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Sequential three-way decisions via multi-granularity

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ABSTRACT

Three-way decisions provide a trisecting-and-acting framework for complex problem solving. For a cost-sensitive decision-making problem under multiple levels of granularity, sequential three-way decisions have come into being. Within this framework, how to act upon the three pair-wise disjoint regions is the most important issue. To this end, we propose a generalized model of sequential three-way decisions via multi-granularity in this paper. Subsequently, we adopt the typical aggregation strategies to implement the following five kinds of multigranulation sequential three-way decisions—the weighted arithmetic mean multigranulation sequential three-way decisions, the optimistic multigranulation sequential three-way decisions, the pessimistic multigranulation sequential three-way decisions, the pessimistic-optimistic multigranulation sequential three-way decisions and the optimistic-pessimistic multigranulation sequential three-way decisions. Furthermore, we discuss the rightness and rationality of the five kinds of multigranulation sequential three-way decisions and also analyze the relationships and differences between them. Finally, the experimental results demonstrate that the first four different multigranulation sequential three-way decisions are effective. These models will accelerate and enrich the development of multigranulation three-way decisions.

1. Introduction

Three-way decision model (3WD) [41,42] is a trisecting-and-acting model for complex problem solving. The basic idea of this model is to divide a universal set of objects into the three pair-disjoint regions and to devise the different effective strategies to act upon those objects of each region. In recent years, there are several successful application results in many fields such as investment management [21], cluster analysis [17,45], face recognition [11], recommendation [47] and the others [2,3,6–8,10,12,14,16,46]. However, how to act upon three probabilistic regions of 3WD is still under an active research [5,38].

By considering the costs of decision results and decision processes, a typical model of 3WD, sequential three-way decision (S3WD), has emerged and become a cost-effective decision-making method [43]. This model aims at achieving a required level of accuracy with a minimal cost in obtaining evidence especially for a problem under multiple levels of granularity. Classical sequential three-way decision models [43] mainly implement a sequential, multistep three-way decision-making using a multilevel granular structure of the universe with respect to a sequence of sets of attributes. In some cases,

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a more reasonable decision-making for complex problem solving is made under multiview granular structures. Therefore, the extended sequential three-way decision models still consider multiple independent granular structures of the universe for decision-making.

As we all know, Qian et al. [23] introduced multigranulation rough set model, where the lower and upper approximations of a concept are characterized under multiple granular structures. The multigranulation rough set models using different multiple binary relations for different information systems have been paid widespread attentions. The researches on multigranulation rough set models mainly focus on the following three aspects.

- The first aspect is the extended model researches on multigranulation rough set models using multiple different binary relations. Using multiple tolerance relations, Qian et al. [24] and Yang et al. [36] discussed incomplete multigranulation rough sets, and Xu et al. [33] constructed and discussed two types of multigranulation tolerance rough set models. Furthermore, Xu et al. [32] constructed multigranulation fuzzy rough sets based on the fuzzy approximation space using multiple fuzzy relations. Lin et al. [18] constructed multigranulation covering fuzzy rough sets in a covering approximation space. Huang et al. [9] discussed an intuitionistic fuzzy multigranulation rough set model. In addition, Qian et al. [25], Xu [34], Yang and Guo [35], Liu et al. [20], Sun et al. [30] and Feng and Mi [4] studied different multigranulation decision-theoretic rough set models.
- The second aspect is knowledge discovery researches on multigranulation rough set model. Liang et al. [15] proposed an efficient feature selection algorithm using a multigranulation view. Lin et al. [19] presented a feature selection method using neighborhood multi-granulation fusion. Li et al. [12], 13] compared multigranulation rough sets with concept lattices via rule acquisition, and further constructed a three-way cognitive concept learning model via multi-granularity. Tan et al. [31] constructed belief structures and characterize knowledge reduction in terms of evidence theory for the multigranulation spaces with decisions.
- The third aspect is topology analysis researches on multigranulation rough set model. Yang et al. [37] studied the hierarchical structures of multigranulation rough sets. She and He [29] discussed the topological and lattice-theoretic properties of multigranulation rough sets. Yao [44] proposed a unified framework to classify and compare two basic models based on the construction of a family of equivalence relations and a combination of the family of approximations, respectively.

With the insightful gain from above multigranulation discussions, one can find a more reasonable decisions for all objects should be made via sequential three-way decisions and multiview granular structures. To this end, we combine with sequential three-way decisions and multigranulation rough sets to build a generalied sequential three-way decision model via multi-granularity. More specifically, we adopt the different aggregation strategies to aggregate the lower approximations and the upper approximations, and construct the weighted arithmetic mean multigranulation sequential three-way decisions (WAMMS3WD), the optimistic multigranultion sequential three-way decisions (OMS3WD), the pessimistic multigranultion sequential three-way decisions (PMS3WD), the pessimistic-optimistic multigranultion sequential three-way decisions (POMS3WD) and the optimistic-pessimistic multigranulation sequential three-way decisions (OPMS3WD). Furthermore, we discuss the rightness and rationality of these models and analyze the corresponding relationships and differences. Finally, real-life experiments are employed to demonstrate their feasibility and effectiveness.

The remainder of this paper is organized as follows. The next section deals with some preliminary concepts and properties regarding the Pawlak's rough sets, multigranulation rough sets and decision-theoretic rough sets. In Section 3, we introduce the five types of multigranulation sequential three-way decisions(MS3WD) and investigate the corresponding properties. In Section 4, we analyze the relationships and rationality among the first four types of MS3WD models. Finally, Section 5 concludes the paper.

2. Preliminary knowledge on decision-theoretic rough set model

In this section, we will review some basic concepts of Pawlak rough set model, multigranulation rough set model and decision-theoretic rough set model. For a detailed description, please refer to papers [22,26,40].

2.1. Pawlak rough set model

For classification tasks, we consider a decision table which is defined as: $S = (U, At = C \cup D, \{V_a | a \in At\}, \{I_a | a \in At\})$, where $U = \{x_1, x_2, \ldots, x_n\}$ is a finite non-empty set of objects, At is a finite nonempty set of attributes, $C = \{c_1, c_2, \ldots, c_M\}$ is a set of conditional attributes describing the objects, and D is a set of decision attributes that indicates the classes of objects. V_a is a nonempty set of values of $a \in At$. I_a is an information function that maps an object x in U to exactly one value v in V_a, that is, $I_a(x) = v$. For simplicity, we assume $D = \{d\}$ in this paper, where d is a decision attribute which has k different decision values, and $V_d = \{1, 2, \ldots, k\}$. A table with multiple decision attributes can be easily transformed into a table with a single decision attribute by considering the Cartesian product of the original decision attributes.

An indiscernibility relation with $A \subseteq At$ is defined as $IND(A) = \{(x, y) \in U \times U | \forall a \in A [I_a(x) = I_a(y)]\}$. The partition generated by $IND(A)$ is denoted as $U/IND(A)$, or simply π_A. The equivalence class containing object x in π_A is given by $[x]_{IND(A)} = \{y \in U | (x, y) \in IND(A)\}$. For simplicity, we write $[x]_A$ instead of $[x]_{IND(A)}$ if $IND(A)$ is understood.

Consider a partition π_D with respect to the decision attribute D and another partition π_A induced by a set of conditional attributes A. These equivalence classes induced by the partition are the basic blocks to construct the Pawlak rough set approximations.

For a decision class $D_q \in \pi_D$, the lower and upper approximations of D_q with respect to a partition π_A are defined by Pawlak [22]:

$$\begin{aligned} \underline{apr}_{\pi_A}(D_q) &= \{x \in U | [x]_A \subseteq D_q\} \\ &= \{x \in U | p(D_q|[x]_A) = 1\}; \end{aligned} \tag{1}$$

$$\begin{aligned} \overline{apr}_{\pi_A}(D_q) &= \{x \in U | [x]_A \cap D_q \neq \emptyset\} \\ &= \{x \in U | p(D_q|[x]_A) > 0\}. \end{aligned} \tag{2}$$

where $p(D_q|[x]_A)$ denotes the conditional probability that an object x belongs to D_q given that the object is in the equivalence class $[x]_A$, i.e., $p(D_q|[x]_A) = \frac{|[x]_A \cap D_q|}{|[x]_A|}$.

The lower and upper approximations of the partition π_D with respect to π_A are the families of the lower and upper approximations of all the equivalence classes of π_D.

$$\begin{aligned} \underline{apr}_{\pi_A}(\pi_D) &= \{\underline{apr}_{\pi_A}(D_1), \underline{apr}_{\pi_A}(D_2), \ldots, \underline{apr}_{\pi_A}(D_k)\}; \\ \overline{apr}_{\pi_A}(\pi_D) &= \{\overline{apr}_{\pi_A}(D_1), \overline{apr}_{\pi_A}(D_2), \ldots, \overline{apr}_{\pi_A}(D_k)\}. \end{aligned} \tag{3}$$

Once we have the lower and upper approximations for each decision class, a positive, boundary and negative region of a partition π_D with respect to a partition π_A is defined as follows:

$$POS_{\pi_A}(\pi_D) = \bigcup_{1 \le q \le k} \underline{apr}_{\pi_A}(D_q); \tag{4}$$

$$\begin{aligned} BND_{\pi_A}(\pi_D) &= \bigcup_{1 \le q \le k} BND_{\pi_A}(D_q) \\ &= \bigcup_{1 \le q \le k} (\overline{apr}_{\pi_A}(D_q) - \underline{apr}_{\pi_A}(D_q)); \end{aligned} \tag{5}$$

$$NEG_{\pi_A}(\pi_D) = U - POS_{\pi_A}(\pi_D) \cup BND_{\pi_A}(\pi_D). \tag{6}$$

For the partition π_D, we can calculate its lower and upper approximations in terms of k two-class problems. $POS_{\pi_A}(\pi_D)$ is the positive region of decision and indicates the union of all the equivalence classes defined by π_A, in which each equivalence class induces a certain decision. $BND_{\pi_A}(\pi_D)$ is the boundary region of decision and is formed by the union of all the equivalence classes defined by π_A which induces the partial decisions. When A equals C, $POS_{\pi_C}(\pi_D)$ denotes the whole positive region of a decision table. If a decision table is consistent, $BND_{\pi_C}(\pi_D) = NEG_{\pi_C}(\pi_D) = \emptyset$ and $POS_{\pi_C}(\pi_D) = U$.

2.2. QIAN's MGRS

In Pawlak rough set model, the target is approximated by one and only one binary relation. In many circumstances, we often need to describe concurrently a target concept through multiple binary relations on the universe according to a user's requirements or targets of problem solving. Qian et al. [23,25] introduced multigranulation rough set theory(MGRS). In these models, a target concept is approximated by a family of indiscernibility relations. In what follows, we briefly review the optimistic and pessimistic multigranulation rough set models in [23].

Definition 1. Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$ and $\forall X \subseteq U$, the optimistic multigranulation lower and upper approximations $\underline{\sum_{i=1}^{m} A_i{}^O(X)}$ and $\overline{\sum_{i=1}^{m} A_i{}^O(X)}$ are defined as follows, respectively.

$$\begin{aligned} \underline{\sum_{i=1}^{m} A_i{}^O(X)} &= \{x \in U | [x]_{A_1} \subseteq X \vee [x]_{A_2} \subseteq X \vee \ldots \vee [x]_{A_m} \subseteq X\}, \\ \overline{\sum_{i=1}^{m} A_i{}^O(X)} &= \sim \underline{\sum_{i=1}^{m} A_i{}^O(\sim X)}. \end{aligned} \tag{7}$$

where $\sim X$ is the complement of set X.

The pair $< \underline{\sum_{i=1}^{m} A_i{}^O(X)}, \overline{\sum_{i=1}^{m} A_i{}^O(X)} >$ is called the optimistic multigranulation rough sets of X. Here the word "optimistic" means that only a single granular structure is needed to satisfy the inclusion condition between an equivalence class and a target concept when the multiple independent granular structures are available.

Definition 2. Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$ and $\forall X \subseteq U$, the pessimistic multigranulation lower and upper approximations $\underline{\sum_{i=1}^{m} A_i^P}(X)$ and $\overline{\sum_{i=1}^{m} A_i^P}(X)$ are defined as follows, respectively.

$$\begin{aligned} \underline{\sum_{i=1}^{m} A_i^P}(X) &= \{x \in U \mid [x]_{A_1} \subseteq X \wedge [x]_{A_2} \subseteq X \wedge \ldots \wedge [x]_{A_m} \subseteq X\}, \\ \overline{\sum_{i=1}^{m} A_i^P}(X) &= \sim \underline{\sum_{i=1}^{m} A_i^P}(\sim X). \end{aligned} \tag{8}$$

where $\sim X$ is the complement of set X.

The pair $< \underline{\sum_{i=1}^{m} A_i^P}(X), \overline{\sum_{i=1}^{m} A_i^P}(X) >$ is called the pessimistic multigranulation rough sets of X. Here the word "pessimistic" means that all granular structures must satisfy the inclusion condition between an equivalence class and a target concept when the multiple independent granular structures are available.

2.3. Decision-theoretic rough sets

In decision-theoretic rough set models, the probabilistic positive, boundary and negative regions are induced using two tolerance threshold parameters α and β $(0 \le \beta < \alpha \le 1)$ by a set of loss functions based on the Bayesian theory procedure [39]. They overcome the weakness in Pawlak rough set model, and are the basis of three-way decisions, which can be described as follows.

Within the framework of three-way decisions, suppose the set of states $\Omega = \{X, \neg X\}$, where $\neg X$ denotes the complement of X, the set of actions $A = \{a_P, a_B, a_N\}$, where a_P, a_B and a_N represent the three actions in classifying an object x, namely, deciding $x \in POS(X)$, deciding x should be further investigated $x \in BND(X)$, and deciding $x \in NEG(X)$. λ_{PP}, λ_{BP} and λ_{NP} denote the loss incurred for taking actions of a_P, a_B and a_N, respectively, when an object belongs to X. Similarly, λ_{PN}, λ_{BN} and λ_{NN} denote the losses incurred for taking the corresponding actions when the object belongs to $\neg X$. By Bayesian decision procedure, for an object x, the expected loss $R(a_\bullet|[x])$ associated with taking the individual action can be expressed as

$$\begin{aligned} R(a_P|[x]) &= \lambda_{PP}P(X|[x]) + \lambda_{PN}P(\neg X|[x]), \\ R(a_N|[x]) &= \lambda_{NP}P(X|[x]) + \lambda_{NN}P(\neg X|[x]), \\ R(a_B|[x]) &= \lambda_{BP}P(X|[x]) + \lambda_{BN}P(\neg X|[x]), \end{aligned}$$

When $0 \le \lambda_{PP} \le \lambda_{BP} < \lambda_{NP}$ and $0 \le \lambda_{NN} \le \lambda_{BN} < \lambda_{PN}$, the Bayesian decision procedure leads to the following three minimum-risk decision rules.

(P1) If $P(X|[x]) \ge \alpha$ and $P(X|[x]) \ge \beta$, decide $x \in POS(X)$,

(N1) If $P(X|[x]) \le \gamma$ and $P(X|[x]) \le \beta$, decide $x \in NEG(X)$,

(B1) If $P(X|[x]) \le \alpha$ and $P(X|[x]) \ge \beta$, decide $x \in BND(X)$, where $\alpha = \frac{\lambda_{PN}-\lambda_{BN}}{(\lambda_{PN}-\lambda_{BN})+(\lambda_{BP}-\lambda_{PP})}$, $\beta = \frac{\lambda_{BN}-\lambda_{NN}}{(\lambda_{BN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{BP})}$, $\gamma = \frac{\lambda_{PN}-\lambda_{NN}}{(\lambda_{PN}-\lambda_{NN})+(\lambda_{NP}-\lambda_{PP})}$.

If $0 \le \beta < \gamma < \alpha \le 1$, (P1)-(B1) can be re-expressed as follows:

(P2) If $P(X|[x]) \ge \alpha$, decide $x \in POS(X)$,

(N2) If $P(X|[x]) \le \beta$, decide $x \in NEG(X)$,

(B2) If $\beta < P(X|[x]) < \alpha$, decide $x \in BND(X)$.

Thus, given two parameters α and β for a decision class D_q with respect to a partition π_A, the probabilistic lower and upper approximations can be defined by:

$$\begin{aligned} \underline{apr}_{\pi_A}^{(\alpha,\beta)}(D_q) &= \{x \in U \mid p(D_q|[x]_A) \ge \alpha\} \\ &= \bigcup_{p(D_q|[x]_A) \ge \alpha} [x]_A; \end{aligned} \tag{9}$$

$$\begin{aligned} \overline{apr}_{\pi_A}^{(\alpha,\beta)}(D_q) &= \{x \in U \mid p(D_q|[x]_A) > \beta\} \\ &= \bigcup_{p(D_q|[x]_A) > \beta} [x]_A. \end{aligned} \tag{10}$$

The above two approximations are the key issue of the decision-theoretic rough set models. We can extend the probabilistic approximations of a single decision D_q to those of the entire partition π_D. The lower and upper approximations of the partition π_D with respect to π_A are the families of the lower and upper approximations of all the equivalence classes of π_D. For simplicity, we assume that the same loss functions are used for all decisions.

$$\begin{aligned} \underline{apr}_{\pi_A}^{(\alpha,\beta)}(\pi_D) &= \{\underline{apr}_{\pi_A}^{(\alpha,\beta)}(D_1), \underline{apr}_{\pi_A}^{(\alpha,\beta)}(D_2), \ldots, \underline{apr}_{\pi_A}^{(\alpha,\beta)}(D_k)\}; \\ \overline{apr}_{\pi_A}^{(\alpha,\beta)}(\pi_D) &= \{\overline{apr}_{\pi_A}^{(\alpha,\beta)}(D_1), \overline{apr}_{\pi_A}^{(\alpha,\beta)}(D_2), \ldots, \overline{apr}_{\pi_A}^{(\alpha,\beta)}(D_k)\}. \end{aligned} \tag{11}$$

In decision-theoretic rough set models, three probabilistic regions (positive, boundary and negative regions) of a partition π_D with respect to a partition π_A can be defined as follows:

$$POS^{(\alpha,\beta)}_{\pi_A}(\pi_D) = \bigcup_{1\le q\le k} \underline{apr}^{(\alpha,\beta)}_{\pi_A}(D_q); \tag{12}$$

$$\begin{aligned} BND^{(\alpha,\beta)}_{\pi_A}(\pi_D) &= \bigcup_{1\le q\le k} BND^{(\alpha,\beta)}_{\pi_A}(D_q) \\ &= \bigcup_{1\le q\le k} (\overline{apr}^{(\alpha,\beta)}_{\pi_A}(D_q) - \underline{apr}^{(\alpha,\beta)}_{\pi_A}(D_q)); \end{aligned} \tag{13}$$

$$NEG^{(\alpha,\beta)}_{\pi_A}(\pi_D) = U - POS^{(\alpha,\beta)}_{\pi_A}(\pi_D) \cup BND^{(\alpha,\beta)}_{\pi_A}(\pi_D). \tag{14}$$

3. Sequential three-way decisions via multi-granularity

As we all know, three-way decision model is a common problem solving strategy within the framework of decision-theoretic rough set models. In general, the existing three-way decision-making methods are classified into two classes—a single one-step three-way decision-making and a sequential, multi-step three-way decision-making. The former may be regarded as a special case of the latter. Thus, we always discuss the sequential three-way decisions. In fact, we often gradually support an acceptance or a rejection at a particular level by modifying different (α, β) parameters. Thus, decision-making at a finer (lower) level needs larger considerations than decision-making at a coarser (higher) level.

3.1. Sequential three-way decisions

In order to facilitate this study, we will introduce the probabilistic (α^l, β^l)-lower and upper approximations for three-way decisions at *lth*-level to modify the original decision-theoretic rough set [27,28].

Definition 3. For a decision table S, a given decision class D^l_q, the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$, for a granular structure $A_i \subseteq C$, the (α^l, β^l)-lower approximation $\underline{apr}^{(\alpha^l,\beta^l)}_{\pi_{A_i}}(D^l_q)$ and (α^l, β^l)-upper approximation $\overline{apr}^{(\alpha^l,\beta^l)}_{\pi_{A_i}}(D^l_q)$ are defined by

$$\begin{aligned} \underline{apr}^{(\alpha^l,\beta^l)}_{\pi_{A_i}}(D^l_q) &= \{x | p(D^l_q | [x]_{A_i}) \ge \alpha^l, x \in U^l\}, \\ \overline{apr}^{(\alpha^l,\beta^l)}_{\pi_{A_i}}(D^l_q) &= \{x | p(D^l_q | [x]_{A_i}) > \beta^l, x \in U^l\}. \end{aligned} \tag{15}$$

where $U^1 = U$, $U^{l+1} = BND^{(\alpha^l,\beta^l)}_{\pi_{A_i}}(D^l_q) = \overline{apr}^{(\alpha^l,\beta^l)}_{\pi_{A_i}}(D^l_q)$ - $\underline{apr}^{(\alpha^l,\beta^l)}_{\pi_{A_i}}(D^l_q)$ is the gradually reduced universe, $[x]_{A_i}$ represents the equivalence class including x in the partition U^l/A_i, and D^l_q represents the equivalence class including x in the partition U^l/D^l.

The pair $<\underline{apr}^{(\alpha^l,\beta^l)}_{\pi_{A_i}}(D^l_q), \overline{apr}^{(\alpha^l,\beta^l)}_{\pi_{A_i}}(D^l_q)>$ is called the *lth*-level lower and upper approximations induced by A_i with respect to D^l_q in U^l. Thus, we can acquire the three probabilistic regions as follows.

$$\begin{aligned} POS^{(\alpha^l,\beta^l)}_{A_i}(D^l_q) &= \underline{apr}^{(\alpha^l,\beta^l)}_{\pi_{A_i}}(D^l_q), \\ &= \{x | p(D^l_q | [x]_{A_i}) \ge \alpha_l, x \in U^l\}; \end{aligned} \tag{16}$$

$$\begin{aligned} BND^{(\alpha^l,\beta^l)}_{A_i}(D^l_q) &= \overline{apr}^{(\alpha^l,\beta^l)}_{\pi_{A_i}}(D^l_q) - \underline{apr}^{(\alpha^l,\beta^l)}_{\pi_{A_i}}(D^l_q), \\ &= \{x | \beta^l < p(D^l_q | [x]_{A_i}) < \alpha^l, x \in U^l\}; \end{aligned} \tag{17}$$

$$\begin{aligned} NEG^{(\alpha^l,\beta^l)}_{A_i}(D^l_q) &= U^l - POS^{(\alpha^l,\beta^l)}_{A_i}(D^l_q) \cup BND^{(\alpha^l,\beta^l)}_{A_i}(D^l_q) \\ &= U^l - \{x | p(D^l_q | [x]_{A_i}) > \beta^l, x \in U^l\}, \\ &= \{x | p(D^l_q | [x]_{A_i}) \le \beta^l, x \in U^l\}. \end{aligned} \tag{18}$$

Similar to the classical decision-theoretic rough sets, when $\alpha^l > \beta^l$, we can obtain the decision rules tie-broke:

(S3WDP) If $P(D^l_q | [x]_{A_i}) \ge \alpha^l$, decide $POS^{(\alpha^l,\beta^l)}_{A_i}(D^l_q)$;

(S3WDN) If $P(D^l_q | [x]_{A_i}) \le \beta^l$, decide $NEG^{(\alpha^l,\beta^l)}_{A_i}(D^l_q)$;

(S3WDB) If $\beta^l < P(D^l_q | [x]_{A_i}) < \alpha^l$, decide $BND^{(\alpha^l,\beta^l)}_{A_i}(D^l_q)$.

In what follows, we construct an algorithm for computing the probabilistic regions of sequential three-way decisions under a granular structure as shown in Algorithm 1. The main idea of Algorithm 1 is that it first deletes those objects belonging to the probabilistic positive region and negative region under the first level of granular, and then obtains the updated universe $U_2 = BND_{A_i}^{(\alpha^1,\beta^1)}(D_q^1)$. In the next level of granular, for the updated universe U_2, delete the objects belonging to the probabilistic positive region and negative region, and update the universe again. This process is repeated until the updated universe becomes an empty set or no level of granular can be computed. It is easy to observe that the time complexity of Algorithm 1 is $O(l|D_q||U|^2)$.

Algorithm 1 Computing the probabilistic regions of sequential three-way decisions under a granular structure. ~
Input: An universal set of objects, U; a dynamic threshold sequence, $(\alpha,\beta)^l = \{(\alpha^1,\beta^1), (\alpha^2,\beta^2), \dots, (\alpha^l,\beta^l)\ \}$; a granular structure, A_i; a decision class, D_q.
Output: Three probabilistic regions, $POS_{A_i}^{(\alpha,\beta)^l}(D_q)$, $BND_{A_i}^{(\alpha,\beta)^l}(D_q)$, and $NEG_{A_i}^{(\alpha,\beta)^l}(D_q)$.

Step 1. $POS_{A_i}^{(\alpha,\beta)^l}(D_q) = \emptyset$, $BND_{A_i}^{(\alpha,\beta)^l}(D_q) = \emptyset$, $NEG_{A_i}^{(\alpha,\beta)^l}(D_q) = \emptyset$;
Step 2. $t = 1$, $U^t = U$;
Step 3. if $U^t = \emptyset$ or $t > l$, turn to Step 9;
Step 4. Compute $POS_{A_i}^{(\alpha^t,\beta^t)}(D_q^t) = \{x | p(D_q^t|[x]_{A_i}) \geq \alpha^t, x \in U^t\}$ and $NEG_{A_i}^{(\alpha^t,\beta^t)}(D_q^t) = \{x | p(D_q^t|[x]_{A_i}) \leq \beta^t, x \in U^t\}$;
Step 5. $POS_{A_i}^{(\alpha,\beta)^l}(D_q) = POS_{A_i}^{(\alpha,\beta)^l}(D_q) \cup POS_{A_i}^{(\alpha^t,\beta^t)}(D_q^t)$; $NEG_{A_i}^{(\alpha,\beta)^l}(D_q) = NEG_{A_i}^{(\alpha,\beta)^l}(D_q) \cup NEG_{A_i}^{(\alpha^t,\beta^t)}(D_q^t)$;
Step 6. $U^{t+1} = \{x | \beta^t < p(D_q^t|[x]_{A_i}) < \alpha^t, x \in U^t\}$;
Step 7. $BND_{A_i}^{(\alpha,\beta)^t}(D_q) = U^{t+1}$;
Step 8. $t = t + 1$; turn to Step 3;
Step 9. Output $POS_{A_i}^{(\alpha,\beta)^l}(D_q)$, $BND_{A_i}^{(\alpha,\beta)^l}(D_q)$, $NEG_{A_i}^{(\alpha,\beta)^l}(D_q)$.

Example 1. Consider a universe set of objects $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, a partition $U/A_1 = \{\{x_1, x_6\}, \{x_2, x_3, x_4, x_{10}\}, \{x_5\}, \{x_7, x_8, x_9\}\}$, and a decision class $D_1 = \{x_1, x_3, x_4, x_6, x_7\}$. Let $(\alpha, \beta)^2 = \{(0.8, 0.4), (0.7, 0.5)\}$, we can obtain the following conditional probabilities under a granular structure A_1 are computed as.

$$p(D_1|[x_1]_{A_1}) = p(D_1|[x_6]_{A_1}) = 1;$$
$$p(D_1|[x_2]_{A_1}) = p(D_1|[x_3]_{A_1}) = p(D_1|[x_4]_{A_1}) = p(D_1|[x_{10}]_{A_1}) = 1/2;$$
$$p(D_1|[x_5]_{A_1}) = 0;$$
$$p(D_1|[x_7]_{A_1}) = p(D_1|[x_8]_{A_1}) = p(D_1|[x_9]_{A_1}) = 1/3;$$

(1) U^1 = U, $D_1^1 = \{x_1, x_3, x_4, x_6, x_7\}$. For the first level of granular, we can compute

$$POS_{A_1}^{(\alpha,\beta)^1}(D_1) = POS_{A_1}^{(\alpha^1,\beta^1)}(D_1^1) = \{x_1, x_6\};$$
$$BND_{A_1}^{(\alpha,\beta)^1}(D_1) = BND_{A_1}^{(\alpha^1,\beta^1)}(D_1^1) = \{x_2, x_3, x_4, x_{10}\};$$
$$NEG_{A_1}^{(\alpha,\beta)^1}(D_1) = NEG_{A_1}^{(\alpha^1,\beta^l)}(D_1^1) = \{x_5, x_7, x_8, x_9\};$$

Thus, we can have $POS_{A_1}^{(\alpha,\beta)^1}(D_1) = \{x_1, x_6\}$, $BND_{A_1}^{(\alpha,\beta)^1}(D_1) = \{x_2, x_3, x_4, x_{10}\}$ and $NEG_{A_1}^{(\alpha,\beta)^1}(D_1) = \{x_5, x_7, x_8, x_9\}$.

(2) Updating the reduced universe $U^2 = \{x_2, x_3, x_4, x_{10}\}$, and the decision class $D_1^2 = \{x_3, x_4, x_7\}$. For the second level of granular, we can compute
$POS_{A_1}^{(\alpha^2,\beta^2)}(D_1^2) = \emptyset$;
$BND_{A_1}^{(\alpha^2,\beta^2)}(D_1^2) = \emptyset$;
$NEG_{A_1}^{(\alpha^2,\beta^2)}(D_1^2) = \{x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}$;
Therefore, we can have $POS_{A_1}^{(\alpha,\beta)^2}(D_1) = POS_{A_1}^{(\alpha^1,\beta^1)}(D_1^1) \cup POS_{A_1}^{(\alpha^2,\beta^2)}(D_1^2) = \{x_1, x_6\}$, $NEG_{A_1}^{(\alpha,\beta)^2}(D_1) = NEG_{A_1}^{(\alpha^1,\beta^1)}(D_1^1) \cup NEG_{A_1}^{(\alpha^2,\beta^2)}(D_1^2) = \{x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}$, and $BND_{A_1}^{(\alpha,\beta)^2}(D_1) = \emptyset$.

3.2. A generalized model of multigranulation sequential three-way decisions

The multigranulation rough sets(MGRS) employ a family of indiscernibility relations to construct the lower and upper approximations. Given multiple granular structures $GS = \{A_1, A_2, \dots, A_m\ \}$ ($A_i \subseteq \mathbf{R}$), the multigranulation decision-theoretic rough sets mainly select a series of actions satisfying that the overall risk is as small as possible, thus one can easily decide an object which belongs to the probabilistic positive region, negative region or boundary region. In these models, there are two kinds of assumptions. One assumes that the values (α^l, β^l) of m granular structures are all equal each other, namely $\alpha_1^l = \alpha_2^l = \dots = \alpha_m^l = \alpha^l$ and $\beta_1^l = \beta_2^l = \dots = \beta_m^l = \beta^l$, and the other assumes that they are not equivalent, in which each granular structure has its independent loss or cost function itself. In this paper, we mainly study the former, and assume all the parameter values of each granular structure are the same. In subsequent sections, we always use $P(D_q^l|[x]_{A_i}) \geq \alpha^l$, $P(D_q^l|[x]_{A_i}) > \beta^l$ and $P(D_q^l|[x]_{A_i}) \leq \beta^l$ to denote $P(D_q^l|[x]_{A_i}) \geq \alpha_i^l$, $P(D_q^l|[x]_{A_i}) > \beta_i^l$ and $P(D_q^l|[x]_{A_i}) \leq \beta_i^l$, respectively.

Given a decision class D_q^l, the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$, for m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, the conditional probability of an object x is denoted as $P(D_q^l|[x]_{A_i})$ $(1 \leq i \leq m)$. Thus, we can construct the $lth-level$ lower and upper approximations $\underline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha^l,\beta^l)}}(D_q^l)$ and $\overline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha^l,\beta^l)}}(D_q^l)$ ($\triangle$ denotes a generalized aggregation strategy) under the dynamic threshold parameter sequence $(\alpha, \beta)^l$ as follows.

Definition 4. Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, and the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$, the lower and upper approximations of the multigranulation sequential three-way decisions are denoted by $\underline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha,\beta)^l}}(D_q)$ and $\overline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha,\beta)^l}}(D_q)$, respectively,

$$\begin{aligned} \underline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha,\beta)^l}(D_q)} &= \bigcup_{1\leq t\leq l} \underline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha^l,\beta^l)}(D_q^t)}; \\ \overline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha,\beta)^l}(D_q)} &= \bigcup_{1\leq t\leq l} \overline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha^l,\beta^l)}(D_q^t)}. \end{aligned} \tag{19}$$

where $U^1 = U$, $U^{t+1} = \overline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha^l,\beta^l)}}(D_q^t)$ - $\underline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha^l,\beta^l)}}(D_q^t)$ is the gradually reduced universe, $[x]_{A_i}$ $(1 \leq i \leq m)$ represents the equivalence class including x in the partition U^t/A_i, D_q^t represents the equivalence class including x in the partition U^t/D, and $P(D_q^t|[x]_{A_i})$ is the conditional probability of the equivalence class $[x]_{A_i}$ with respect to D_q^t.

Similar to the decision-theoretic rough sets, we further extend the probabilistic approximations and regions of a decision class D_q to a partition π_D. For simplicity, we assume that the same threshold parameters are used for all the decision classes.

Definition 5. Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$ and a decision partition $\pi_D = \{D_1, D_2, \ldots, D_k\}$, the lower and upper approximations of multigranulation sequential three-way decisions with respect to π_D are defined as

$$\begin{aligned} \underline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha,\beta)^l}}(\pi_D) &= (\underline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha,\beta)^l}}(D_1), \underline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha,\beta)^l}}(D_2), \ldots, \underline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha,\beta)^l}}(D_k)), \\ \overline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha,\beta)_l}}(\pi_D) &= (\overline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha,\beta)^l}}(D_1), \overline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha,\beta)^l}}(D_2), \ldots, \overline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha,\beta)^l}}(D_k)). \end{aligned} \tag{20}$$

According to the above definitions, the probabilistic positive, boundary and negative regions with respect to target decision partition π_D can be denoted as follows:

$$POS_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D) = \bigcup_{1\leq q\leq k} \underline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha,\beta)^l}(D_q)} \tag{21}$$

$$\begin{aligned} BND_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D) &= \bigcup_{1\leq q\leq k} BND^{\triangle,(\alpha^l,\beta^l)}(D_q) \\ &= \bigcup_{1\leq q\leq k} \left(\overline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha^l,\beta^l)}}(D_q^l) - \underline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha^l,\beta^l)}(D_q^l)} \right) \end{aligned} \tag{22}$$

$$NEG_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D) = U - POS^{\triangle,(\alpha,\beta)^l}(\pi_D) \cup BND^{\triangle,(\alpha,\beta)^l}(\pi_D) \tag{23}$$

As discussed above, we can conclude the monotonicity of a decision partition π_D as follows.

Proposition 1. *Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$, and a decision partition π_D, then*

(1) $POS_{GS}^{\triangle,(\alpha,\beta)^1}(\pi_D) \subseteq POS_{GS}^{\triangle,(\alpha,\beta)^2}(\pi_D) \subseteq \ldots \subseteq POS_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D)$;
(2) $NEG_{GS}^{\triangle,(\alpha,\beta)^1}(\pi_D) \subseteq NEG_{GS}^{\triangle,(\alpha,\beta)^2}(\pi_D) \subseteq \ldots \subseteq NEG_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D)$;
(3) $BND_{GS}^{\triangle,(\alpha,\beta)^i}(\pi_D) \supseteq BND_{GS}^{\triangle,(\alpha,\beta)^j}(\pi_D) (i \leq j)$.

Proposition 2. *Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, and the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$. Then*

(1) *If $0 \leq \beta^l \leq \beta^{l'} < \alpha^{l'} \leq \alpha^l \leq 1$, then*

$$\underline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha^l,\beta^l)}(D_q^l)} \subseteq \underline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha^{l'},\beta^{l'})}(D_q^l)}; \overline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha^l,\beta^l)}}(D_q^l) \supseteq \overline{\sum_{i=1}^m A_i{}^{\triangle,(\alpha^{l'},\beta^{l'})}}(D_q^l)$$

.

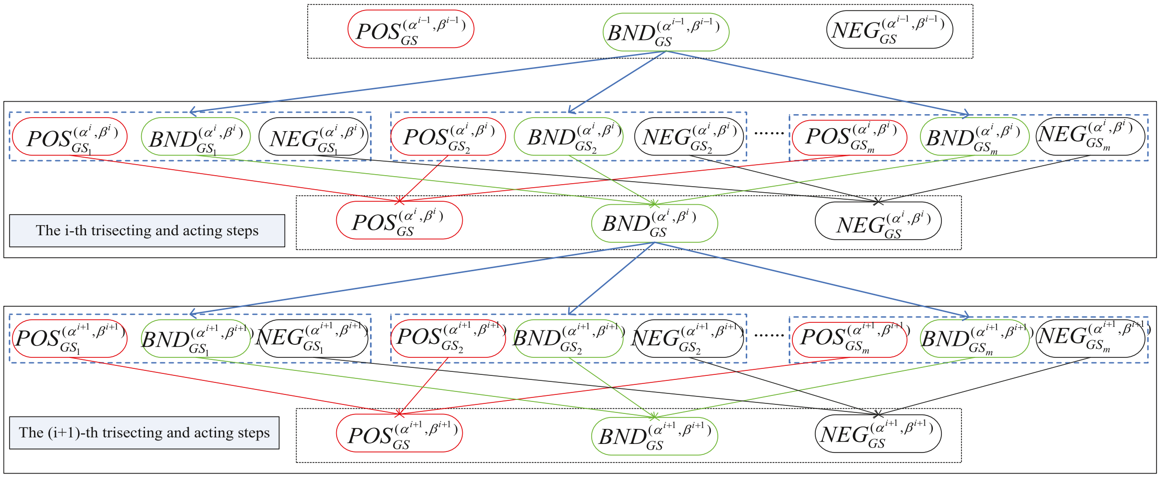


Fig. 1. The generalized model of multigranulation sequential three-way decisions.

(2) *If* $\forall D_q^{l'} \subseteq D_q^l \subseteq U_l$, *then*

$$\underline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha^l,\beta^l)}(D_q^{l'}) \subseteq \underline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha^l,\beta^l)}(D_q^l);\ \overline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha^l,\beta^l)}(D_q^{l'}) \subseteq \overline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha^l,\beta^l)}(D_q^l)$$

.

Proposition 2 shows that the bigger the value of α^l, the smaller the lower approximation, and the smaller the value of β^l, the bigger the upper approximation. In addition, the more the objects of a target concept, the larger the size of the lower and upper approximations.

In what follows, we design a generalized model of multigranulation sequential three-way decisions as shown in Fig. 1. In this model, the probabilistic boundary region under the coarse level of granular is regarded as the reduced universe under the fine level of granular. However, how to aggregate the probabilistic positive and negative regions induced from multiple granular structures is an important research issue.

Here we present an algorithm for computing the probabilistic regions of a generalized multigranulation sequential three-way decision model as shown in Algorithm 2. The main idea of Algorithm 2 is that it first deletes those objects belonging to the probabilistic positive region and negative region under the first level of granular, and then obtain the updated universe $U^2 = BND_{GS}^{\triangle,(\alpha^1,\beta^1)}(\pi_D)$. In the next level of granular, for the updated universe U^2, delete the objects belonging to the probabilistic positive region and negative region, and update the universe again. This process is repeated until the updated universe becomes an empty set or no level of granular can be computed. It is easy to observe that the time complexity of Algorithm 2 is $O(l\sum_{i=1}^{m} A_i|U|^2)$.

Algorithm 2 Computing the probabilistic regions under different multigranulation sequential three-way decisions. ~
Input: An universal set of objects, U; a dynamic threshold sequence $(\alpha,\beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$; m granular structures, $GS = \{A_1, A_2, \ldots, A_m\}$
Output: Three probabilistic regions, $POS_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D)$, $BND_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D)$, and $NEG_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D)$.

Step 1. $POS_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D) = \emptyset$, $BND_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D) = \emptyset$; $NEG_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D) = \emptyset$;
Step 2. $t = 1$, $U^t = U$;
Step 3. if $U^t = \emptyset$ or $t > l$, turn to Step 9;
Step 4. Compute $POS_{GS}^{\triangle,(\alpha^t,\beta^t)}(\pi_D) = \bigcup_{1\le q\le k} \underline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha^t,\beta^t)}(D_q^t)$ and $NEG_{GS}^{\triangle,(\alpha^t,\beta^t)}(\pi_D) = U^t - \bigcup_{1\le q\le k} \overline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha^t,\beta^t)}(D_q^t)$;
Step 5. $POS_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D) = POS_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D) \cup POS_{GS}^{\triangle,(\alpha^t,\beta^t)}(\pi_D)$; $NEG_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D) = NEG_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D) \cup NEG_{GS}^{\triangle,(\alpha^t,\beta^t)}(\pi_D)$;
Step 6. $U^{t+1} = \bigcup_{1\le q\le k} (\overline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha^t,\beta^t)}(D_q^t) - \underline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha^t,\beta^t)}(D_q^t))$;
Step 7. $BND_{GS}^{\triangle,(\alpha,\beta)^t}(\pi_D) = U^{t+1}$;
Step 8. $t = t + 1$; turn to Step 3;
Step 9. Output $POS_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D)$, $BND_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D)$, $NEG_{GS}^{\triangle,(\alpha,\beta)^l}(\pi_D)$.

3.3. Five typical models of multigranulation sequential three-way decisions using different aggregation strategies

In this subsection, we mainly focus on five typical models of multigranulation sequential three-way decisions. In order to facilitate this study, we define Pr_{max} and Pr_{min} to denote the maximum and minimum probability of D_q^l with respect to $[x]_{A_1}$ as follows.

$$\begin{cases} Pr_{max} = max\{P(D_q^l|[x]_{A_1}), P(D_q^l|[x]_{A_2}), \ldots, P(D_q^l|[x]_{A_m})\}, \\ Pr_{min} = min\{P(D_q^l|[x]_{A_1}), P(D_q^l|[x]_{A_2}), \ldots, P(D_q^l|[x]_{A_m})\}. \end{cases} \tag{24}$$

In order to apply to more multigranulation environments, we mainly discuss five different multigranulation sequential three-way decision models using different aggregation strategies. The WAMMS3WD model implements the idea of the weighted arithmetic mean to construct the lower and upper approximations. Besides the WAMMS3WD model, we adopt the Optimistic-Optimistic (OO), Pessimistic-Pessimistic (PP), Pessimistic-Optimistic (PO) and Optimistic-Pessimistic (OP) aggregation strategies to generate the OMS3WD, PMS3WD, POMS3WD and OPMS3WD models. Specifically, the OMS3WD model adopts the same aggregation strategies “seeking common ground while reserving differences (SCRD)” to deal with the lower and upper approximations, while the PMS3WD model employs “seeking common ground while eliminating differences (SCED)” to process those approximations. The two hybrid models—POMS3WD and OPMS3WD, use the different strategies “SCED” /“SCRD” for aggregating the lower/upper approximations, respectively.

3.3.1. Weighted arithmetic mean multigranulation sequential three-way decisions

In decision-making analysis, we often use the “average thought” to judge something. As illustrated in the mean multigranulation decision-theoretic rough set model [25], we can employ this idea to define a kind of mean multigranulation sequential three-way decisions for U^l. In this mean multigranulation decision-theoretic rough set, when the loss function is fixed, judging the conditional probability of an object x within a target concept in m granular structures can be computed by its mathematic expectation. That is to say,

$$E(P(X|x)) = (P(X|[x]_{A_1}) + P(X|[x]_{A_2}) + \ldots + P(X|[x]_{A_m}))/m. \tag{25}$$

The joint probability is estimated by the mean value of m conditional probabilities. Thus, the lower approximation contains those objects in which each object requires the corresponding mean value of m conditional probabilities satisfying the probability constraint ($\geq \alpha$) between its equivalence class and the approximate target, while its upper approximation collects those objects in which each object requires the corresponding mean value of m conditional probabilities satisfying the probability constraint ($> \beta$) between its equivalence class and the approximate target. We here usethe idea of the weighted arithmetic mean(for short WAM) to construct a weighted arithmetic mean multigranulation model of sequential three-way decisions. Thus, we use “WAMP”, “WAMB” and “WAMN” to denote the three kinds of decision rules.

Definition 6. Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$, and a weighting vector $\omega = \{w_1, w_2, \ldots, w_m\}$ which satisfies that $w_i \in [0, 1]$ and $\sum_{i=1}^m w_i = 1$, the lower and upper approximations of the weighted arithmetic mean multigranulation sequential three-way decisions are defined by $\underline{\sum_{i=1}^m A_i^{WAM,(\alpha^l,\beta^l)}}(D_q^l)$ and $\overline{\sum_{i=1}^m A_i^{WAM,(\alpha^l,\beta^l)}}(D_q^l)$,

$$\begin{aligned} \underline{\sum_{i=1}^m A_i^{WAM,(\alpha^l,\beta^l)}}(D_q^l) &= \{x | w_1 P(D_q^l|[x]_{A_1}) + w_2 P(D_q^l|[x]_{A_2}) + \ldots + w_m P(D_q^l|[x]_{A_m}) \geq \alpha^l, x \in U^l\} \\ &= \{x | \textstyle\sum_{i=1}^m w_i P(D_q^l|[x]_{A_i}) \geq \alpha^l, x \in U^l\}; \\ \overline{\sum_{i=1}^m A_i^{WAM,(\alpha^l,\beta^l)}}(D_q^l) &= \{x | w_1 P(D_q^l|[x]_{A_1}) + w_2 P(D_q^l|[x]_{A_2}) + \ldots + w_m P(D_q^l|[x]_{A_m}) > \beta^l, x \in U^l\} \\ &= U^l - \{x | \textstyle\sum_{i=1}^m w_i P(D_q^l|[x]_{A_i}) \leq \beta^l, x \in U^l\}. \end{aligned} \tag{26}$$

where $U^1 = U$, $U^{l+1} = \overline{\sum_{i=1}^m A_i^{WAM,(\alpha^l,\beta^l)}}(D_q^l)$ - $\underline{\sum_{i=1}^m A_i^{WAM,(\alpha^l,\beta^l)}}(D_q^l)$ is the gradually reduced universe, $[x]_{A_i}$ $(1 \leq i \leq m)$ represents the equivalence class including x in the partition U^l/A_i and $P(D_q^l|[x]_{A_i})$ is the conditional probability of the equivalence class $[x]_{A_i}$ with respect to D_q^l.

The pair $< \underline{\sum_{i=1}^m A_i^{WAM,(\alpha^l,\beta^l)}}(D_q^l), \overline{\sum_{i=1}^m A_i^{WAM,(\alpha^l,\beta^l)}}(D_q^l) >$ is called the lth-level weighted arithmetic mean multigranulation sequential three-way decisions. By the lower and upper approximations $\underline{\sum_{i=1}^m A_i^{WAM,(\alpha^l,\beta^l)}}(D_q^l)$ and

$\overline{\sum_{i=1}^{m} A_i}^{WAM,(\alpha^l,\beta^l)}(D_q^l)$, the weighted arithmetic mean multigranulation boundary region of D_q^l is

$$BND^{WAM,(\alpha^l,\beta^l)}(D_q^l) = \overline{\sum_{i=1}^{m} A_i}^{WAM,(\alpha^l,\beta^l)}(D_q^l) - \underline{\sum_{i=1}^{m} A_i}^{WAM,(\alpha^l,\beta^l)}(D_q^l). \tag{27}$$

Here we denote the lower and upper approximations under the mean multigranulation rough sets [25] as $\underline{\sum_{i=1}^{m} A_i}^{M,(\alpha^l,\beta^l)}(D_q^l)$ and $\overline{\sum_{i=1}^{m} A_i}^{M,(\alpha^l,\beta^l)}(D_q^l)$. According to the definitions of the weighted arithmetic mean multigranulation sequential three-way decisions, we can have the following propositions.

Proposition 3. *Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$, if for all $w_i = \frac{1}{m}$ ($i = 1, \ldots, m$), then*

(1) $\underline{\sum_{i=1}^{m} A_i}^{WAM,(\alpha^l,\beta^l)}(D_q^l) = \underline{\sum_{i=1}^{m} A_i}^{M,(\alpha^l,\beta^l)}(D_q^l)$;

(2) $\overline{\sum_{i=1}^{m} A_i}^{WAM,(\alpha^l,\beta^l)}(D_q^l) = \overline{\sum_{i=1}^{m} A_i}^{M,(\alpha^l,\beta^l)}(D_q^l)$.

Similar to the classical three-way decisions, when $\alpha^l > \beta^l$, we can obtain the decision rules tie-broke:

(WAMP) If $w_1P(D_q^l|[x]_{A_1}) + w_2P(D_q^l|[x]_{A_2}) + \ldots + w_mP(D_q^l|[x]_{A_m}) \geq \alpha^l$, decide $POS^{WAM,(\alpha^l,\beta^l)}(D_q^l)$;

(WAMN) If $w_1P(D_q^l|[x]_{A_1}) + w_2P(D_q^l|[x]_{A_2}) + \ldots + w_mP(D_q^l|[x]_{A_m}) \leq \beta^l$, decide $NEG^{WAM,(\alpha^l,\beta^l)}(D_q^l)$;

(WAMB) If $\beta^l < w_1(P(D_q^l|[x]_{A_1}) + w_2P(D_q^l|[x]_{A_2}) + \ldots + w_mP(D_q^l|[x]_{A_m}) < \alpha^l$, decide $BND^{WAM,(\alpha^l,\beta^l)}(D_q^l)$.

3.3.2. Optimistic multigranulation sequential three-way decisions

In decision making analysis, we sometimes use the “Seeking common ground while reserving differences(SCRD)” strategy [25] to evaluate the projects or professional titles. In order to facilitate this study, we suppose the parameter values (α^l, β^l) of each granular structure are equal each other. Here we define the lower and upper approximations of the optimistic multigranulation sequential three-way decisions at *lth*-level.

Definition 7. Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, and the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$, the lower and upper approximations of the optimistic multigranulation sequential three-way decisions are denoted by $\underline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l)$ and $\overline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l)$,

$$\begin{aligned}
\underline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l) &= \{x|P(D_q^l|[x]_{A_1}) \geq \alpha^l \vee P(D_q^l|[x]_{A_2}) \geq \alpha^l \vee \ldots \vee P(D_q^l|[x]_{A_m}) \geq \alpha^l, x \in U^l\};\\
\overline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l) &= \{x|P(D_q^l|[x]_{A_1}) > \beta^l \vee P(D_q^l|[x]_{A_2}) > \beta^l \vee \ldots \vee P(D_q^l|[x]_{A_m}) > \beta^l, x \in U^l\},\\
&= U^l - \{x|P(D_q^l|[x]_{A_1}) \leq \beta^l \wedge P(D_q^l|[x]_{A_2}) \leq \beta^l \wedge \ldots \wedge P(D_q^l|[x]_{A_m}) \leq \beta^l, x \in U^l\}.
\end{aligned} \tag{28}$$

where $U^1 = U$, $U^{l+1} = \overline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l)$ - $\underline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l)$ is the gradually reduced universe, $[x]_{A_i}$ $(1 \leq i \leq m)$ represents the equivalence class including x in the partition U^l/A_i and $P(D_q^l|[x]_{A_i})$ is the conditional probability of the equivalence class $[x]_{A_i}$ with respect to D_q^l.

The pair $< \underline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l), \overline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l) >$ is called the *lth*-level optimistic multigranulation sequential three-way decisions. According to the lower and upper approximations $\underline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l)$ and $\overline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l)$, the optimistic multigranulation boundary region of D_q^l is

$$BND^{O,(\alpha^l,\beta^l)}(D_q^l) = \overline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l) - \underline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l). \tag{29}$$

According to these definitions of the optimistic multigranulation sequential three-way decisions, one can conclude these propositions as follows.

Proposition 4. *Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, and the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$. Then, the following properties hold.*

(1)

$$\underline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l) = \bigcup_{1\le i\le m} \underline{apr}_{\pi_{A_i}}^{(\alpha^l,\beta^l)}(D_q^l);$$

(2)

$$\overline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l) = \bigcup_{1\le i\le m} \overline{apr}_{\pi_{A_i}}^{(\alpha^l,\beta^l)}(D_q^l).$$

When $\alpha^l \in (\beta^l, Pr_{max}]$, we can decide $x \in POS^{O,(\alpha^l,\beta^l)}(D_q^l)$. As well, if $\beta^l \in [Pr_{max}, \alpha^l)$, we can decide $x \in NEG^{O,(\alpha^l,\beta^l)}(D_q^l)$. Similar to the classical three-way decisions, we can obtain the decision rules tie-broke:

(OP) If $\exists i \in \{1, 2, \ldots, m\}$ such that $P(D_q^l|[x]_{A_i}) \ge \alpha^l$, decide $POS^{O,(\alpha^l,\beta^l)}(D_q^l)$;

(ON) If $\forall i \in \{1, 2, \ldots, m\}$ such that $P(D_q^l|[x]_{A_i}) \le \beta^l$, decide $NEG^{O,(\alpha^l,\beta^l)}(D_q^l)$;

(OB) Otherwise, decide $BND^{O,(\alpha^l,\beta^l)}(D_q^l)$.

3.3.3. Pessimistic multigranulation sequential three-way decisions

As we all know, "Seeking common ground while eliminating differences (SCED)" is one of usual decision strategies for decision making [25]. This strategy argues that one reserves common decisions while deleting inconsistent decisions, which can be seen as a conservative decision strategy. Based on this idea, we define the lower and upper approximations of the pessimistic multigranulation sequential three-way decisions at *lth*-level as follows.

Definition 8. Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, and the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$, the lower and upper approximations of the pessimistic multigranulation sequential three-way decisions are denoted by $\underline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l)$ and $\overline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l)$, respectively,

$$\begin{aligned}
\underline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l) &= \{x|P(D_q^l|[x]_{A_1}) \ge \alpha^l \wedge P(D_q^l|[x]_{A_2}) \ge \alpha^l \wedge \ldots \wedge P(D_q^l|[x]_{A_m}) \ge \alpha^l, x \in U^l\};\\
\overline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l) &= \{x|P(D_q^l|[x]_{A_1}) > \beta^l \wedge P(D_q^l|[x]_{A_2}) > \beta^l \wedge \ldots \wedge P(D_q^l|[x]_{A_m}) > \beta^l, x \in U^l\},\\
&= U^l - \{x|P(D_q^l|[x]_{A_1}) \le \beta^l \vee P(D_q^l|[x]_{A_2}) \le \beta^l \vee \ldots \vee P(D_q^l|[x]_{A_m}) \le \beta^l, x \in U^l\}.
\end{aligned} \tag{30}$$

where $U^1 = U$, $U^{l+1} = \overline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l)$ - $\underline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l)$ is the gradually reduced universe, $[x]_{A_i}$ $(1 \le i \le m)$ represents the equivalence class including x in the partition U^l/A_i and $P(D_q^l|[x]_{A_i})$ is the conditional probability of the equivalence class $[x]_{A_i}$ with respect to D_q^l.

The pair $< \underline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l), \overline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l) >$ is called the *lth*-level pessimistic multigranulation sequential three-way decisions. By the lower and upper approximations $\underline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l)$ and $\overline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l)$, the pessimistic multigranulation boundary region of D_q^l is

$$BND^{P,(\alpha^l,\beta^l)}(D_q^l) = \overline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l) - \underline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l). \tag{31}$$

According to these definitions of the pessimistic multigranulation sequential three-way decisions, we have the propositions as follows.

Proposition 5. *Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, and the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$. Then, the following properties hold.*

(1)

$$\underline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l) = \bigcap_{1\le i\le m} \underline{apr}_{\pi_{A_i}}^{(\alpha^l,\beta^l)}(D_q^l);$$

(2)

$$\overline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l) = \bigcap_{1\le i\le m} \overline{apr}_{\pi_{A_i}}^{(\alpha^l,\beta^l)}(D_q^l).$$

When $\alpha^l \in (\beta^l, Pr_{min}]$, we can decide $x \in POS^{P,(\alpha^l,\beta^l)}(D_q^l)$. As well, if $\beta^l \in [Pr_{min}, \alpha^l)$, we can decide $x \in NEG^{P,(\alpha^l,\beta^l)}(D_q^l)$. Similar to the classical three-way decisions, when $\alpha^l > \beta^l$, we can obtain the decision rules tie-broke:

(PP) If $\forall i \in \{1, 2, \ldots, m\}$ such that $P(D_q^l|[x]_{A_i}) \geq \alpha^l$, decide $POS^{P,(\alpha^l,\beta^l)}(D_q^l)$;

(PN) If $\exists i \in \{1, 2, \ldots, m\}$ such that $P(D_q^l|[x]_{A_i}) \leq \beta^l$, decide $NEG^{P,(\alpha^l,\beta^l)}(D_q^l)$;

(PB) Otherwise, decide $BND^{P,(\alpha^l,\beta^l)}(D_q^l)$.

3.3.4. Pessimistic-Optimistic multigranulation sequential three-way decisions

For the optimistic and pessimistic three-way decisions, they use the same strategies to aggregate the lower and upper approximations. Besides, we can adopt the conservative strategy for the lower approximation and use the aggressive strategy for the upper approximations [48]. Thus, we can define the pessimistic-optimistic multigranulation sequential three-way decisions as follows.

Definition 9. Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, and the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$, the lower and upper approximations of the pessimistic-optimistic multigranulation sequential three-way decisions are denoted by $\underline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l)$ and $\overline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l)$, respectively,

$$\begin{aligned} \underline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l) &= \{x|P(D_q^l|[x]_{A_1}) \geq \alpha^l \wedge P(D_q^l|[x]_{A_2}) \geq \alpha^l \wedge \ldots \wedge P(D_q^l|[x]_{A_m}) \geq \alpha^l, x \in U^l\}; \\ \overline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l) &= \{x|P(D_q^l|[x]_{A_1}) > \beta^l \vee P(D_q^l|[x]_{A_2}) > \beta^l \vee \ldots \vee P(D_q^l|[x]_{A_m}) > \beta^l, x \in U^l\} \\ &= U^l - \{x|P(D_q^l|[x]_{A_1}) \leq \beta^l \wedge P(D_q^l|[x]_{A_2}) \leq \beta^l \wedge \ldots \wedge P(D_q^l|[x]_{A_m}) \leq \beta^l, x \in U^l\}. \end{aligned} \tag{32}$$

where $U^1 = U$, $U^{l+1} = \overline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l)$ - $\underline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l)$ is the gradually reduced universe, $[x]_{A_i}$ $(1 \leq i \leq m)$ represents the equivalence class including x in the partition U^l/A_i and $P(D_q^l|[x]_{A_i})$ is the conditional probability of the equivalence class $[x]_{A_i}$ with respect to D_q^l.

The pair $< \underline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l), \overline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l) >$ is called the *lth*-level pessimistic-optimistic multigranulation sequential three-way decisions. By the lower and upper approximations $\underline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l)$ and $\overline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l)$, the pessimistic-optimistic multigranulation boundary region of D_q^l is

$$BND^{PO,(\alpha^l,\beta^l)}(D_q^l) = \overline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l) - \underline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l). \tag{33}$$

According to these definitions, we have the propositions as follows.

Proposition 6. *Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, and the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$. Then, the following properties hold.*

(1)

$$\underline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l) = \bigcap_{1 \leq i \leq m} \underline{apr}_{\pi_{A_i}}^{(\alpha^l,\beta^l)}(D_q^l);$$

(2)

$$\overline{\sum_{i=1}^m A_i^{PO,(\alpha^l,\beta^l)}}(D_q^l) = \bigcup_{1 \leq i \leq m} \overline{apr}_{\pi_{A_i}}^{(\alpha^l,\beta^l)}(D_q^l).$$

When $\alpha^l \in (\beta^l, Pr_{min}]$, we can decide $x \in POS^{PO,(\alpha^l,\beta^l)}(D_q^l)$. As well, if $\beta^l \in [Pr_{max}, \alpha^l)$, we can decide $x \in NEG^{PO,(\alpha^l,\beta^l)}(D_q^l)$. Similar to the classical three-way decisions, when $\alpha^l > \beta^l$, we can obtain the decision rules tie-broke:

(POP) If $\forall i \in \{1, 2, \ldots, m\}$ such that $P(D_q^l|[x]_{A_i}) \geq \alpha^l$, decide $POS^{PO,(\alpha^l,\beta^l)}(D_q^l)$;

(PON) If $\forall i \in \{1, 2, \ldots, m\}$ such that $P(D_q^l|[x]_{A_i}) \leq \beta^l$, decide $NEG^{PO,(\alpha^l,\beta^l)}(D_q^l)$;

(POB) Otherwise, decide $BND^{PO,(\alpha^l,\beta^l)}(D_q^l)$.

3.3.5. Optimistic-pessimistic multigranulation sequential three-way decisions

As illustrated in [48], we seem to use the aggregative strategy and the conservative strategy for the lower approximation and the upper approximation, respectively. Here we define optimistic-pessimistic multigranulation sequential three-way decisions.

Definition 10. Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, and the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$, the lower and upper approximations of the optimistic-pessimistic multigranulation sequential three-way decisions are denoted by $\underline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l)$ and $\overline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l)$, respectively,

$$\begin{aligned}\underline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l) &= \{x|P(D_q^l|[x]_{A_1}) \geq \alpha^l \vee P(D_q^l|[x]_{A_2}) \geq \alpha^l \vee \ldots \vee P(D_q^l|[x]_{A_m}) \geq \alpha^l, x \in U^l\};\\ \overline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l) &= \{x|P(D_q^l|[x]_{A_1}) > \beta^l \wedge P(D_q^l|[x]_{A_2}) > \beta^l \wedge \ldots \wedge P(D_q^l|[x]_{A_m}) > \beta^l, x \in U^l\}\\ &= U^l - \{x|P(D_q^l|[x]_{A_1}) \leq \beta^l \vee P(D_q^l|[x]_{A_2}) \leq \beta^l \vee \ldots \vee P(D_q^l|[x]_{A_m}) \leq \beta^l, x \in U^l\}.\end{aligned} \tag{34}$$

where $U^1 = U$, $U^{l+1} = \overline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l)$ - $\underline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l)$ is the gradually reduced universe, $[x]_{A_i}$ $(1 \leq i \leq m)$ represents the equivalence class including x in the partition U^l/A_i and $P(D_q^l|[x]_{A_i})$ is the conditional probability of the equivalence class $[x]_{A_i}$ with respect to D_q^l.

The pair $< \underline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l), \overline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l) >$ is called the *lth*-level optimistic-pessimistic multigranulation sequential three-way decisions. By the lower and upper approximations $\underline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l)$ and $\overline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l)$, the optimistic-pessimistic multigranulation boundary region of D_q^l is

$$BND^{OP,(\alpha^l,\beta^l)}(D_q^l) = \overline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l) - \underline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l). \tag{35}$$

According to these definitions, we have the propositions as follows.

Proposition 7. *Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, and the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$. Then, the following properties hold.*

(1)
$$\underline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l) = \bigcup_{1\leq i\leq m} \underline{apr}_{\pi_{A_i}}^{(\alpha^l,\beta^l)}(D_q^l);$$

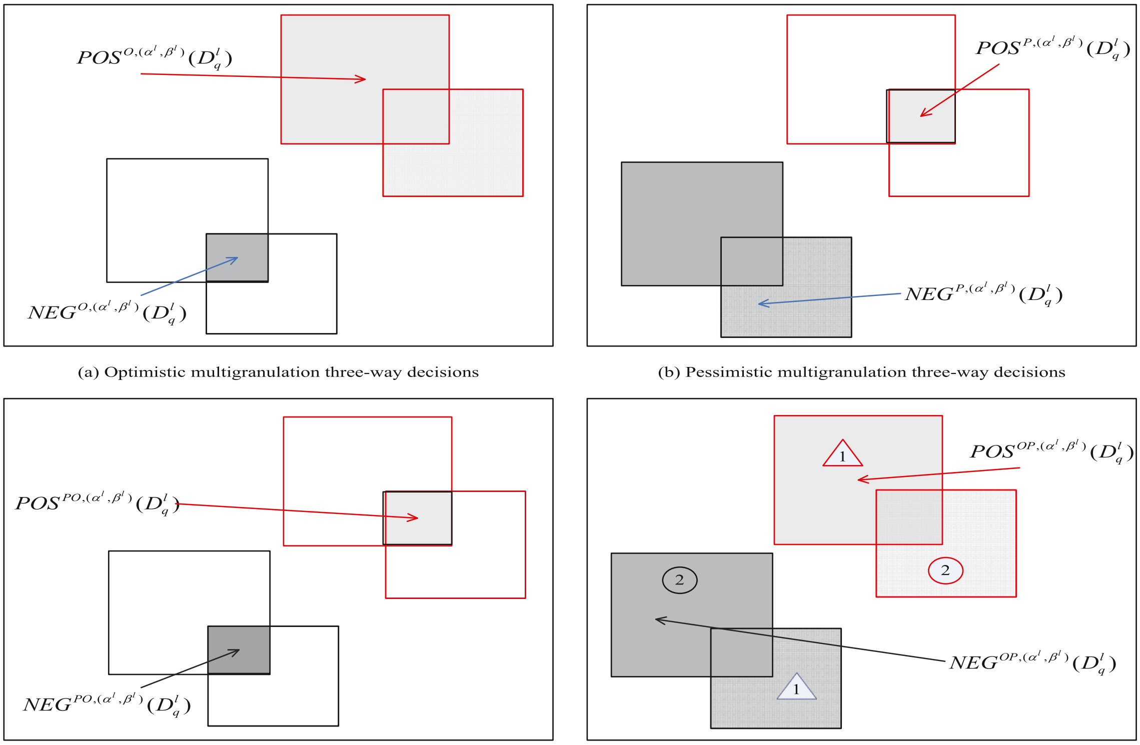
(2)
$$\overline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l) = \bigcap_{1\leq i\leq m} \overline{apr}_{\pi_{A_i}}^{(\alpha^l,\beta^l)}(D_q^l).$$

Similar to the classical three-way decisions, when $\alpha^l > \beta^l$, we can obtain the decision rules tie-broke:
(OPP) If $\exists i \in \{1, 2, \ldots, m\}$ such that $P(D_q^l|[x]_{A_i}) \geq \alpha^l$, decide $POS^{OP,(\alpha^l,\beta^l)}(D_q^l)$;
(OPN) If $\exists j \in \{1, 2, \ldots, m\}$ such that $P(D_q^l|[x]_{A_j}) \leq \beta^l$, decide $NEG^{OP,(\alpha^l,\beta^l)}(D_q^l)$;
(OPB) Otherwise, decide $BND^{OP,(\alpha^l,\beta^l)}(D_q^l)$.

Remark 1. This optimistic-pessimistic multigranulation sequential three-way decisions may be **wrong** in some cases. For example, suppose an object x satisfying $P(D_q^l|[x]_{A_i}) \geq \alpha^l$ for some $A_i \subseteq C$ and $P(D_q^l|[x]_{A_j}) \leq \beta^l$ for some $A_j \subseteq C$, we cannot judge that $x \in POS^{OP,(\alpha^l,\beta^l)}(D_q^l)$ or $x \in NEG^{OP,(\alpha^l,\beta^l)}(D_q^l)$. If $\alpha^l \leq Pr_{max}$ and $\beta^l < Pr_{min}$, we can decide $x \in POS^{OP,(\alpha^l,\beta^l)}(D_q^l)$. If $\alpha^l > Pr_{max}$ and $\beta^l \geq Pr_{min}$, we can decide $x \in NEG^{OP,(\alpha^l,\beta^l)}(D_q^l)$. However, these two conditions do not always hold. In addition, we cannot guarantee that $\overline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l) \supseteq \underline{\sum_{i=1}^m A_i}^{OP,(\alpha^l,\beta^l)}(D_q^l)$ holds. Fortunately, this model is right in the multigranulation Pawlak rough set models. Fig. 2 illustrates the four kind models of multigranulation three-way decisions using optimistic and/or pessimistic strategies. In Fig. 2(d), we can find objects 1 and 2 may belong to the different probabilistic positive regions, meanwhile they also belong to the negative regions.

Example 2. (Continued with Example 1) Consider $U/A_1 = \{\{x_1, x_6\}, \{x_2, x_3, x_4, x_{10}\}, \{x_5\}, \{x_7, x_8, x_9\}\}$, $U/A_2 = \{\{x_1, x_{10}\}, \{x_2, x_3, x_4, x_8\}, \{x_5\}, \{x_6, x_7, x_9\}\}$, $D_1 = \{x_1, x_3, x_4, x_6, x_7\}$, $w_1 = \frac{1}{2}$, $w_2 = \frac{1}{2}$, and $(\alpha, \beta) = (0.75, 0.45)$ and $m = 2$. In what follows, we compute the conditional probabilities with respect to different partitions under two granular structures.

(1) For a granular structure A_1, the conditional probabilities with respect to π_{A_1} are computed as
$p(D_1|[x_1]_{A_1}) = p(D_1|[x_6]_{A_1}) = 1$;
$p(D_1|[x_5]_{A_1}) = 0$;
$p(D_1|[x_2]_{A_1}) = p(D_1|[x_3]_{A_1}) = p(D_1|[x_4]_{A_1}) = p(D_1|[x_{10}]_{A_1}) = 1/2$;
$p(D_1|[x_7]_{A_1}) = p(D_1|[x_8]_{A_1}) = p(D_1|[x_9]_{A_1}) = 1/3$;



(a) Optimistic multigranulation three-way decisions

(b) Pessimistic multigranulation three-way decisions

(c) Pessimistic-Optimistic multigranulation three-way decisions

(d) Optimistic-Pessimistic multigranulation three-way decisions (✓)

Fig. 2. Four multigranulation models of S3WD using optimistic and/or pessimistic strategies.

Table 1
Comparisons of the probabilistic regions of five MS3WD models under (α, β)=(0.75,0.45).

No.	$\triangle$	$\underline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha,\beta)}(D_1)$	$\overline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha,\beta)}(D_1)$	$POS^{\triangle,(\alpha,\beta)}(D_1)$	$BND^{\triangle,(\alpha,\beta)}(D_1)$	$NEG^{\triangle,(\alpha,\beta)}(D_1)$
1	WAM(✓)	$\{x_1, x_6\}$	$\{x_1, x_2, x_3, x_4, x_6, x_7, x_9, x_{10}\}$	$\{x_1, x_6\}$	$\{x_2, x_3, x_4, x_7, x_9, x_{10}\}$	$\{x_5, x_8\}$
2	O(✓)	$\{x_1, x_6\}$	$\{x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_9, x_{10}\}$	$\{x_1, x_6\}$	$\{x_2, x_3, x_4, x_7, x_8, x_9, x_{10}\}$	$\{x_5\}$
3	P(✓)	$\{\emptyset\}$	$\{x_1, x_2, x_3, x_4, x_6, x_{10}\}$	$\{\emptyset\}$	$\{x_1, x_2, x_3, x_4, x_6, x_{10}\}$	$\{x_5, x_7, x_8, x_9\}$
4	PO(✓)	$\{\emptyset\}$	$\{x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_9, x_{10}\}$	$\{\emptyset\}$	$\{x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_9, x_{10}\}$	$\{x_5\}$
5	OP(✓)	$\{x_1, x_6\}$	$\{x_1, x_2, x_3, x_4, x_6, x_{10}\}$	$\{x_1, x_6\}$	$\{x_2, x_3, x_4, x_{10}\}$	$\{x_5, x_7, x_8, x_9\}$

(2) For a granular structure A_2, the conditional probabilities with respect to π_{A_2} are computed as
$p(D_1|[x_1]_{A_2}) = p(D_1|[x_{10}]_{A_2}) = 1/2$;
$p(D_1|[x_2]_{A_2}) = p(D_1|[x_3]_{A_2}) = p(D_1|[x_4]_{A_2}) = p(D_1|[x_8]_{A_2}) = 1/2$;
$p(D_1|[x_5]_{A_2}) = 0$;
$p(D_1|[x_6]_{A_2}) = p(D_1|[x_7]_{A_2}) = p(D_1|[x_9]_{A_2}) = 2/3$;

According to these above definitions, we can have the three probabilistic regions of five MS3WD models as shown in Table 1. □

However, when we set $(\alpha, \beta) = (0.75, 0.5)$, the OPMS3WD model easily results in inaccurate results. As illustrated in Table 2, one can not decide whether x_1 belongs to the probabilistic positive region or the probabilistic negative region for OPMS3WD model.

Remark 2. For OPMS3WD model, different thresholds may lead to the ambiguous results. This conflict may be solved through the consultation or the voting mechanism. This model may be applied in these cases such as the controversial project and postgraduate recruitment, and so on. For the convenience, we mainly discuss the first four types of multigranulation sequential three-way decision models in the remainder of this paper.

Table 2
Comparisons of the probabilistic regions of five MS3WD models under (α, β)=(0.75,0.5).

No.	$\triangle$	$\underline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha,\beta)}(D_1)$	$\overline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha,\beta)}(D_1)$	$POS^{\triangle,(\alpha,\beta)}(D_1)$	$BND^{\triangle,(\alpha,\beta)}(D_1)$	$NEG^{\triangle,(\alpha,\beta)}(D_1)$
1	$WAM(\checkmark)$	$\{x_1, x_6\}$	$\{x_1, x_6\}$	$\{x_1, x_6\}$	$\emptyset$	$\{x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}$
2	$O(\checkmark)$	$\{x_1, x_6\}$	$\{x_1, x_6, x_7, x_9\}$	$\{x_1, x_6\}$	$\{x_7, x_9\}$	$\{x_2, x_3, x_4, x_5, x_8, x_{10}\}$
3	$P(\checkmark)$	$\{\emptyset\}$	$\{x_6\}$	$\{\emptyset\}$	$\{x_6\}$	$\{x_1, x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}$
4	$PO(\checkmark)$	$\{\emptyset\}$	$\{x_1, x_6, x_7, x_9\}$	$\{\emptyset\}$	$\{x_1, x_6, x_7, x_9\}$	$\{x_2, x_3, x_4, x_5, x_8, x_{10}\}$
5	$OP(\times)$	$\{x_1, x_6\}$	$\{x_6\}$	$\{x_1, x_6\}$	$\emptyset$	$\{x_1, x_2, x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}$

3.3.6. Relationship among the first four types of multigranulation sequential three-way decisions

In this section, we investigate the relationships among the weighted arithmetic mean multigranulation sequential three-way decisions, the optimistic multigranulation sequential three-way decisions, the pessimistic multigranulation sequential three-way decisions, and the pessimistic-optimistic multigranulation sequential three-way decisions.

Theorem 1. *Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, and the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$ where $0 \leq \beta^l < \alpha^l \leq 1$, we have*

(1) *if*

$$\alpha^l = 1, \underline{\sum_{i=1}^{m} A_i}^{WAM,(\alpha^l,\beta^l)}(D_q^l) = \underline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l) = \underline{\sum_{i=1}^{m} A_i}^{PO,(\alpha^l,\beta^l)}(D_q^l) \subseteq \underline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l);$$

(2) *if*

$$\beta^l = 0, \overline{\sum_{i=1}^{m} A_i}^{WAM,(\alpha^l,\beta^l)}(D_q^l) = \overline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l) = \overline{\sum_{i=1}^{m} A_i}^{PO,(\alpha^l,\beta^l)}(D_q^l) \supseteq \overline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l).$$

Theorem 2. *Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, and the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$ where $0 \leq \beta^l \leq \alpha^l \leq 1$, we have*

(1)

$$\underline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l) = \underline{\sum_{i=1}^{m} A_i}^{PO,(\alpha^l,\beta^l)}(D_q^l) \subseteq \underline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l);$$

(2)

$$\overline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l) \subseteq \overline{\sum_{i=1}^{m} A_i}^{PO,(\alpha^l,\beta^l)}(D_q^l) = \overline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l).$$

Theorem 3. *Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, a decision class D_q^l, and the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$ where $0 \leq \beta^l < \alpha^l \leq 1$, we have*

(1)

$$\underline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l) \subseteq \underline{\sum_{i=1}^{m} A_i}^{WAM,(\alpha^l,\beta^l)}(D_q^l) \subseteq \underline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l);$$

(2)

$$\overline{\sum_{i=1}^{m} A_i}^{P,(\alpha^l,\beta^l)}(D_q^l) \subseteq \overline{\sum_{i=1}^{m} A_i}^{WAM,(\alpha^l,\beta^l)}(D_q^l) \subseteq \overline{\sum_{i=1}^{m} A_i}^{O,(\alpha^l,\beta^l)}(D_q^l).$$

Theorem 4. *Given m granular structures $GS = \{A_1, A_2, \ldots, A_m\}$, the dynamic threshold parameter sequence $(\alpha, \beta)^l = \{(\alpha^1, \beta^1), (\alpha^2, \beta^2), \ldots, (\alpha^l, \beta^l)\}$, and a decision partition π_D, then*

(1)

$$POS_{GS}^{P,(\alpha,\beta)^l}(\pi_D) = POS_{GS}^{PO,(\alpha,\beta)^l}(\pi_D) \subseteq POS_{GS}^{WAM,(\alpha,\beta)^l}(\pi_D) \subseteq POS_{GS}^{O,(\alpha,\beta)^l}(\pi_D);$$

(2)

$$NEG_{GS}^{PO,(\alpha,\beta)^l}(\pi_D) = NEG_{GS}^{O,(\alpha,\beta)^l}(\pi_D) \subseteq NEG_{GS}^{WAM,(\alpha,\beta)^l}(\pi_D) \subseteq NEG_{GS}^{P,(\alpha,\beta)^l}(\pi_D);$$

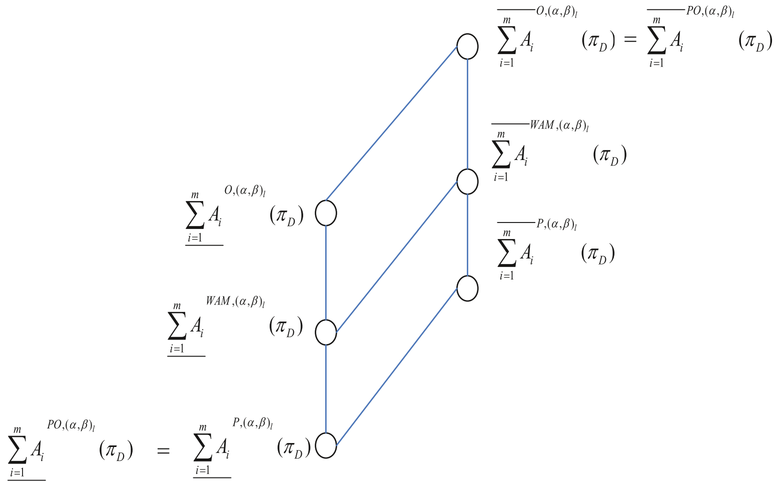


Fig. 3. Relationships of lower and upper approximations under four kinds of multigranulation sequential three-way decisions.

Table 3
Comparisons of the probabilistic regions of π_D under four multigranulation sequential three-way decisions.

No.	$\triangle$	$\underline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha,\beta)}(\pi_D)$	$\overline{\sum_{i=1}^{m} A_i}^{\triangle,(\alpha,\beta)}(\pi_D)$	$POS^{\triangle,(\alpha,\beta)}(\pi_D)$	$BND^{\triangle,(\alpha,\beta)}(\pi_D)$	$NEG^{\triangle,(\alpha,\beta)}(\pi_D)$
1	WAM	$\{x_1, x_5, x_6\}$	$\{x_1, x_5, x_6, x_8\}$	$\{x_1, x_5, x_6\}$	$\{x_8\}$	$\{x_2, x_3, x_4, x_7, x_9, x_{10}\}$
2	O	$\{x_1, x_5, x_6\}$	$\{x_1, x_5, x_6, x_7, x_8, x_9\}$	$\{x_1, x_5, x_6\}$	$\{x_7, x_8, x_9\}$	$\{x_2, x_3, x_4, x_{10}\}$
3	P	$\{x_5\}$	$\{x_5, x_6\}$	$\{x_5\}$	$\{x_6\}$	$\{x_1, x_2, x_3, x_4, x_7, x_8, x_9, x_{10}\}$
4	PO	$\{x_5\}$	$\{x_1, x_5, x_6, x_7, x_8, x_9\}$	$\{x_5\}$	$\{x_1, x_6, x_7, x_8, x_9\}$	$\{x_2, x_3, x_4, x_{10}\}$

(3)

$$BND_{GS}^{O,(\alpha,\beta)^l}(\pi_D) \subseteq BND_{GS}^{PO,(\alpha,\beta)^l}(\pi_D), BND_{GS}^{P,(\alpha,\beta)^l}(\pi_D) \subseteq BND_{GS}^{PO,(\alpha,\beta)^l}(\pi_D), BND_{GS}^{WAM,(\alpha,\beta)^l}(\pi_D) \subseteq BND_{GS}^{PO,(\alpha,\beta)^l}(\pi_D).$$

Fig. 3 illustrates the relationships of the lower and upper approximations under four kinds of multigranulation sequential three-way decision model. One can check that the size of the probabilistic boundary region under the pessimistic-optimistic multigranulation sequential three-way decision model is largest. However, the sizes of the probabilistic boundary regions under the other three multigranulation sequential three-way decisions cannot be compared, which can also be verified by Example 2. On the other hand, one can check the size of the probabilistic positive region under the optimistic view is largest, while that under the pessimistic view is smallest.

Example 3. (Continued with Example 2) Suppose $\pi_D = \{\{x_1, x_3, x_4, x_6, x_7\}, \{x_2, x_5, x_8, x_9, x_{10}\}\}$, $w_1 = \frac{1}{2}$ and $w_2 = \frac{1}{2}$. Here we compute the probabilistic positive, boundary and negative regions with respect to π_D as shown in Table 3. It is obvious that $|POS^{O,(0.75,0.5)}(\pi_D)| = 3 \geq |POS^{P,(0.75,0.5)}(\pi_D)| = 1$ and $|BND^{PO,(0.75,0.5)}(\pi_D)| = 5$, which is biggest under four kinds of multigranulation sequential three-way decisions. □

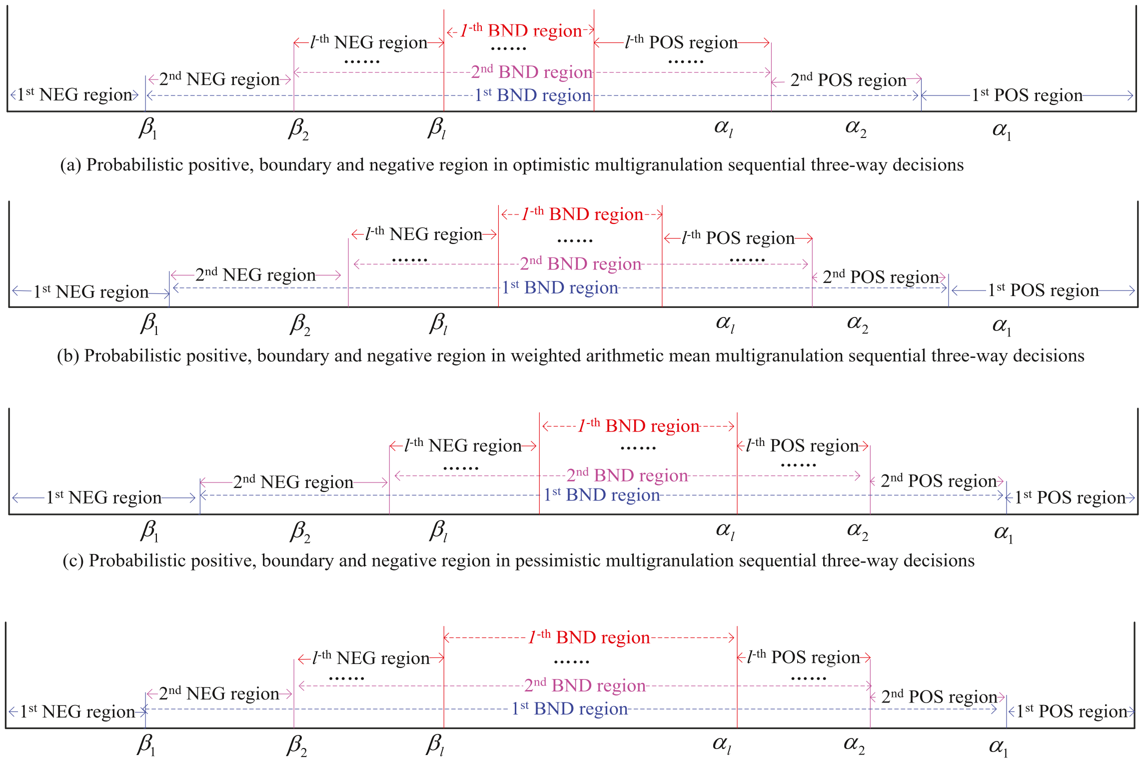
In what follows, we illustrate the computing process from multiple levels of granularity through an example.

Example 4. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, $U/A_1 = \{\{x_1, x_9\}, \{x_2, x_3, x_4, x_7\}, \{x_5\}, \{x_6, x_8, x_{10}\}\}$, $U/A_2 = \{\{x_1, x_{10}\}, \{x_2, x_4, x_8\}, \{x_3, x_6, x_7, x_9\}, \{x_5\}\}$, $\pi_D = \{\{x_1, x_3, x_4, x_6, x_7\}, \{x_2, x_5, x_8, x_9, x_{10}\}\}$, $w_1 = \frac{1}{2}$, $w_2 = \frac{1}{2}$, and $(\alpha, \beta)_2 = \{(0.85,0.5),(0.75,0.6)\}$. Here we compute the probabilistic positive, boundary and negative regions with respect to π_D as shown in Table 4. □

From Table 4, one can check that the probabilistic boundary region becomes smaller, while the corresponding positive and negative ones gradually turn larger. Fig. 4 illustrates the increasing trends of the three probabilistic regions under four models of multigranulation sequential three-way decisions. In general, the size of the probabilistic positive region under the optimistic multigranulation sequential three-way decision model is largest as illustrated in Fig. 4(a). One may increase the value of α to decrease the risk of three-way decision-making at different levels of granularity. On the other hand, the size of the probabilistic positive region under the pessimistic multigranulation sequential three-way decision model is smallest in Fig. 4(c). One may decrease the value of α under the evaluation criteria at different levels of granularity to acquire the alternative solutions. In such case, the pessimistic-optimistic multigranulation three-way decisions will be applied more broadly in Fig. 4(d). In addition, in Fig. 4(b), the weighted arithmetic mean multigranulation three-way decisions are also a good choice for some voting cases.

Table 4
Comparisons of the probabilistic regions of π_D under four multigranulation sequential three-way decisions.

$\triangle$	$POS^{\triangle,(\alpha,\beta)^1}(\pi_D)$	$BND^{\triangle,(\alpha,\beta)^1}(\pi_D)$	$NEG^{\triangle,(\alpha,\beta)^1}(\pi_D)$	$POS^{\triangle,(\alpha,\beta)^2}(\pi_D)$	$BND^{\triangle,(\alpha,\beta)^2}(\pi_D)$	$NEG^{\triangle,(\alpha,\beta)^2}(\pi_D)$
WAM	$\{x_5\}$	$\{x_2, x_3, x_4, x_6, x_7, x_8, x_9, x_{10}\}$	$\{x_1\}$	$\{x_3, x_5, x_7\}$	$\{x_8, x_9\}$	$\{x_1, x_2, x_4, x_6, x_{10}\}$
O	$\{x_5\}$	$\{x_2, x_3, x_4, x_6, x_7, x_8, x_9, x_{10}\}$	$\{x_1\}$	$\{x_2, x_3, x_4, x_5, x_6, x_7, x_9\}$	$\{x_8, x_{10}\}$	$\{x_1\}$
P	$\{x_5\}$	$\{x_3, x_7, x_8\}$	$\{x_1, x_2, x_4, x_6, x_9, x_{10}\}$	$\{x_3, x_5, x_7\}$	$\{x_8\}$	$\{x_1, x_2, x_4, x_6, x_9, x_{10}\}$
PO	$\{x_5\}$	$\{x_2, x_3, x_4, x_6, x_7, x_8, x_9, x_{10}\}$	$\{x_1\}$	$\{x_3, x_5, x_7\}$	$\{x_2, x_4, x_6, x_8, x_9, x_{10}\}$	$\{x_1\}$



(a) Probabilistic positive, boundary and negative region in optimistic multigranulation sequential three-way decisions

(b) Probabilistic positive, boundary and negative region in weighted arithmetic mean multigranulation sequential three-way decisions

(c) Probabilistic positive, boundary and negative region in pessimistic multigranulation sequential three-way decisions

(d) Probabilistic positive, boundary and negative region in pessimistic-optimistic multigranulation sequential three-way decisions

Fig. 4. Comparisons of the three probabilistic regions under four multigranulation sequential three-way decisions.

Table 5
Description of the datasets.

No.	Dataset	$\|U\|$	$\|C\|$	$\|V_d\|$
1	Zoo	101	16	7
2	Credit Approval	690	14	2
3	German Credit Data	1000	20	2
4	Chess-kr-vs-kp	3,196	36	2
5	Nursery	12,960	8	5
6	Connect-4	67,757	42	3

4. Experiment analysis

In order to evaluate our algorithms, we implement some experiments on a personal computer with Windows7, 2.40 GHz CPU and 16GB memory. The software is Visual C# 2012. The objective of the following experimental results is to compare the sizes of the probabilistic regions under four multigranulation sequential three-way decisions. For sake of clarification, the weighted artimatic mean multigranulation sequential three-way decisions, optimistic multigranulation sequential three-way decisions, pessimistic multigranulation sequential three-way decisions and pessimistic-optimistic multigranulation sequential three-way decisions are abbreviated as WAMMS3WD, OMS3WD, PMS3WD and POMS3WD, respectively. Since our approaches only deal with discrete attributes, we employ Rosetta software (http://www.lcb.uu.se/tools/rosetta/) to fill in some missing values and transform the numerical and continuous data into the discrete ones. We perform the experiments on six datasets from UCI Repository of machine learning databases [1]. The characteristics of six datasets are summarized in Table 5.

4.1. Comparisons of the three probabilistic regions under different multigranulation sequential three-way decisions

In what follows, we compare the numbers of the probabilistic positive and negative regions, and analyze the uncertainty of the probabilistic boundary regions under different multigranulation sequential three-way decisions on six datasets. We use $\alpha^t = \alpha\ (1 - (t-1)\cdot\delta)$ and $\beta^t = \beta\ (1 + (t-1)\cdot\delta)$ $(1 \le t \le l)$ to denote these threshold values under different levels of granularity, where α^t, β^t and δ represent the t^{th}-level threshold values and step length, respectively. If $\alpha_t \ge 1$, let $\alpha^t =$

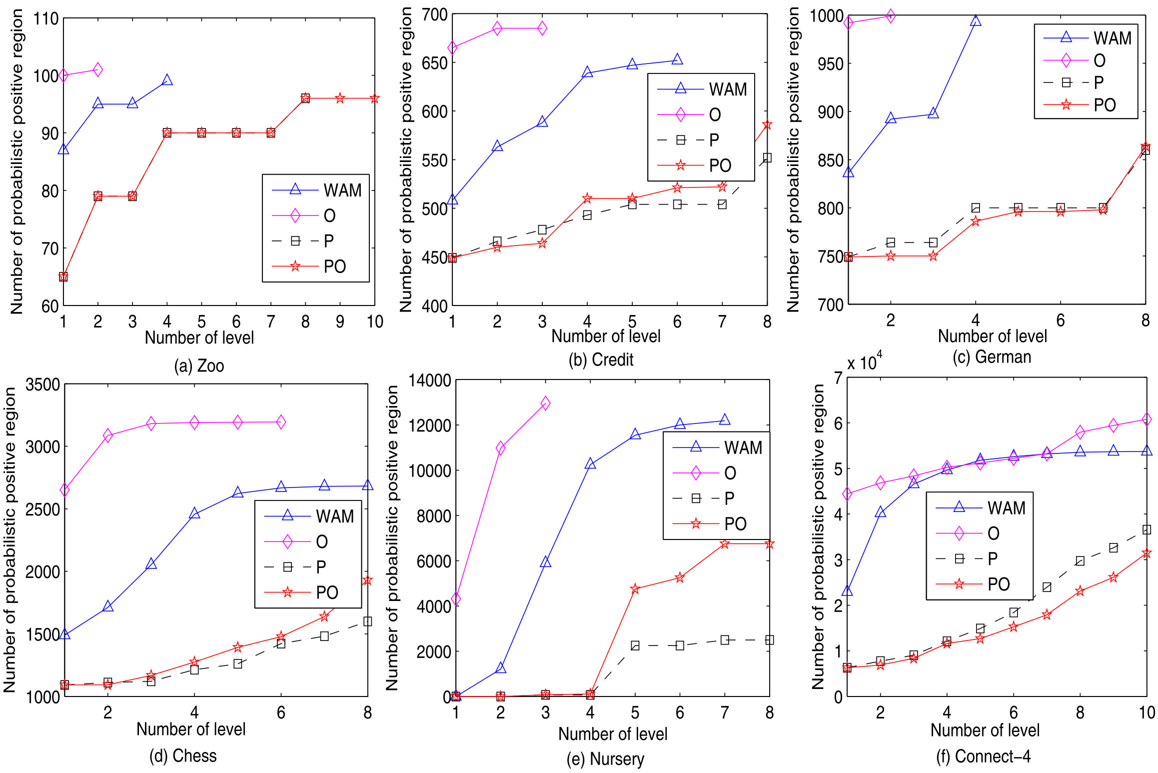


Fig. 5. Comparisons of the probabilistic positive regions on six datasets.

1. As well, if $\beta^t \le 0$, we set $\beta^t = 0$. For all the datasets, we divide the whole condition attribute set into two parts, namely two granular structure and set $\delta = 0.02$.

4.1.1. Comparisons of the numbers of the probabilistic positive and negative regions under different multigranulation sequential three-way decisions

For such two datasets Zoo and Nursery, we set $(\alpha, \beta) = (0.8, 0.4)$ and $(\alpha, \beta) = (0.7, 0.4)$, respectively. Figs. 5 and 6 illustrate the number of the probabilistic positive and negative regions under different multigranulation sequential three-way decisions. Through empirical testing, we can conclude as follows.

- For OMS3WD, it is obvious that the size of the probabilistic positive region is larger than that of the other ones in Fig. 5, even equal to the number of the objects. For PMS3WD, one can check that the size of the probabilistic negative region is largest in general. In some cases, the number of the negative regions are zero in Fig. 6(a) and (e) for OMS3WD and POMS3WD.
- For OMS3WD, the maximal numbers of the probabilistic positive regions are acquired at lower levels. However, POMS3WD and PMS3WD need higher levels to reach the maximum of the probabilistic positive regions. For WAMS3WD, the required numbers of granular levels are between those of the other MS3WD models. In Fig. 5(a), the numbers of granular levels are 2, 4, 8 and 10 for OMS3WD, WAMS3WD, PMS3WD and POMS3WD, respectively. In Fig. 5(b)–(f), the numbers of granular levels are the same for PMS3WD and POMS3WD.

4.1.2. Comparisons of uncertainty of the probabilistic boundary regions under different multigranulation sequential three-way decisions

We employ the deferment rate to evaluate the quality of the probabilistic boundary regions under different multigranulation sequential three-way decisions as follows:

$$deferment\ rate: DR^t = \frac{|BND_{GS}^{\triangle,(\alpha,\beta)^t}(\pi_D)|}{|U|} \quad (36)$$

The experiment results under different multigranulation sequential three-way decisions on six datasets are shown in Fig. 7. One can easily check that the boundary regions monotonously decrease to zeros for OMS3WD in most cases quickly. For POMS3WD, the uncertainty of the probabilistic boundary region is much larger than those of the other MS3WD models.

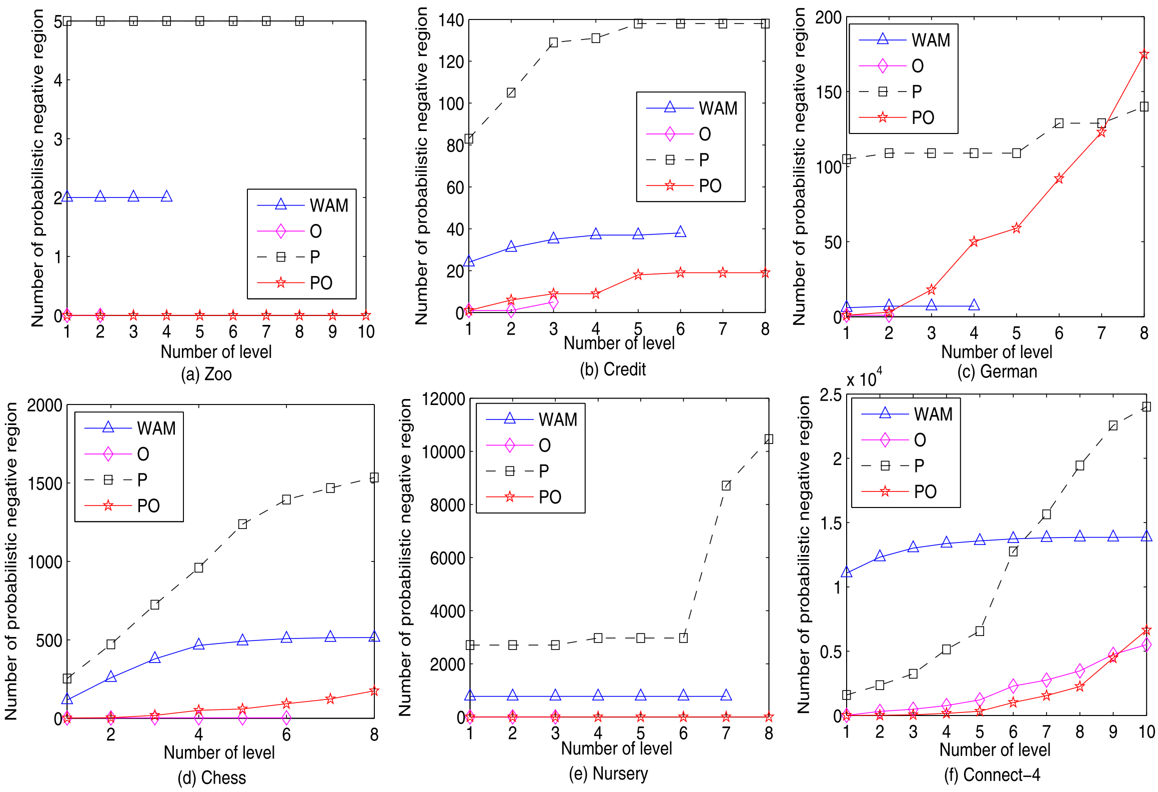


Fig. 6. Comparisons of the probabilistic negative regions on six datasets.

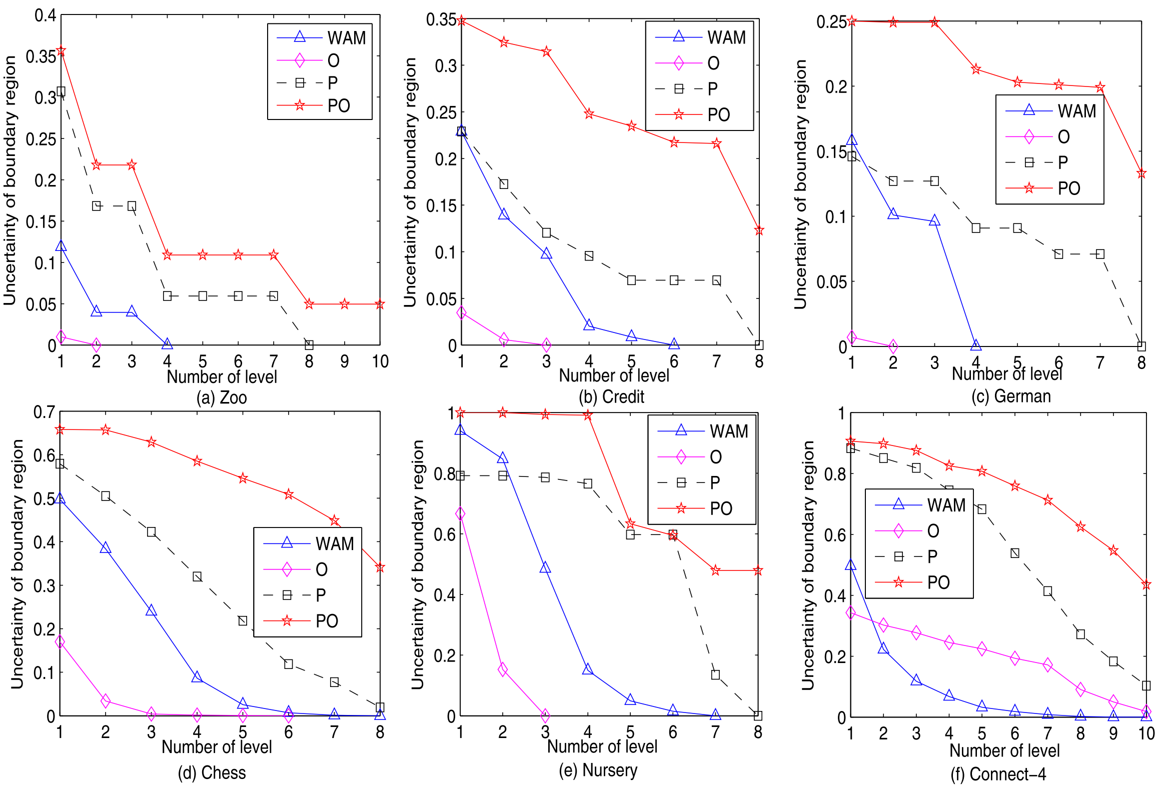


Fig. 7. Comparisons of uncertainty of the probabilistic boundary regions on six datasets.

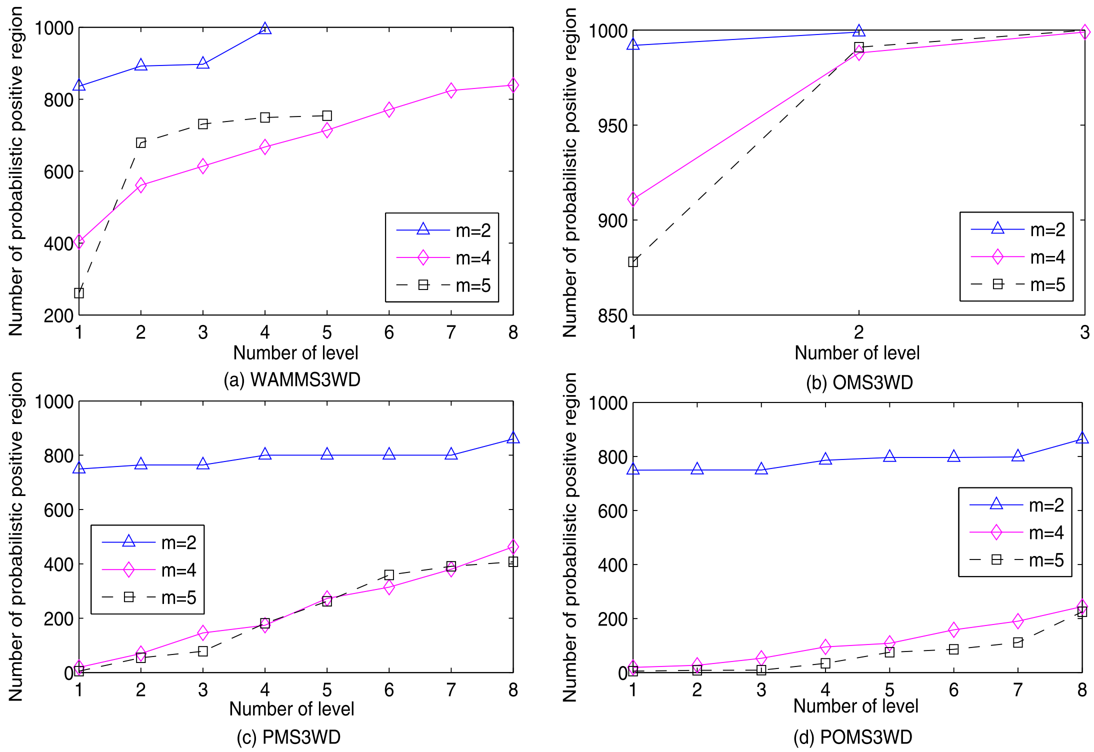


Fig. 8. Comparisons of the probabilistic positive regions under different multigranulation sequential three-way decisions on German.

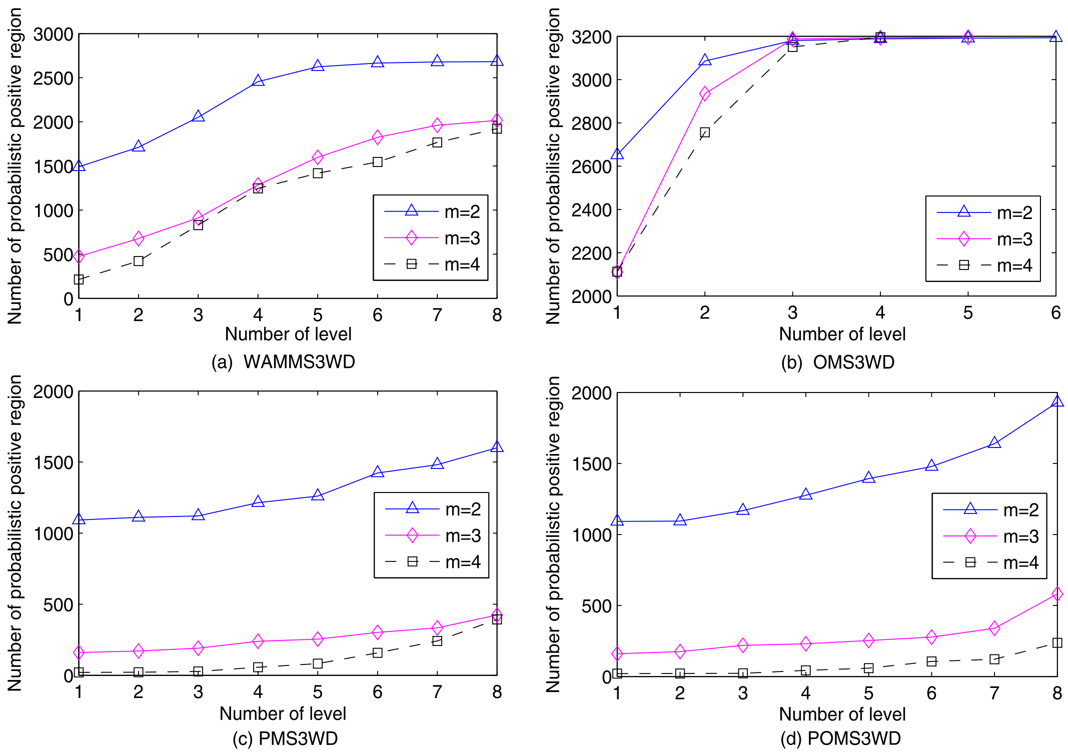


Fig. 9. Comparisons of the probabilistic positive regions under different multigranulation sequential three-way decisions on Chess-kr-vs-kp.

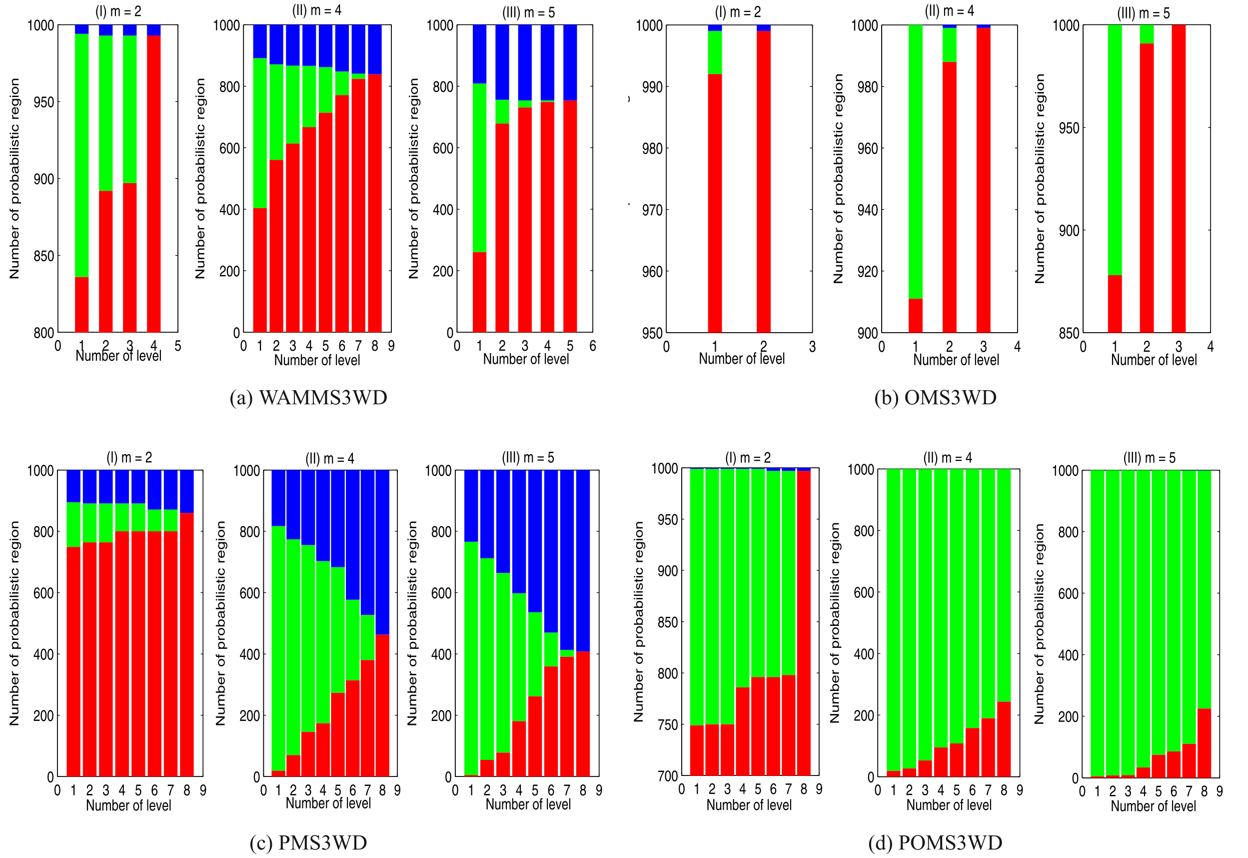


Fig. 10. Four multigranulaiton sequential three-way decisions on German.

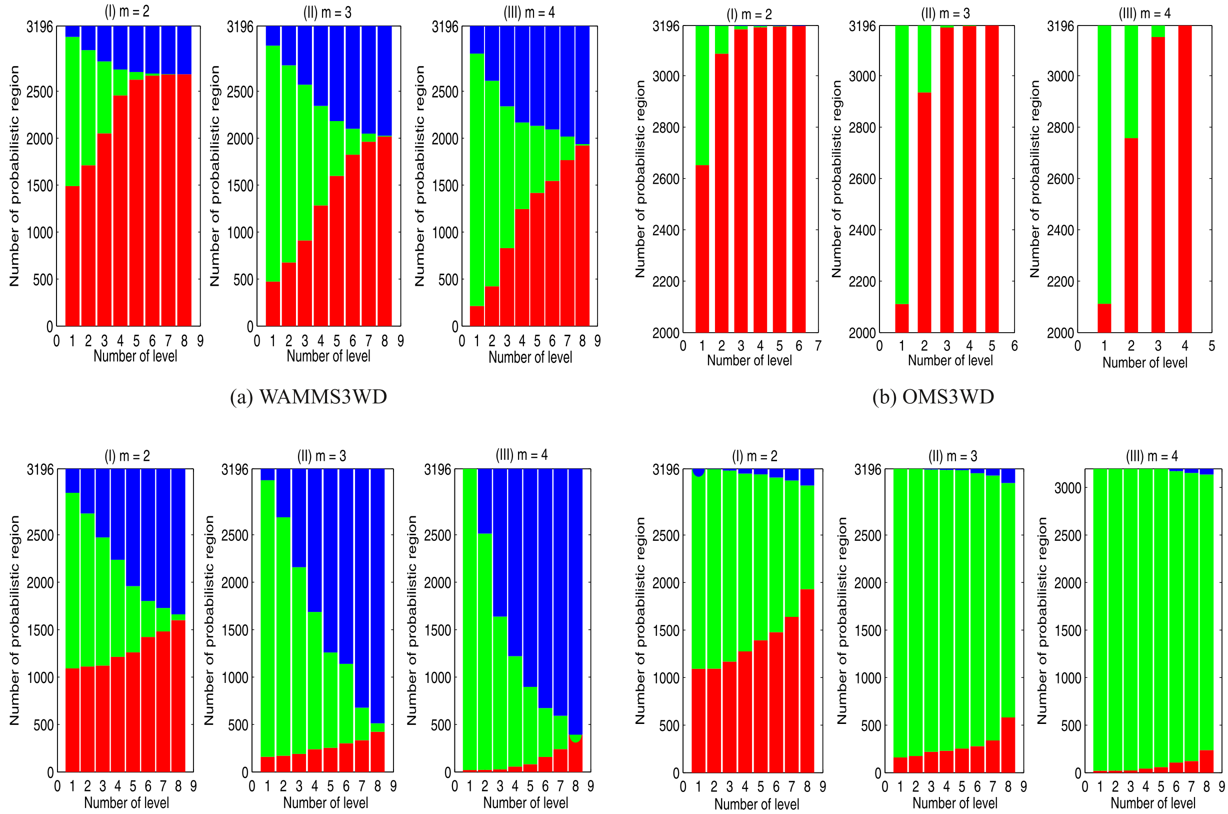
4.2. Comparisons of the size of the probabilistic positive regions under different number of granular structures

In this subsection, we mainly compare the number of the probabilistic positive regions under different multiview granular structures. We divide the number of the whole condition attribute set M into m group. The first granular structure is the attribute sequence $\{c_1, c_2, \ldots, c_{\frac{M}{m}}\}$, the second granular structure the attribute sequence $\{c_{\frac{M}{m}+1}, c_{\frac{M}{m}+2}, \ldots, c_{\frac{2M}{m}}\}$, and so on. For German, we set the number of granular structures m to 2, 4 and 5 in Fig. 8. For Chess-kr-vs-kp, we let m be equal to 2, 3 and 4 as shown in Fig. 9. One can easily find that the sizes of the probabilistic positive regions decrease with the numbers of granular structures increase in most cases under different multigranulation sequential three-way decisions. This is because the number of the objects in the probabilistic positive regions that induced by the less attribute subsets become smaller. Figs. 10 and 11 illustrate the detailed change trends of the three probabilistic regions under different multigranualtion sequential three-way decisions with the increasing level of granularity. It is obvious that the more the granular structures, the smaller the probabilistic positive regions, especially for POMS3WD.

5. Conclusions

In this paper, we propose a generalized multigranulation sequential three-way decisions. Within this framework, we mainly adopt different aggregation strategies for fuzing the lower approximations and the upper approximations to construct the five models of multigranulation sequential three-way decisions. Furthermore, we analyze the properties and relationships of different multigranulation sequential three-way decisions. Finally, the experiments show that the proposed MS3WD models are effective and efficient.

In future research, we will focus on the study of multigranulation sequential three-way decisions using different values of α and β for each granular structure. Extending the proposed multigranulation three-way decisions for different decision systems will be another research direction.



(a) WAMMS3WD (b) OMS3WD

(c) PMS3WD (d) POMS3WD

Fig. 11. Four multigranulaiton sequential three-way decisions on Chess-kr-vs-kp.

Conflict of interest

The authors declare no conflict of interest.

Acknowledgments

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