ is a redundant dependency relative to $B \rightarrow D$, that is, $B_1 \rightarrow D_1$ is redundant relatively; Let $D \subseteq B$, then we say $B \rightarrow D$ is redundant absolutely.

Table 11 A dependency set in Table 1 with $\varpi = (0.6, 0.6)$

$bd \rightarrow c$	$ce \rightarrow b$	$ab \rightarrow cde$	$ac \rightarrow bde$
$ade \rightarrow bc$	$cde \rightarrow ab$	$bde \rightarrow ac$	

By removing all all redundant dependencies, we can further get a smaller dependency set. For instance, when $\varpi = (0.6, 0.6)$ and $\varpi = (0.5, 0.6)$, the corresponding dependency sets in Table 1 are shown in Tables 11 and 12 separately. In above tables, for any $B \rightarrow D$, which is simplified as $B \rightarrow D/B$.

6 Summary and outlook

In recent years, concept lattice and rough set find wide applications in a variety of fields successfully. They are two different type of theories, but they share many similarities. So studying the combination theory between concept lattice and rough set will surely help us better understand and analyze complicated problems. Therefore, the paper brings concept lattice into rough set, and presents a now rough set model based on concept lattice. In the model, one general binary relation on universe can be decomposed into several block-relations, which can be viewed as granules to study algebraic structure and offer solutions to problems such as reduction and core in complex information systems. The so-called algebraic structure refers to the block-relation lattice, namely $(\mathscr{B}(S_{\varpi}), \leq)$, which can organize all granulation results $\{\kappa(R_B^{\varpi})\,|B \subseteq AT\}$ in the form of lattice structure. It is known that concept lattice is very suitable for discovering IF-THEN rules, so $(\mathscr{B}(S_{\varpi}), \leq)$ also can be used to acquire rules in complex information systems. In addition, in some complicated and large-scale data, the computing process based on lattice concept is often accompanied by high time and space complexity, aiming at the problem, the paper overcomes it by introducing the granularity of attribute domain, and further convert complex information systems into relatively simple ones.

In short, the paper gives the definition of relation granulation, and further offers an reasonable idea for the expansion of rough set by introducing concept lattice and GrC. Meanwhile, it also helps to the deep combination of concept lattice and rough set. The focus of our research in the next step will

Table 12 A dependency set in Table 1 with $\varpi = (0.5, 0.6)$

$bc \rightarrow d$	$bd \rightarrow c$	$ac \rightarrow e$	$ae \rightarrow c$
$ce \rightarrow a$	$ab \rightarrow cde$	$acd \rightarrow be$	$ade \rightarrow bc$
$bce \rightarrow ad$	$bde \rightarrow ac$	$cde \rightarrow ab$	

be how to apply GrC to further overcoming the high time and space complexities in large-scale data.

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