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ORIGINAL ARTICLE

A knowledge acquisition method based on concept lattice and inclusion degree for ordered information systems

Yong Liu[1,2,3] · Xiangping Kang[4,5] · Duoqian Miao[4] · Deyu Li[6]

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Abstract
In some information system with order features, when users consider "greater than" or "less than" relations to a certain degree rather than in the full sense, using traditional methods may face great limitations. In light of natural connections among concept lattice, inclusion degree, order relations, and the feasibility of mutual integration among the three (concept lattice is essentially a type of data analysis tool using binary relations as research objects, while inclusion degree is a type of powerful tool for measuring uncertain order relations), the paper attempts to analyze uncertain order relations quantitatively within the framework of integration theory of concept lattice and inclusion degree. By which, the research scope of order relations undergoes an expansion-to-contraction process. Namely, certain order relations are first expanded to fuzzy or uncertain relations, and then the fuzzy or uncertain relations are allowed to contract to a degree of certainty by setting threshold parameters. Clearly, by properly widening the research scope of order relations, the model not only has good robustness and generalization ability, but also can meet actual needs flexibly. On this basis, solutions for algebraic structure, reduction, core, dependency, et al. are further studied deeply in ordered information systems. In short, the paper, as a meaningful try and exploration, is conducive to the integration of theories, and may offer some new and feasible ways for the study of order relations and ordered information systems.

Keywords Concept lattice · Inclusion degree · Ordered information systems · Uncertain order relations

✉ Xiangping Kang
tongji_kangxp@sina.com

Yong Liu
liuyong@sdu.edu.cn

Duoqian Miao
miaoduoqian@163.com

Deyu Li
lidy@sxu.edu.cn

1 School of Information Engineering, Lingnan Normal University, Zhanjiang 524048, China

2 Department of Computer Engineering, Changji University, Changji 831100, China

3 School of Software, Shandong University, Jinan 250101, China

4 Key Laboratory of Embedded System and Service Computing, Ministry of Education, Tongji University, Shanghai 201804, China

5 China National Tobacco Corporation, Shanxi Branch, Taiyuan 030000, Shanxi, China

6 Key Laboratory of Computational Intelligence and Chinese Information Processing of Ministry of Education, Shanxi University, Taiyuan 030006, China

1 Introduction

In the cognitive course, humans always tend to classify same or similar subjects into one class based on similar features, or close distances, or functional convergence, and then mark this class as a word to form an abstract concept. In this case, a concept is a unit of thought, and a large conceptual system can be gradually established for humans to cognize the objective world. By simulating humans' conceptual cognitive process, Wille pioneered concept lattice in 1982, also known as formal concept analysis [40], which, as an important application branch of order and lattice theory, is essentially a type of data analysis tool using binary relations as the main research objects. On the one hand, by defining a pair of intent and extent operators, it can obtain concepts of rich semantics from data, and any concept consists of two parts-intent and extent; on the other hand, with help of the lattice algebra structure, it can intuitively present order relationships among concepts. In recent years, theories, methods, and tools for concept lattices are under continuous update [18, 22, 23, 27, 34, 53], especially the integration of concept lattice with rough sets [2, 3, 11, 12, 35, 36, 39, 49], fuzzy sets [7, 15, 19, 32], granular computing [13, 16, 17, 29, 38, 41, 58], et al., making the theoretical system increasingly sophisticated and also greatly improving the analysis capability to complex-datas.

1.1 Concept lattice and order relations

It is known that, as a special type of binary relations, order relations are universal, and can intuitively show the sequence or size relationship among objects. For instance, being early versus being late, being superior versus being inferior, or being tall versus being short are all concrete manifestations of order relations, and in particular, transitivity is the most basic feature of order relations.

Definition 1 "$\preccurlyeq$" is called an order relation on the set L, if it satisfies following conditions for all elements $x, y, z \in L$

1. reflexivity: $x \preccurlyeq x$;
2. antisymmetry: $x \preccurlyeq y$ and $x \neq y \Rightarrow$ not $y \preccurlyeq x$;
3. transitivity: $x \preccurlyeq y$ and $y \preccurlyeq z \Rightarrow x \preccurlyeq z$.

In the case, we say $(L, \preccurlyeq)$ is an order set.

There is a natural connection between one-valued contexts and order relations. Generally, order relations can always be divided into two different types, namely internal ones and external ones. Here, an internal one refers to the order relation existing within a single set, for instance, "$\preccurlyeq$" is internal in $(V, \preccurlyeq)$. An external one refers to the order relation among sets, for instance, "$\preccurlyeq$" is external in $(V \cup W, \preccurlyeq)$, where "$\preccurlyeq$" represents the order relation from the set V to the set W.

(1) An internal order relation is a special form of the external one, i.e., although $(V \cup V, \preccurlyeq)$ and $(V, \preccurlyeq)$ have some differences in the form of expression, they are essentially the same.

(2) Any order relations, whether internal or external, always can be uniformly expressed in a formal context, i.e., by the rule

$$v \preccurlyeq w \Leftrightarrow (v, w) \in R_{\preccurlyeq}, \quad v \in V,\ w \in W$$

$(V \cup W, \preccurlyeq)$ and $(V, W, R_{\preccurlyeq})$ can be converted to each other, where $(V, W, R_{\preccurlyeq})$ is a formal context. That means we can use concept lattice to analyze order relations. In the following, $(V, W, R_{\preccurlyeq})$ is simplified as $(V, W, \preccurlyeq)$.

For instance, for any internal order relation, it always can be shown in two ways, one is in the form of formal context shown in Table 1, the other is represented as the order set shown in Fig. 1.

Considering that concept lattice is a data analysis tool using formal contexts as research objects, and both $(V \cup W, \preccurlyeq)$ and $(V, W, R_{\preccurlyeq})$ are two different manifestations of the same subject, in the following, to facilitate a unified formalization, for any order relation, whether internal or external, it will be represented as a formal context, namely, $(V, \preccurlyeq)$ will be expressed in the form of $(V, V, \preccurlyeq)$, meanwhile, $(V \cup W, \preccurlyeq)$ will be formalized as $(V, W, \preccurlyeq)$.

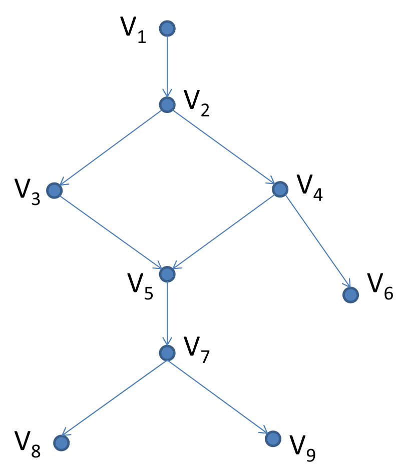

Fig. 1 A typical order set

Table 1 A typical formal context

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	$\preccurlyeq$								
v_2	$\preccurlyeq$	$\preccurlyeq$							
v_3	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$						
v_4	$\preccurlyeq$	$\preccurlyeq$		$\preccurlyeq$					
v_5	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$				
v_6	$\preccurlyeq$	$\preccurlyeq$		$\preccurlyeq$		$\preccurlyeq$			
v_7	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$		$\preccurlyeq$		
v_8	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$		$\preccurlyeq$	$\preccurlyeq$	
v_9	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$		$\preccurlyeq$		$\preccurlyeq$

1.2 Inclusion degree and order relations

It is known that boundary ambiguity of concepts is universal, and even lots of opposite concepts such as "beautiful" versus "ugly", and "good" versus "bad" have no absolutely clear-cut distinction. In the case, to define a fuzzy concept accurately, it is necessary to quantitatively describe its extent. For this reason, Zadeh proposed fuzzy sets theory in 1965 [52]. It does not simply approve or disapprove for an object whether belongs to or not belongs to a set. Instead, it can use values of some membership function to express the uncertainty.

It is also known that classical order relations are clear, definite, and unambiguous, namely, they merely consider "greater than" or "less than" in a full sense, and allowing only one of the two relations to be valid. In fact, the oversimplification of "relations" usually prevents the discovery of potentially valuable knowledge from seemingly unrelated things. Especially, when some user allows subtle errors, or considers a certain degree of order relations, the use of traditional certain order relation analysis logic will face great limitations and may even lead to completely erroneous conclusions.

To solve problems similar to the above, Zhang et al. proposed inclusion degree theory [55, 56]. The theory, as a new method for measuring uncertain relations, is of great theoretical significance in advancing the research of relations from the certainty stage to the uncertainty stage.

Essentially, fuzzy sets broadens the research scope of sets by means of membership functions [37, 44, 48], while inclusion degree broaden the research scope of relations by means of inclusion degree functions. It is noteworthy that inclusion degree can not only quantitatively describe the inclusion relations among sets, but also provide an effective quantitative analysis method for uncertain order relations.

In recent years, inclusion degree has been extensively studied [4, 45, 50], and applied to a variety of fields such as pattern recognition [14], neural networks [33], uncertain reasoning [57]. Young and Fan defined inclusion degree from an axiomatic perspective [6, 51]. Zhang et al. stated that inclusion degree in a broad sense is the generalization of a variety of uncertain reasoning methods including uncertain probabilistic reasoning, evidential reasoning, fuzzy reasoning, and information reasoning. That is to say, inclusion degree provides a unified theoretical framework and general principle for uncertain reasoning [55, 56]. Liang and Xu revealed relations between inclusion degree and various metrics in rough sets and proved that relevant metrics in rough sets could be attributed to inclusion degree [20, 46]. Qian et al. established relationships among the consistency, inclusion degree and fuzzy measure in some different types of decision tables [24]. Zhang et al. proposed a framework for comparing two interval sets by inclusion measures, which can be used to three-way decisions [54].

1.3 Inclusion degree, concept lattice and order relations

Being able to deal with uncertainty is an important ability for humans to achieve cognition and reasoning. Here, considering that inclusion degree is an effective measurement method to describe uncertain order relations while concept lattice is essentially a type of data analysis tool using binary relations as research objects, including order relations, so there is a significant connection among inclusion degree, concept lattice, and order relations. In this case, concept lattice and inclusion degree, as special relation analysis tools, will certainly help to deal with order relations. That is to say, exploring the integration theory among the three is not only rational to some extent, but is of important theoretical significance. From a macro-perspective, the integration theory helps to abstract a new data analysis framework, which may provides a more stronger theoretical basis for the uncertain order relations processing; from a micro-perspective, the integration theory can not only expand the data-analysis scope of concept lattice from certain order relations to uncertain order relations but also may provide some new valuable ideas for the expansion of concept intent and extent.

At present, the integration studies on inclusion degree and concept lattice are rare and still in their early stage. Qu et al. introduced inclusion degree to FCA, and proved intents, extents and implications can be reconstructed by inclusion degree. These results will be helpful to understand the essence of concepts and the structure of concept lattice, and can be regarded as the main foundation of quantitative measures for FCA [28]. When processing large-scale data or solving some complicated problem, the size of the lattice may be too large to be handled. To solve the problem, Xie et al. provided a new method from the perspective of inclusion degree [43]. Xiao and He applied concept lattice and inclusion degree to the field of integrating multi-source geo-ontologies [42]. For obtaining a concept lattice of appropriate complexity and size, on the basis of inclusion degree and neighborhood system, Ma et al. proposed a method which could effectively reduce the number of concepts while conserving the main formal structure [21].

1.4 Knowledge acquisition models in ordered information systems

To date, for ordered information systems, although scholars have proposed many types of dominance-based rough set models [1, 5, 9, 10, 25, 26, 30, 47], most ones are based on certain order relations, and thus may be greatly limited in actual application. It is known that compared to equivalence relations, similarity relations are not concerned with "equal" or "unequal" but with the degree of similarity. Similarly, compared with classical order relations, uncertain order relations are not concerned with "greater than" or "less than" in a full sense, but with the degree to which the relation "greater than" or "less than" is valid. Clearly, when a certain degree of error is allowed or a certain degree of some order relation is considered, lots of traditional dominance-based rough set models will be incapable. In the case, how to advance the research scope of order relations from certainty to uncertainty, or how to reasonably analyze uncertain order relations, has gradually become research focus in recent years.

In fact, uncertain order relation always plays an important role, which is more effective to meet and explain the actual human decision-making process. To date, research results in the area are very few. For the reason, the study mainly focuses on the integration of concept lattice and inclusion degree to solve the problems in ordered information systems, and further attempts to offer some feasible idea for the analysis of uncertain order relations as well as the common problems of algebraic structure, reduction, core and dependency in ordered information systems.

Following sections are arranged as follows: Sect. 2 recalls basic notions of concept lattice and inclusion degree briefly; Sect, 3 discusses the integration of concept lattice and inclusion degree; Sect. 4 widens the research scope of order relations from certainty to uncertainty, and then introduces threshold parameters to contract the research scope of order relations from uncertainty to a certain degree of certainty; Sect. 5 details corresponding solution methods to the problems of algebraic structure, reduction, core, and dependency within the integration framework of concept lattice and inclusion degree; Sect. 6 discusses perspectives for further works.

2 Basic notions of concept lattice and inclusion degree

To facilitate follow-up research, the section primarily serves as an introduction to some basic notions [8, 57].

Definition 2 [28] In $(L, L, \preccurlyeq)$, for all $x, y \in L$, if $\mathscr{D}(y/x)$ meets following conditions

(1) $0 \leq \mathscr{D}(y/x) \leq 1$;
(2) if $x \preccurlyeq y$, then $\mathscr{D}(y/x) = 1$;
(3) if $x \preccurlyeq y \preccurlyeq z$, then $\mathscr{D}(x/z) \leq \mathscr{D}(x/y)$;
(4) if $x \preccurlyeq y$, then for each $z \in L$ there exists $\mathscr{D}(x/z) \leq \mathscr{D}(y/z)$.

then we say $\mathscr{D}$ is an inclusion degree of $(L, L, \preccurlyeq)$.

$\mathscr{D}(y/x)$ reflects the degree of x being less than y; it not only reflect certain order relations, such as $x \preccurlyeq y \Leftrightarrow \mathscr{D}(y/x) = 1$, but can quantitatively reflect uncertain order relations, such as $0 \leq \mathscr{D}(y/x) \leq 1$.

A formal context (G, M, I) consists of two sets G and M and the binary relation $I \subseteq G \times M$ from the set G to the set M. Especially, If I is a kind of order relation, we usually say (G, M, I) is an order context. Note that, in the following, for any formal context (G, M, I), if no otherwise specified, it refers to an order context denoted as $(G, M, \preccurlyeq)$.

In $K = (G, M, \preccurlyeq)$, for any $A \subseteq G$, we define

$$A' = \{m \in M | g \preccurlyeq m, \forall g \in A\}$$

Correspondingly, for any $B \subseteq M$, we define

$$B' = \{g \in G | g \preccurlyeq m, \forall m \in B\}$$

(A, B) is called a concept in K, if $A' = B$ and $B' = A$. Let (A, B) be a concept, we say A and B are concept extent and concept intent separately. Further more, for concepts (A_1, B_1) and (A_2, B_2), if there exists the following order relationship

$$(A_1, B_1) \preccurlyeq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2 \Leftrightarrow B_2 \subseteq B_1$$

then $(\mathscr{B}(K), \preccurlyeq)$ is a complete lattice, where $\mathscr{B}(K)$ is the set of all concepts in K.

Proposition 1 *In* $(G, M, \preccurlyeq)$, *let* $A, A_1, A_2 \subseteq G, B, B_1, B_2 \subseteq M$, *then*

(1) $A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$ (2) $B_1 \subseteq B_2 \Rightarrow B_2' \subseteq B_1'$
(3) $A \subseteq A''; B \subseteq B''$ (4) $A' = A'''; B' = B'''$

3 The integration theory of concept lattice and inclusion degree

It is known that inclusion degree is mainly used to measure the degree of inclusion relationship among sets. A common inclusion degree model is as follows:

$$\mathbf{c}(Y/X) = \frac{|X \cap Y|}{|X|}$$

It is easy to observe that $\mathbf{c}(Y/X)$ denotes the extent to which the set X is included in the set Y. $\mathbf{c}(Y/X)$ is derived from the set theory, and does not have complicated external forms of expression, nor does it have complex internal logic, thereby making it very easy for people to recognize and understand the essence of inclusion degree.

In recent years, although scholars have proposed many complex inclusion degree models for measuring uncertain inclusion relations or uncertain order relations, those models are essentially derived from the extension and expansion of $\mathbf{c}(Y/X)$, with the core ideas not having undergone dramatic change. Similarly, the inclusion degree model $\mathscr{D}$ that will be constructed in this study is also derived from $\mathbf{c}(Y/X)$. In the following, we will discuss in-depth the link between $\mathbf{c}(Y/X)$ and $\mathscr{D}$ in Definition 2 to provide a modeling basis for constructing an inclusion degree function specific to uncertain order relations.

Definition 3 In $(V, V, \preccurlyeq)$, for any $v \in V$ we define

$$v^{\uparrow} = \{w \in V | v \preccurlyeq w\}, \ \ v^{\downarrow} = \{w \in V | w \preccurlyeq v\}$$

$v^{\uparrow}$ and $v^{\downarrow}$ are called the upper bound and the lower bound of v separately.

For instance, for the nodes V_3 and V_4 in Fig. 1, the corresponding upper bounds and lower bounds are shown in Fig. 2.

Lemma 1 *In* $(V, V, \preccurlyeq)$, *for* $v, w \in V$, *there exists*

$$v \preccurlyeq w \Leftrightarrow w^{\uparrow} \subseteq v^{\uparrow} \Leftrightarrow v^{\downarrow} \subseteq w^{\downarrow}$$

Here, we employ the nodes V_3 and V_7 in Fig. 1 as examples. In Fig. 1, we can easily obtain $V_7 \preccurlyeq V_3$ as well as the conclusion shown in Fig. 3. In the case, we have $V_7 \preccurlyeq V_3 \Leftrightarrow V_3^{\uparrow} \subseteq V_7^{\uparrow} \Leftrightarrow V_7^{\downarrow} \subseteq V_3^{\downarrow}$ immediately, that is also can be used to verify the conclusion in Lemma 1.

As shown by Lemma 1, there is a close relationship between certain order relations and certain inclusion relations, and these can be transformed into each other. Does this mean that there is also a close connection between inclusion degree functions specific to order relations and inclusion degree functions specific to inclusion relations? The answer to this question is "yes" in this study, which means that there must be a close connection of $\mathbf{c}(Y/X)$ to $\mathscr{D}$ in Definition 2. In the case, this study proposes the following rationalization criterias, which will provide a modeling basis for the construction of $\mathscr{D}$.

Criterion 1: The higher the degree to which $w^{\uparrow}$ is included in $v^{\uparrow}$, if and only if the higher the confidence level of $w \in V$ being greater than $v \in V$. Namely,

$$\mathbf{c}(v^{\uparrow}/w^{\uparrow}) \uparrow \text{, if and only if } \mathscr{D}(w/v) \ \uparrow$$

In what follows, any $\mathscr{D}$ constructed by upper bounds is denoted as **int**_.

Criterion 2: The higher the degree to which $w^{\downarrow}$ is included in $v^{\downarrow}$, if and only if the higher the confidence level of $v \in V$ being greater than $w \in V$. Namely,

$$\mathbf{c}(v^{\downarrow}/w^{\downarrow}) \uparrow \text{, if and only if } \mathscr{D}(v/w) \ \uparrow$$

In what follows, any $\mathscr{D}$ constructed by lower bounds is denoted as **ext**_.

In essence, both **int**_ and **ext**_ can be used to measure the uncertain order relations in $(V, V, \preccurlyeq)$, but from different perspectives of data analysis. Of course, it is also possible to expand from a single perspective to a collaborative perspective, that is, to conduct fusion on the above-mentioned inclusion degree models of different perspectives to avoid the limitations and one-sidedness under a single perspective. Next, from the perspective of concept lattice, this study will construct two different types of inclusion degree models that matches above viewpoints.

Definition 4 In $(G, M, \preccurlyeq)$, for any $m_1, m_2 \in M$ we define

$$\mathbf{ext}\_(m_2/m_1) = \frac{|\neg m'_1|}{|m'_1|} \times \frac{|m'_1 \cap m'_2|}{|\neg (m'_1 \cap m'_2)|}$$

where $\neg m'_1 = G - m'_1$, $\neg (m'_1 \cap m'_2) = G - (m'_1 \cap m'_2)$. Meanwhile, for any $g_1, g_2 \in G$ we define

$$\mathbf{int}\_(g_2/g_1) = \frac{|g'_1|}{|\neg g'_1|} \times \frac{|\neg(g'_1 \cup g'_2)|}{|g'_1 \cup g'_2|}$$

where $\neg g'_1 = M - g'_1$, $\neg (g'_1 \cap g'_2) = M - (g'_1 \cap g'_2)$.

In Definition 4, $\mathbf{ext}\_(m_2/m_1)$ is constructed mainly on the basis of following considerations:

(1) $\mathbf{ext}\_(m_2/m_1)$ is proportional to $\frac{|\neg m'_1|}{|m'_1|}$, which is mainly used to measure the size of $|m'_1|$. That is,

$$\frac{|\neg m'_1|}{|m'_1|} \uparrow \text{, if and only if } |m'_1| \downarrow$$

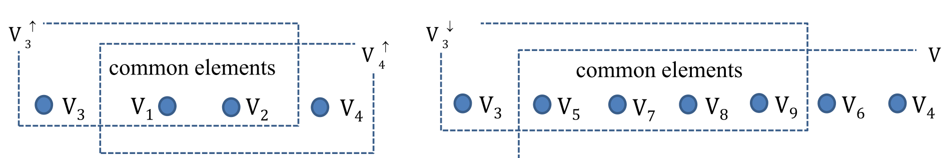

Fig. 2 Upper bounds and lower bounds of V_3 and V_4 in Fig. 1

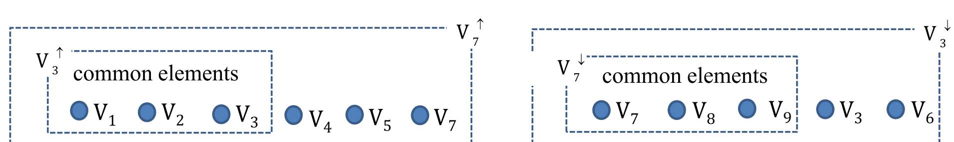

Fig. 3 Upper bounds and lower bounds of V_3 and V_7 in Fig. 1

That also means, when $|m_1'|$ becomes smaller, then the probability of $m_1 \preccurlyeq m_2$ or $m_1' \subseteq m_2'$ may be greater.

(2) $\mathbf{ext\_}(m_2/m_1)$ is proportional to $\frac{|m_1' \cap m_2'|}{|\neg(m_1' \cap m_2')|}$, which is mainly used to measure the size of $|m_1' \cap m_2'|$. That is,

$$\frac{|m_1' \cap m_2'|}{|\neg(m_1' \cap m_2')|} \uparrow, \text{ if and only if } |m_1' \cap m_2'| \uparrow$$

In the case, there may come to such a conclusion, namely, on the premise of $|m_1'|$ with small value, if $|m_1' \cap m_2'|$ changes from small to big, then the credibility of $m_1 \preccurlyeq m_2$ or $m_1' \subseteq m_2'$ may be becoming greater.

Here, what is worth mentioning is that $\mathbf{ext\_}(m_2/m_1)$ is essentially a product of two inclusion degrees, namely, both $\frac{|m_1' \cap m_2'|}{|m_1'|}$ and $\frac{|\neg m_1'|}{|\neg m_1' \cup \neg m_2'|}$ are all inclusion degree functions met to Definition 2.

Similarly, based on above viewpoints, we can also construct the function $\mathbf{int\_}(g_2/g_1)$, here, we will not elaborate.

Definition 5 In $(G, M, \preccurlyeq)$, we define

$$g_1 \preccurlyeq g_2 \Leftrightarrow {g'}_2 \subseteq {g'}_1, \quad g_1, g_2 \in G$$

the corresponding order set is denoted as $(G, G, \preccurlyeq)$. Similarly, we can define the order set $(M, M, \preccurlyeq)$, in which "$\preccurlyeq$" is described as

$$m_1 \preccurlyeq m_2 \Leftrightarrow {m'}_1 \subseteq {m'}_2, \quad m_1, m_2 \in M$$

Theorem 1 $\mathbf{ext\_}$ *is an inclusion degree of* $(M, M, \preccurlyeq)$, *and* $\mathbf{int\_}$ *is an inclusion degree of* $(G, G, \preccurlyeq)$.

Proof First, we prove that $\mathbf{ext\_}$ is an inclusion degree of $(M, M, \preccurlyeq)$.

It can be easily verified that $0 \le \mathbf{ext\_}(m_2/m_1) \le 1$ holds from the following formula

$$\mathbf{ext\_}(m_2/m_1) = \frac{|m_1' \cap m_2'|}{|m_1'|} \times \frac{|\neg m_1'|}{|\neg m_1' \cup \neg m_2'|}$$

Further more, according to Proposition 1, we have following conclusions

If $m_1 \preccurlyeq m_2$, then $m_1' \subseteq m_2'$ and $\neg m_2' \subseteq \neg m_1'$ hold. And further, it is easy to see that $\mathbf{ext\_}(m_2/m_1) = 1$.

If $m_1 \preccurlyeq m_2 \preccurlyeq m_3$, then there exist $m_1' \subseteq m_2' \subseteq m_3'$ and $\neg m_3' \subseteq \neg m_2' \subseteq \neg m_1'$, and further we can obtain

$$\mathbf{ext\_}(m_1/m_3) = \frac{|m_1'|}{|m_3'|} \times \frac{|\neg m_3'|}{|\neg m_1'|} \le \frac{|m_1'|}{|m_2'|} \times \frac{|\neg m_2'|}{|\neg m_1'|} = \mathbf{ext\_}(m_1/m_2)$$

It follows that $\mathbf{ext\_}(m_1/m_3) \le \mathbf{ext\_}(m_1/m_2)$.

If $m_1 \preccurlyeq m_2$, then $m_1' \subseteq m_2'$ and $\neg m_2' \subseteq \neg m_1'$ hold. And further, for any $m_3 \in M$, we can obtain

$$\frac{|m_1' \cap m_3'|}{|m_3'|} \times \frac{|\neg m_3'|}{|\neg m_1' \cup \neg m_3'|} \le \frac{|m_2' \cap m_3'|}{|m_3'|} \times \frac{|\neg m_3'|}{|\neg m_2' \cup \neg m_3'|}$$

i.e., $\mathbf{ext\_}(m_1/m_3) \le \mathbf{ext\_}(m_2/m_3)$ holds.

From above conclusions, we can conclude that $\mathbf{ext\_}$ is an inclusion degree of $(M, M, \preccurlyeq)$. Next, we show that $\mathbf{int\_}$ is an inclusion degree of $(G, G, \preccurlyeq)$.

First, $0 \le \mathbf{int\_}(g_2/g_1) \le 1$ is due to the fact,

$$0 \le \frac{|g_1'|}{|g_1' \cup g_2'|} \times \frac{|\neg g_1' \cap \neg g_2'|}{|\neg g_1'|} \le 1$$

Next, we show that $\mathbf{int\_}$ meets following properties by Proposition 1.

If $g_1 \preccurlyeq g_2$, then $g_2' \subseteq g_1'$ and $\neg g_1' \subseteq \neg g_2'$ hold, and further we can obtain $\mathbf{int\_}(g_2/g_1) = 1$ easily.

If $g_1 \preccurlyeq g_2 \preccurlyeq g_3$, then we have $g_3' \subseteq g_2' \subseteq g_1'$ and $\neg g_1' \subseteq \neg g_2' \subseteq \neg g_3'$, and further there exists following conclusion

$$\mathbf{int\_}(g_1/g_3) = \frac{|g_3'|}{|g_1'|} \times \frac{|\neg g_1'|}{|\neg g_3'|} \le \frac{|g_2'|}{|g_1'|} \times \frac{|\neg g_1'|}{|\neg g_2'|} = \mathbf{int\_}(g_1/g_2)$$

If $g_1 \preccurlyeq g_2$, then $g_2' \subseteq g_1'$ and $\neg g_1' \subseteq \neg g_2'$ hold, and further for any $g_3 \in G$, we have

$$\mathbf{int\_}(g_1/g_3) = \frac{|g_3'|}{|g_1' \cup g_3'|} \times \frac{|\neg g_1' \cap \neg g_3'|}{|\neg g_3'|} \le \frac{|g_3'|}{|g_2' \cup g_3'|} \times \frac{|\neg g_2' \cap \neg g_3'|}{|\neg g_3'|} = \mathbf{int\_}(g_2/g_3)$$

From above conclusions, we know $\mathbf{int\_}$ is an inclusion degree of $(G, G, \preccurlyeq)$. □

For example, for the nodes V_3 and V_4 in Table 1, by $\mathbf{int\_}$ we have

$$\mathbf{int\_}(V_3/V_4) = \frac{|V_4'|}{|\neg V_4'|} \times \frac{|\neg(V_3' \cup V_4')|}{|V_3' \cup V_4'|} = 0.625$$

where $V_3' = \{V_1, V_2, V_3\}$, $V_4' = \{V_1, V_2, V_4\}$. Meanwhile, by $\mathbf{ext\_}$ we have

$$\mathbf{ext\_}(V_3/V_4) = \frac{|\neg {V'}_4|}{|V_4'|} \times \frac{|V_3' \cap V_4'|}{|\neg(V_3' \cap V_4')|} = 0.4$$

where $V_3' = \{V_3, V_5, V_7, V_8, V_9\}$, $V_4' = \{V_4, V_5, V_6, V_7, V_8, V_9\}$.

For other nodes V_i and V_j, the corresponding $\mathbf{int\_}(V_i/V_j)$ and $\mathbf{ext\_}(V_i/V_j)$ can be found in Tables 2 and 3 separately.

Lemma 2 *In $(G, M, \preccurlyeq)$, let $g \in G, m \in M$, then $g' = g^{\uparrow}$, $m' = m^{\downarrow}$.*

Lemma 3 *In $(G, M, \preccurlyeq)$, let $g_1^{\uparrow} \subseteq g_2^{\uparrow} \subseteq g_3^{\uparrow}$, then following conclusions hold simultaneously.*

(1) $\mathbf{c}(g_1^{\uparrow}/g_3^{\uparrow}) \le \mathbf{c}(g_1^{\uparrow}/g_2^{\uparrow})$;
(2) $\mathbf{int\_}(g_3/g_1) \le \mathbf{int\_}(g_2/g_1)$.

Lemma 4 *In $(G, M, \preccurlyeq)$, let $m_1^{\downarrow} \subseteq m_2^{\downarrow} \subseteq m_3^{\downarrow}$, then following conclusions hold simultaneously.*

(1) $\mathbf{c}(m_1^{\downarrow}/m_3^{\downarrow}) \le \mathbf{c}(m_1^{\downarrow}/m_2^{\downarrow})$;
(2) $\mathbf{ext\_}(m_1/m_3) \le \mathbf{ext\_}(m_1/m_2)$.

Lemmas 3 and 4 further verify that the construction of **ext_** and **int_** can well meet the Criterion 1 and the Criterion 2, which can also indirectly indicate the rationality and feasibility of the two criteria.

Theorem 2 *In $(G, M, \preccurlyeq)$, if $V = G = M$, then **inc_** is an inclusion degree of $(V, V, \preccurlyeq)$, where **inc_** is defined by*

$$\begin{aligned}\mathbf{inc\_}(v_2/v_1) =& \alpha \times \mathbf{ext\_}(v_2/v_1) + (1-\alpha) \\ & \times \mathbf{int\_}(v_2/v_1),\ \ v_1, v_2 \in V\end{aligned}$$

Proof It can be easily verified that $0 \le \mathbf{inc\_}(v_2/v_1) \le 1$ from the definitions of **ext_** and **int_** . Moreover, by Theorem 1, we can easily complete the following reasoning.

Let $v_1 \preccurlyeq v_2$, then we can see $\mathbf{inc\_}(v_2/v_1) = 1$ from $\mathbf{int\_}(v_2/v_1) = 1$ and $\mathbf{int\_}(v_2/v_1) = 1$.

Let $v_1 \preccurlyeq v_2 \preccurlyeq v_3$, since $\mathbf{ext\_}(v_1/v_3) \le \mathbf{ext\_}(v_1/v_2)$ and $\mathbf{int\_}(v_1/v_3) \le \mathbf{int\_}(v_1/v_2)$, we can obtain $\mathbf{inc\_}(v_1/v_3) \le \mathbf{inc\_}(v_1/v_2)$ immediately.

Let $v_1 \preccurlyeq v_2$, then for any $v_3 \in V$, since $\mathbf{ext\_}(v_1/v_3) \le \mathbf{ext\_}(v_2/v_3)$ and $\mathbf{int\_}(v_1/v_3) \le \mathbf{int\_}(v_2/v_3)$, we have $\mathbf{inc\_}(v_1/v_3) \le \mathbf{inc\_}(v_2/v_3)$.

Form above conclusions, we can finally see **inc_** is an inclusion degree of $(V, V, \preccurlyeq)$. □

In the similar way, we can define the order set $(2^M, 2^M, \preccurlyeq)$ and the corresponding inclusion degree **set_ext_**.

Definition 6 In $(G, M, \preccurlyeq)$, let 2^M be the power set of M, we define

$$B_1 \preccurlyeq B_2 \Leftrightarrow B_1' \subseteq B_2',\ \ B_1, B_2 \in 2^M$$

the corresponding order set is denoted as $(2^M, 2^M, \preccurlyeq)$.

Table 2 The quantitative analysis result of uncertain order relation in Table 1 by **int_**

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	$\preccurlyeq$	0.4375	0.2500	0.2500	0.1000	0.1563	0.0625	0.0357	0.0357
v_2	$\preccurlyeq$	$\preccurlyeq$	0.5714	0.5714	0.2286	0.3571	0.1429	0.0816	0.0816
v_3	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.6250	0.4000	0.4000	0.2500	0.1429	0.1429
v_4	$\preccurlyeq$	$\preccurlyeq$	0.6250	$\preccurlyeq$	0.4000	0.6250	0.2500	0.1429	0.1429
v_5	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.6250	0.6250	0.3571	0.3571
v_6	$\preccurlyeq$	$\preccurlyeq$	0.6400	$\preccurlyeq$	0.4000	$\preccurlyeq$	0.2286	0.1000	0.1000
v_7	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.5714	$\preccurlyeq$	0.5714	0.5714
v_8	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.4375	$\preccurlyeq$	$\preccurlyeq$	0.4375
v_9	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.4375	$\preccurlyeq$	0.4375	$\preccurlyeq$

Table 3 The quantitative analysis result of uncertain order relation in Table 1 by **ext_**

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	$\preccurlyeq$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
v_2	$\preccurlyeq$	$\preccurlyeq$	0.1563	0.2500	0.1000	0.0156	0.0625	0.0156	0.0156
v_3	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.6400	0.6400	0.0000	0.4000	0.1000	0.1000
v_4	$\preccurlyeq$	$\preccurlyeq$	0.4000	$\preccurlyeq$	0.4000	0.0625	0.2500	0.0625	0.0625
v_5	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.0000	0.6250	0.1563	0.1563
v_6	$\preccurlyeq$	$\preccurlyeq$	0.0000	$\preccurlyeq$	0.0000	$\preccurlyeq$	0.0000	0.0000	0.0000
v_7	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.0000	$\preccurlyeq$	0.2500	0.2500
v_8	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.0000	$\preccurlyeq$	$\preccurlyeq$	0.0000
v_9	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.0000	$\preccurlyeq$	0.0000	$\preccurlyeq$

Theorem 3 **set_ext_** *is an inclusion degree of* $(2^M, 2^M, \preccurlyeq)$, *where* **set_ext_** *is defined as*

$$\mathbf{set\_ext\_}(B_2/B_1) = \frac{|\neg B_1'|}{|B_1'|} \times \frac{|B_1' \cap B_2'|}{|\neg (B_1' \cap B_2')|},\ B_1, B_2 \in \mathscr{P}(M)$$

Proof The proving process is similar to that in Theorem 1, here which will not be discussed in detail. □

In essence, above-mentioned inclusion degree models are all derived from $\mathbf{c}(X/Y)$, that is, despite the fact that above models are quite different than $\mathbf{c}(X/Y)$ in terms of forms, the core modeling ideas remain the same.

4 Study on order relations from certainty to uncertainty to a degree of certainty

In practical application, by properly widening the research scope of order relations, It will be helpful to get more hidden knowledge from data. In this regard, inclusion degree maybe provide a feasible solution method. Here, by means of inclusion degree, the paper presents the following solution process:

The first step is to expand classical order relations to the uncertain ones, where the uncertain ones are essentially a special type of fuzzy order relations. For a given data set, the classical order relation is usually known while the inclusion degree function is unknown, thereby making it necessary to construct a reasonable inclusion function from the classical order relation and then make a quantitative description of uncertain order relations. For instance, suppose that we start with a classical order relation $\preccurlyeq$ and derive an inclusion degree function **inc_**. If **inc_** is defined as the membership function $\mu_{\preccurlyeq}(v, w)$, namely $\mu_{\preccurlyeq}(v, w) = \mathbf{inc\_}(w/v)$, then we can obtain a fuzzy order relation "$\preccurlyeq_{\sim}$", here, $\mu_{\preccurlyeq}(v, w)$ essentially reflects the degree of v being less than w.

For instance, in Fig. 1, let us suppose that V_i represents the i-th solution to a problem, and that the order relation represents the relative superiority (or inferiority) relationship among the solutions. In this scenario, the relative superiority (or inferiority) relationship among some solutions is clear, but for others the relationship is unclear. In the case, when it comes to selecting three relatively good solutions, it is necessary to conduct a quantitative analysis of the relationship between solutions V_3 and V_4. Here, let $\alpha = 0.5$, then by means of **inc_**, we have $\mathbf{inc\_}(V3/V_4) = 0.5125$ and $\mathbf{inc\_}(V4/V_3) = 0.6325$. Namely, the degree of V_4 being superior to V_3 is greater than the degree of V_3 being superior to V_4. In the case, people may tend to believe that V_4 is superior to V_3 to some extent.

The second step is to contract the order relations from uncertainty to a degree of certainty by artificially setting threshold parameters. In other words, it is to transform infinite uncertainty into finite certainty and transform the complex fuzzy relation into simple certain relation. In particular, let $0 \le \sigma \le 1$, then the fuzzy order relation "$\preccurlyeq_{\sim}$" with $\mu_{\preccurlyeq}(v, w) = \mathbf{inc\_}(w/v)$ can be converted to the new certain relation

$$\preccurlyeq_{\sigma} = \{(v, w) | \mathbf{inc\_}(w/v) \ge \sigma\}$$

Follow the example in the first step, if $\sigma = 0.55$, then users believe that V_4 is superior to V_3. However, if $\sigma = 0.5$, then we see V_3 and V_4 are all good solutions, namely, for users, they are indistinguishable.

Obviously, the first step can employ **inc_** to widen the classical order relation "$\preccurlyeq$" to the fuzzy order relation "$\preccurlyeq_{\sim}$". In fact, the "$\preccurlyeq_{\sim}$" can also be described as a fuzzy order context defined as follows

A fuzzy order context is of the form $(G, M, \preccurlyeq_{\sim})$, where "$\preccurlyeq_{\sim}$" is a fuzzy order relation between G and M, the corresponding membership function is defined as $\mu_{\preccurlyeq} : G \times M \to [0, 1]$ with $\mu_{\preccurlyeq}(g, m) = \mathbf{inc\_}(m/g)$. That is, for any $(g, m) \in G \times M$, $\mu_{\preccurlyeq}(g, m)$ means the degree that "m" is greater than "g".

Meanwhile, the second step can further contract the fuzzy order relation "$\preccurlyeq_{\sim}$" into the simple certain relation "$\preccurlyeq_{\sigma}$". In fact, the "$\preccurlyeq_{\sigma}$" can also be described as a σ-order context defined as follows

Definition 7 Let $(G, M, \preccurlyeq_{\sim})$ be a fuzzy order context such that $\mu_{\preccurlyeq}(g, m) = \mathbf{inc}(m/g)$, $0 \le \sigma \le 1$, if

$$\preccurlyeq_{\sigma} = \left\{(g, m) \in G \times M | \mu_{\preccurlyeq}(g, m) \ge \sigma\right\}$$

we say $(G, M, \preccurlyeq_{\sigma})$ is a σ-order context.

For example, let $\alpha = 0.5$, then by means of the method in Theorem 2, the certain order relation in Table 1 can be expanded to the fuzzy order context shown in Table 4. And further, let $\sigma = 0.55$, then Table 4 can be contract to the σ -order context shown in Table 5.

5 Application of the integration theory in ordered information systems

In this section, we try to introduce the integration theory to ordered information systems. Here, the reason for which we introduce concept lattice into information systems, mainly in light of the natural connection between concept lattice and binary relations. Since any binary relation can be uniformly expressed as the form of one-valued formal context, In this case, concept lattice, as the special analysis tool for relations, will certainly help to deal with order relations and tolerance relations. In addition, how

Table 4 A fuzzy order context deducted from Table 1

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	$\preccurlyeq$	0.2188	0.1250	0.1250	0.0500	0.0782	0.0313	0.0179	0.0179
v_2	$\preccurlyeq$	$\preccurlyeq$	0.3639	0.4107	0.1643	0.1864	0.1027	0.0486	0.0486
v_3	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.6325	0.5200	0.2000	0.3250	0.1215	0.1215
v_4	$\preccurlyeq$	$\preccurlyeq$	0.5125	$\preccurlyeq$	0.4000	0.3438	0.2500	0.1027	0.1027
v_5	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.3125	0.6250	0.2567	0.2567
v_6	$\preccurlyeq$	$\preccurlyeq$	0.3200	$\preccurlyeq$	0.2000	$\preccurlyeq$	0.1143	0.0500	0.0500
v_7	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.2857	$\preccurlyeq$	0.4107	0.4107
v_8	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.2188	$\preccurlyeq$	$\preccurlyeq$	0.2188
v_9	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.2188	$\preccurlyeq$	0.2188	$\preccurlyeq$

Table 5 A σ-order context deducted from Table 4

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9
v_1	$\preccurlyeq$								
v_2	$\preccurlyeq$	$\preccurlyeq$							
v_3	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$					
v_4	$\preccurlyeq$	$\preccurlyeq$		$\preccurlyeq$					
v_5	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$		$\preccurlyeq$		
v_6	$\preccurlyeq$	$\preccurlyeq$		$\preccurlyeq$		$\preccurlyeq$			
v_7	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$		$\preccurlyeq$		
v_8	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$		$\preccurlyeq$	$\preccurlyeq$	
v_9	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$		$\preccurlyeq$		$\preccurlyeq$

to expand classical concept lattice and further apply to complex information systems, will surely be one of the mainstream directions of the development of concept lattice in the future, so that is another reason why we apply concept lattice to complex information systems.

5.1 One-valued formal contexts derived from ordered information systems

An information system is $(U, \mathscr{A}, V, f)$, where $V = \cup_{a\in\mathscr{A}} V_a$, $f : U \times \mathscr{A} \to V$ is a mapping such that $f(x, a) \in V_a$ for each $a \in \mathscr{A}$ and $x \in U$. Normally, members of U are called objects, members of $\mathscr{A}$ are called attributes; V_a is called the domain of attribute a; U is called the universe of discourse.

In classic definition on information systems, for any $m \in \mathscr{A}$, there is no further description of the relationship between any $v \in V_m$ and $w \in V_m$. But, in reality, there may exist some complex relationship among values in V_m, which often plays a determining role in scientifically and effectively acquiring knowledge. At this point, we can see that requiring no prior knowledge is both the advantage and disadvantage of rough sets, considering that, the study attempts to introduce prior knowledge, namely scales, to further supplement and expand the classical information system.

Note that the ordered information system in the paper refer to the one containing two types of attributes, one kind is nominal (the value set of any attribute consists of several

Table 6 A typical ordered information system

	a	b	c	d	e
1	Good	Big	v_1	Light	High
2	Good	Small	v_2	Light	High
3	Poor	Small	v_6	Light	Medium
4	Poor	Small	v_8	Heavy	Low
5	Good	Big	v_4	Light	High
6	Good	Small	v_3	Light	High
7	Poor	Small	v_5	Light	Medium
8	Poor	Small	v_7	Heavy	Low

discrete values, and different values are mutually independent), another kind is order symbolic (the value set of any attribute consists of several discrete values, and among values there may exist order relationship).

As an example, an ordered information system about cars is given in Table 6, where $U = \{1, 2, \ldots, 8\}$ is the set of objects and $\mathscr{A} = \{a, b, c, d, e\}$ is the set of attributes with a = acceleration performance, b = inner space, c = performance/price ratio, d = weight, and e = maximum speed. In addition, attributes a, b, d and e are all normal, and c is order symbolic. Here, as the priori knowledge, $(V_c, V_c, \preccurlyeq)$ is given in the form of order set shown in Fig. 4. In the case, let $\alpha = 0.5$, then by means of the method in Theorem 2,

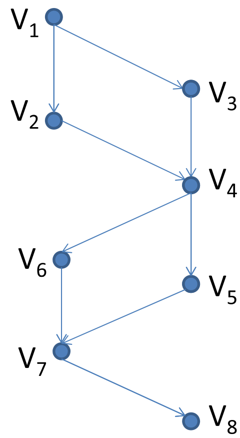

Fig. 4 An order set

the result of quantitative analysis of the uncertain order relation in Fig. 4 is shown in Table 7, its σ-order context is shown in Table 8, where $\sigma = 0.55$.

Definition 8 In $(U, \mathscr{A}, V, f)$, we say $S_m = (V_m, V_m, I_m)$ is a scale of $m \in \mathscr{A}$, if $I_m \subseteq V_m \times V_m$.

In above definition, S_m essentially describes the relationship among different values in V_m. Meanwhile, as the auxiliary prior knowledge, scales are useful supplement to the classical definition of information systems.

In fact, on the basis of scales, an ordered information system S can converted into an one-valued context by scaling. The basic idea of scaling can be simply understood as the strategy for converting an information system into an one-valued formal context on the basis of scales. Generally speaking, there are many kind of ways of scaling, here, we will introduce a simple one [11, 31, 40].

Definition 9 Let $S_m = (V_m, V_m, I_m)$ be a σ-order context as well as a scale of $m \in \mathscr{A}$, then by the rule

$$\big((x, y), m\big) \in J_\sigma \Leftrightarrow (v, w) \in I_m,\ \ v = f(x, m) \text{ and } w = f(y, m)$$

the ordered information system S can be transformed to an one-valued context

$$K_\sigma = (U^2, \mathscr{A}, J_\sigma)$$

where I_m is defined as

- when m is normal, $I_m = \big\{(v, v) | v \in V_m\big\}$;
- when m is order symbolic, $I_m = \big\{(v, w) | v \preccurlyeq_\sigma w,\ v, w \in V_m\big\}$.

Clearly, in the transformation process, the scales $S_m, m \in \mathscr{A}$ are essentially σ-order contexts, which only play intermediary roles, rather the final derivative context. Further more, for any values $v, w \in V_m$, K_σ no longer contains their own value information, and only reflects whether they meet $(v, w) \in I_m$. Based on the transformation idea, K_σ is not only more simple than the original ordered information system S, but also can better embody

Table 7 Results of quantitative analysis of the uncertain order relation in Fig. 4

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	$\preccurlyeq$	0.2143	0.2143	0.0715	0.0429	0.0429	0.0102	0.0000
v_2	$\preccurlyeq$	$\preccurlyeq$	0.5556	0.4445	0.2000	0.2000	0.0794	0.0238
v_3	$\preccurlyeq$	0.5556	$\preccurlyeq$	0.4445	0.2000	0.2000	0.0794	0.0238
v_4	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.4800	0.4800	0.1715	0.0429
v_5	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.5556	0.3969	0.1191
v_6	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.5556	$\preccurlyeq$	0.3969	0.1191
v_7	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	0.2143
v_8	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$

Table 8 A σ-order context deducted from Table 7

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	$\preccurlyeq$							
v_2	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$					
v_3	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$					
v_4	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$				
v_5	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$		
v_6	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$		
v_7	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	
v_8	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$	$\preccurlyeq$

Table 9 An one-valued formal context derived from Table 6

	a	b	c	d	e
(1, 1)	×	×	×	×	×
(1, 2)	×			×	×
(1, 3)				×	
(1, 4)					
(1, 5)	×	×		×	×
(1, 6)	×			×	×
(1, 7)				×	
⋮	⋮	⋮	⋮	⋮	
(8, 6)		×	×		
(8, 7)	×	×	×		
(8, 8)	×	×	×	×	×

the relationship between objects intuitively. For example, by the transformation rule in Definition 9 and the scale shown in Table 8, an one-valued formal context shown in Table 9 can be derived from Table 6.

5.2 Algebraic structure in ordered information systems

Concept lattice helps to endow an ordered information system S with stronger algebraic structure. In this section, we present the σ-concept lattice, which can organize all binary relations in S in the form of a lattice, which is very suitable for rules finding, and hierarchy and visualization of the knowledge.

In an information system, for any subset $B \subseteq \mathscr{A}$, it can always determine a binary relation R_B. In the following, the corresponding binary relation relative to B is defined as

$$\begin{aligned} R_B^{\delta} =& \{(x, y) \in U^2 \mid \forall m \in B, f(x, m) \\ &= f(y, m) \text{ or } f(x, m) \preccurlyeq_{\sigma} f(y, m)\} \end{aligned}$$

In $K_\sigma = (U^2, \mathscr{A}, J_\sigma)$, let $R \subseteq U^2, B \subseteq \mathscr{A}$, then the operators in Section 2 will be formally represented as

$$R' = \{m \in \mathscr{A} \mid \big((x, y), m\big) \in J_\sigma, \quad \forall (x, y) \in R\}$$

and

$$B' = \{(x, y) \in U^2 \mid \big((x, y), m\big) \in J_\sigma, \quad \forall m \in B\}$$

In the case, we say $(R, B) \in \mathscr{B}(K_\sigma)$ is a σ-concept, while $\big(\mathscr{B}(K_\sigma), \preccurlyeq\big)$ is a σ-concept lattice.

Theorem 4 *In* $(U^2, \mathscr{A}, J_\sigma)$, *let* $B \subseteq \mathscr{A}$, *then* $B' = R_B^{\sigma}$.

Proof The result is straightforward and the proof is omitted. □

Lemma 5 *Let* (R, B) *be a* σ*-concept,* $D \subseteq \mathscr{A}$. *If* $D'' = B$, *then*

$$R_D^{\sigma} = R_B^{\sigma} = R$$

From above discusses, It is easy to observe that $(\mathscr{B}(K_\sigma), \preccurlyeq)$ can endow $(U, \mathscr{A}, V, f)$ with a stronger algebraic structure. Namely, it can organize all the binary relations in $(U, \mathscr{A}, V, f)$ in the form of a lattice. In practical applications, users can flexibly adjust parameters to meet their actual needs. For example, in Table 6, let $\alpha = 0.5$, $\sigma = 0.55$, then by Theorem 2, the corresponding lattice structure is shown in Fig. 5, where for lattice nodes only concept intents are given (concept extents are not easy to show and are therefore omitted here). For another

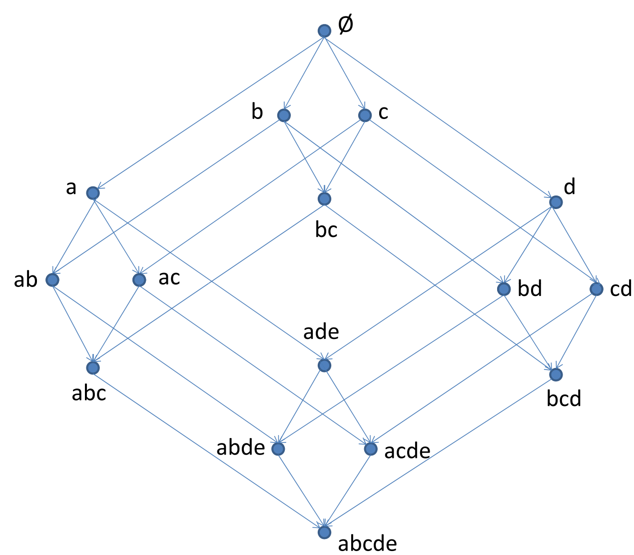

Fig. 5 A lattice structure deducted from Table 6 with $\sigma = 0.55$

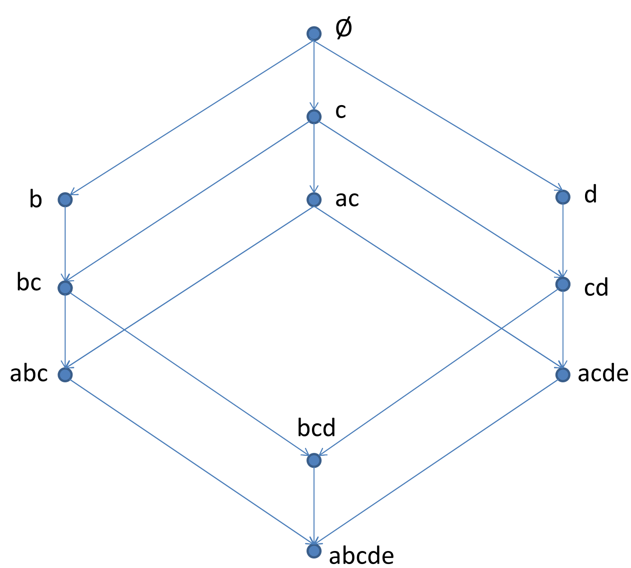

Fig. 6 A concept lattice deducted from Table 6 with $\sigma = 0.07$

example, when $\sigma = 0.07$, the corresponding lattice structure is shown in Fig. 6.

5.3 Reduction, core and dependency in ordered information systems

Suppose $(U, \mathscr{A}, V, f)$ is an ordered information system, if $R_B^\sigma \neq R_{B-m}^\sigma$, then we say $m \in B$ is indispensable in $B \subseteq \mathscr{A}$; Further if every $m \in B$ is indispensable, we say B is independent. The set of all indispensable attributes in B is called the core of B denoted as $CORE(B)$. If $C \subseteq D \subseteq \mathscr{A}$ and C is independent and $R_D^\sigma = R_C^\sigma$, then C is called a reduction of D.

Here, we can see that the order set $(2^{\mathscr{A}}, 2^{\mathscr{A}}, \preccurlyeq)$ and the corresponding inclusion degree **set_inc_**, defined in Section 3, can also be deducted from K_δ. In the following, by means of **set_inc_**, the solutions to reduction, core, et al. will be studied.

Theorem 5 *Let $B, L \subseteq \mathscr{A}$, if $L \in \triangle(K_\sigma)$, then*

$$B'' \subseteq L \Leftrightarrow \textbf{set\_inc\_}(B/L) = 1$$

where $\triangle(K_\sigma)$ is the set of all intents of concepts in K_σ.

Proof By Proposition 1 and Theorem 3, we have $L \in \triangle(K_\sigma) \Leftrightarrow L = L''$, and further implement following reasoning processes

$$B'' \subseteq L \Leftrightarrow B'' \subseteq L'' \Leftrightarrow L' \subseteq B' \Leftrightarrow \text{set\_inc\_}(B/L) = 1$$

which completes the proof. □

Theorem 6 *Let $B, C \subseteq \mathscr{A}$, following statements are equivalent*

(1) $R_B^\sigma \subseteq R_C^\sigma$
(2) $\forall L \in \triangle(K_\sigma), \textbf{set\_inc\_}(B/L) \neq 1$ or $\textbf{set\_inc\_}(C/L) = 1$ holds

Proof It can be easily verified that following deduction processes are true based on Proposition 1, Theorems 3 and 5.

(2)→(1):

start: $\forall L \in \triangle(K_\sigma)$, $\textbf{set\_inc\_}(B/L) \neq 1$ or $\textbf{set\_inc\_}(C/L) = 1$ holds

⇒ $\forall L \in \triangle(K_\sigma)$, $B'' \nsubseteq L$ or $C'' \subseteq L$ holds
⇒ Especially, for $B'' \in \triangle(K_\sigma)$, $B'' \nsubseteq B''$ or $C'' \subseteq B''$ holds
⇒ $C'' \subseteq B'' \Rightarrow B' \subseteq C' \Rightarrow$ **end** : $R_B^\sigma \subseteq R_C^\sigma$

(1)→(2):

Suppose: $\forall L \in \triangle(K_\sigma)$, $\textbf{set\_inc\_}(B/L) = 1$ and $\textbf{set\_inc\_}(C/L) \neq 1$ hold.

start: Suppose

⇒ $\forall L \in \triangle(K_\sigma)$, $B'' \subseteq L$ and $C'' \nsubseteq L$ hold
⇒ Especially, for the intent $B'' \in \triangle(K_\sigma)$, $B'' \subseteq B''$ and $C'' \nsubseteq B''$ hold.
⇒ **end:** $C'' \nsubseteq B''$ holds.

Moreover, from $R_B^\sigma \subseteq R_C^\sigma$, we can implement following deduction processes

$$R_B^\sigma \subseteq R_C^\sigma \Rightarrow B' \subseteq C' \Rightarrow C'' \subseteq B''$$

Obviously, $C'' \subseteq B''$ contradicts the earlier result $C'' \nsubseteq B''$. This also means above **Suppose** is false. On account of this, we can immediately find that if $R_B^\sigma \subseteq R_C^\sigma$, then for any $L \in \triangle(K_\sigma)$, $\textbf{set\_inc\_}(B/L) \neq 1$ or $\textbf{set\_inc\_}(C/L) = 1$ holds. Hence, when $R_B^\sigma \subseteq R_C^\sigma$, then the conclusion (2) is true. □

Theorem 7 *Let $B \subseteq \mathscr{A}$, $m \in B$ is indispensable in B, if*

$$\exists L \in \triangle(K_\sigma) \text{ such that } \textbf{set\_inc\_}\big((B-m)/L\big) = 1 \text{ and } \textbf{set\_inc\_}(B/L) \neq 1$$

Proof From Theorem 6, it follows that $R_B^\sigma \nsubseteq R_{B-m}^\sigma$, this also implies that $R_B^\sigma \neq R_{B-m}^\sigma$. In the case, we can see m is indispensable in B. □

Theorem 7 also states that for each $m \in B$, if there always exists corresponding $L \in \triangle(K_\sigma)$ such that $\textbf{set\_inc\_}(B/L) = 1$ and $\textbf{set\_inc\_}\big((B-m)/L\big) \neq 1$, then it will be verified easily that B is independent.

Theorem 8 *Let $m \in B \subseteq AT$, if*

$$\exists L \in \triangle(K_\delta) \text{ such that } \textbf{set\_inc\_}\big((B-m)/L\big) = 1 \text{ and } \textbf{set\_inc\_}(B/L) \neq 1$$

then $m \in CORE(B)$.

Proof From Theorem 6, it follows immediately that

start: $\exists L \in \triangle(K_\delta)$ such that $\textbf{set\_inc\_}(B - m/L) = 1$ and $\textbf{set\_inc\_}(B/L) \neq 1$

⇒ $R_{B-m}^\sigma \nsubseteq R_B^\sigma \Rightarrow R_{B-m}^\sigma \neq R_B^\sigma \Rightarrow$ m is indispensable in B
⇒ **end:** $m \in CORE(B)$

□

Theorem 9 *Let $C \subseteq B \subseteq \mathscr{A}$. C is a reduction of B, if following conditions are met*

(1) $\forall L \in \triangle(K_\sigma), \textbf{set\_inc\_}(C/L) \neq 1$ or $\textbf{set\_inc\_}(B/L) = 1$ holds;
(2) for any $m \in C$, there always exists corresponding $L \in \triangle(K_\sigma)$ such that

$$\textbf{set\_inc\_}(C-m/L)=1 \text{ and } \textbf{set\_inc\_}(B/L)\neq 1$$

Proof By Theorem 6, it follows that $R_C^\sigma \subseteq R_B^\sigma$ from the condition (1). Moreover, starting from $C \subseteq B$, one can realize following reasoning process based on Proposition 1 and Theorem 4.

$$C \subseteq B \Rightarrow B' \subseteq C' \Rightarrow R_C^\sigma \subseteq R_B^\sigma$$

Obviously, one can know that $R_C^\sigma = R_B^\sigma$ holds.

And further, by Theorems 4, 6 and $R_C^\sigma = R_B^\sigma$, starting from the condition (2), there exists following induce process.

start: condition (2)

$\Rightarrow$ $\forall m \in C$, $R_{C-m}^\sigma \nsubseteq R_B^\sigma$ holds
$\Rightarrow$ $\forall m \in C$, $R_{C-m}^\sigma \neq R_B^\sigma$ holds
$\Rightarrow$ by $R_C^\sigma = R_B^\sigma$, we have $\forall m \in C$, $R_{C-m}^\sigma \neq R_C^\sigma$
$\Rightarrow$ **end:** C is a reduction of B

□

For example, in Table 6, let $\alpha = 0.5$, $\sigma = 0.55$, $B = \{a, c, d, e\}$, then by referring to Table 10, we can see that there exists intent $L = \{a, d, e\}$ such that $\textbf{set\_inc\_}(B-c/L)=1$ and $\textbf{set\_inc\_}(B/L)\neq 1$, then by means of Theorem 8, we can judge $c \in CORE(B)$. In the similar way, we can further judge $a, d, e \notin CORE(B)$, therefore, we have $CORE(B) = \{c\}$. In addition, by means of Theorem 9, we can see that $\{a, c, d\}$ and $\{c, e\}$ are all reductions of B. Here, for any $L \in \triangle(K_\sigma)$ and any subset $D \subseteq B$, the corresponding $\textbf{set\_inc\_}(D/L)$ can be found in Table 10.

5.4 Dependency in ordered information systems

In complex information systems, there may be multiple forms of dependency between $B \subseteq \mathscr{A}$ and $D \subseteq \mathscr{A}$, such as function dependency, order dependency. In the paper, the dependency is defined by means of $R_B^\sigma \subseteq R_D^\sigma$, that is, if $R_B^\sigma \subseteq R_D^\sigma$, then $B \xrightarrow{\sigma} D$ is called a certain σ-dependency; if $R_B^\sigma \nsubseteq R_D^\sigma$, then we say $B \xrightarrow{\sigma} D$ is a uncertain σ-dependency. Note that if there is no special description, any dependency mentioned below refers to the certain one.

It is known that the scale of dependencies extracted from a data set is often very large and finding valuable ones may be a lengthy process. In the case, how to remove worthless and redundant ones from mass dependencies has become an important issue. For the reason, the study proposes an analytical approach based on the inclusion-inference, which can eliminate a large number of redundant rules to finally obtain a smaller dependency set referred to as a dependency generation set. Namely, if we know a dependency generation set, then we can obtain all dependency by means of the inclusion-inference.

Theorem 10 *Let $B, D \subseteq \mathscr{A}$, then following statements are equivalent*

(1) $\textbf{set\_inc\_}(D/B) = 1$;
(2) $B \xrightarrow{\sigma} D$;
(3) $\forall L \in \triangle(K_\sigma)$, $\textbf{set\_inc\_}(B/L) \neq 1$ or $\textbf{set\_inc\_}(D/L) = 1$ holds.

Proof Conclusions can be inferred immediately from Theorem 6 and the following facts

Table 10 Quantitative analysis result of some uncertain order relation by **set_ext_**

$\triangle(K_\sigma)$	acde	acd	ace	ade	cde	ac	ad	ae	cd	ce	de	c	e
Ø	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
a	0.3913	0.3913	0.3913	0.6000	0.3913	0.5238	0.6000	0.6000	0.3913	0.3913	0.6000	0.5238	0.6000
b	0.1680	0.1680	0.1680	0.2000	0.1680	0.2348	0.2000	0.2000	0.2348	0.1680	0.2000	0.4105	0.2000
c	0.2677	0.2677	0.2677	0.2677	0.2677	0.3584	0.2677	0.2677	0.4681	0.2677	0.2677	1.0000	0.2677
d	0.2348	0.2348	0.2348	0.3600	0.2348	0.2348	0.3600	0.3600	0.4105	0.2348	0.3600	0.4105	0.3600
ab	0.4667	0.4667	0.4667	0.5556	0.4667	0.6522	0.5556	0.5556	0.4667	0.4667	0.5556	0.6522	0.5556
ac	0.7470	0.7470	0.7470	0.7470	0.7470	1.0000	0.7470	0.7470	0.7470	0.7470	0.7470	1.0000	0.7470
bc	0.4092	0.4092	0.4092	0.4092	0.4092	0.5719	0.4092	0.4092	0.5719	0.4092	0.4092	1.0000	0.4092
bd	0.4667	0.4667	0.4667	0.5556	0.4667	0.4667	0.5556	0.5556	0.6522	0.4667	0.5556	0.6522	0.5556
cd	0.5719	0.5719	0.5719	0.5719	0.5719	0.5719	0.5719	0.5719	1.0000	0.5719	0.5719	1.0000	0.5719
abc	0.7156	0.7156	0.7156	0.7156	0.7156	1.0000	0.7156	0.7156	0.7156	0.7156	0.7156	1.0000	0.7156
ade	0.6522	0.6522	0.6522	1.0000	0.6522	0.6522	1.0000	1.0000	0.6522	0.6522	1.0000	0.6522	1.0000
bcd	0.7156	0.7156	0.7156	0.7156	0.7156	0.7156	0.7156	0.7156	1.0000	0.7156	0.7156	1.0000	0.7156
abde	0.8400	0.8400	0.8400	1.0000	0.8400	0.8400	1.0000	1.0000	0.8400	0.8400	1.0000	0.8400	1.0000
acde	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
abcde	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

$$B \xrightarrow{\sigma} D \Leftrightarrow R_B^\sigma \subseteq R_D^\sigma \Leftrightarrow B' \subseteq D' \Leftrightarrow \textbf{set\_inc\_}(D/B) = 1$$

□

Lemma 6 *For every $D \subseteq \mathscr{A}$, there always exists* $\textbf{set\_inc\_}(D''/D) = 1$.

Definition 10 Let $D \subseteq \mathscr{A}$ be independent, we say $\textbf{set\_inc\_}(D''/D) = 1$ is basic.

Lemma 7 *Let* $\textbf{set\_inc\_}(D''/D) = 1$ *and* $\textbf{set\_inc\_}(B/D) = 1$, *then there always exists $B \subseteq D''$.*

Lemma 8 *If D is a reduction of some subset in $\mathscr{A}$, then it must be independent.*

inclusion-inference: Let $E \subseteq E_1$, $F_1 \subseteq F$, then $\textbf{set\_inc\_}(F_1/E_1) = 1$ can be inferred from **set_inc_**(F/E)=1. To understand intuitionally and easily, the inclusion-inference can also be formally represented as

$$\frac{\textbf{set\_inc\_}(F/E) = 1,\ E \subseteq E_1,\ F_1 \subseteq F}{\textbf{set\_inc\_}(F_1/E_1) = 1}$$

Theorem 11 *Any* $\textbf{set\_inc\_}(C/B) = 1$ *can be inferred from some basic one by means of above inclusion-inference.*

Proof For any $\textbf{set\_inc\_}(C/B) = 1$, there must exist $\textbf{set\_inc\_}(D''/D) = 1$ such that D is a reduction of $B \subseteq \mathscr{A}$, and then by Lemma 8, we see D is also independent. In the case, from $\textbf{set\_inc\_}(D''/D) = 1$ and $\textbf{set\_inc\_}(C/B) = 1$, it follows that $C \subseteq D''$ by Lemma 7. In the case, since $\textbf{set\_inc\_}(D''/D) = 1$ and $D \subseteq B$ and $C \subseteq D''$ are all true, $\textbf{set\_inc\_}(C/B) = 1$ can be further inferred from $\textbf{set\_inc\_}(D''/D) = 1$ by means of the inclusion-inference. Since $\textbf{set\_inc\_}(D''/D) = 1$ is basic, so we can see the conclusion is true. □

Here, Theorem 11 states that for any $\textbf{set\_inc\_}(C/B) = 1$, it needn't to be calculated, but can be inferred from some basic one by means of the inclusion-inference. Namely, what we provide to users is only the set of basic ones, from which users can selectively derive others to meet their specific needs. In the following theorem, we say $D \xrightarrow{\sigma} D''$ is basic, if $\textbf{set\_inc\_}(D''/D) = 1$ is basic. In the case, we can easily prove the following theorem.

Theorem 12 *Any dependency can be inferred from some basic dependency.*

Proof The conclusion can be inferred immediately from Theorems 10 and 11. □

For example, in Table 6, let $\alpha = 0.5$, $\sigma = 0.55$, $B = \{c, d, e\}$ and $C = \{a\}$. Then, by means of Theorems 7 and 9, and by referring to Fig. 5, we can easily see that $D = \{c, e\}$ is a reduction of B, while it is also independent. Meanwhile, by Theorem 10, we have $B \xrightarrow{\sigma} C$. In the case, since $D'' = \{a, c, d, e\}$, $\textbf{set\_inc\_}(D''/D) = 1$, $D \subseteq B$ and $C \subseteq D''$ are all true, $\textbf{set\_inc\_}(C/B) = 1$ can be inferred from $\textbf{set\_inc\_}(D''/D) = 1$ by the inclusion-inference. Here, $D \xrightarrow{\sigma} D''$ is basic while $\textbf{set\_inc\_}(D''/D) = 1$ is basic. That also means $B \xrightarrow{\sigma} C$ can be inferred from the basic dependency $D \xrightarrow{\sigma} D''$.

As an effective complement to the rule-type knowledge, confidence degree plays an important role, which is essentially a quantitative description. In the following, for every uncertain dependency $B \xrightarrow{\sigma} D$, the corresponding confidence degree is defined as

$$\mathrm{Conf}(B \xrightarrow{\sigma} D) = \textbf{set\_inc\_}(D/B)$$

It is easy to observe that the lower conf($B \xrightarrow{\sigma} D$) is, the weaker relationship between B and D is, and the lower valuable of $B \xrightarrow{\sigma} D$ we believe. In the case, we always introduce some parameter such as $0 \le \omega \le 1$ to meet uses' actual need. Namely, if conf($B \xrightarrow{\sigma} D$) $\ge \omega$, then we say the uncertain dependency $B \xrightarrow{\sigma} D$ is credible. For example, in Table 6, let $\alpha = 0.5, \sigma = 0.55, \omega = 0.6, B = \{a, c, e\}$ and $D = \{b, d\}$. Then, for any subset of B and subset D, we can calculate Conf($B \xrightarrow{\sigma} D$), and further judge whether $B \xrightarrow{\sigma} D$ is credible. Here, the related result is shown in Table 11.

To some extent, the methods, context, objectives in the paper is similar to the ones in the literature [12], for better understanding them totally, here, we give the comparison and analysis to reveal the similarities and differences.

In terms of macroscopic, there are some similarities, specifically,

(1) both help to expand the application scope of concept lattice, and help to understand the essence of rough sets from the view of concept lattice.
(2) both are oriented to information systems, and are employed to solve similar problems such as algebraic structure, core, reduction, et al.
(3) both need to transform information systems into one-valued formal contexts, and then, the one-valued formal contexts can be further severed as new data sets to solve problems on the basis of concept lattice.

From microscopic angle, there exist significant differences, specially,

(1) in terms of research objects, this paper emphasizes order relation rather than tolerance relations.
(2) in terms of meeting users' needs, there exists some major differences, namely, what is emphasized in the paper is how to satisfy the situation that if users con-

Table 11 Quantitative analysis result of some uncertain dependencies by confidence degree

	b		d		bd	
	Conf	Credible	Conf	Credible	Conf	Credible
a	0.6000	$a \xrightarrow{\sigma} b$	0.6000	$a \xrightarrow{\sigma} d$	0.3333	
c	0.4681		0.4681		0.2677	
e	0.5556		1.0000	$e \xrightarrow{\sigma} d$	0.5556	
ac	0.7470	$ac \xrightarrow{\sigma} b$	0.7470	$ac \xrightarrow{\sigma} d$	0.5345	
ae	0.5556		1.0000	$ae \xrightarrow{\sigma} d$	0.5556	
ce	0.7156	$ce \xrightarrow{\sigma} b$	1.0000	$ce \xrightarrow{\sigma} d$	0.7156	$ce \xrightarrow{\sigma} bd$
ace	0.7156	$ace \xrightarrow{\sigma} b$	1.0000	$ace \xrightarrow{\sigma} d$	0.7156	$ace \xrightarrow{\sigma} bd$

sider "greater than" or "less than" relations to a certain degree rather than in the full sense, while the literature [12] emphasizes the situation that users just require "most" rather than "all" elements in a class are similar to each other.

(3) in terms of technology path, both need to first transform information systems into one-valued formal contexts by means of scales, but as the most crucial, important and basic elements during the process of transformation, scales in this paper and ones in [12] are essential differences.

(4) in terms of algebraic structure, both of this paper and the literature [12] all can organize relations in the form of lattice, but the former is primarily concerned with order relations and the latter is more oriented towards tolerance relations.

(5) for how to solve issues in information systems such as reduction, core, dependency, et al., the literature [12], on the basis of concept lattice, gives some new feasible ideas, while this paper emphasizes the solution within the framework of integration theory.

In general, although there are many similarities in macroscopical, there are significant differences in terms of research background, or solution ideas, or internal modeling mechanism. These differences, while enriching the theory of concept lattice, will certainly lay a solid foundation for the deep expansion of application scope of concept lattice, and the deep integration of concept lattice, inclusion degree and rough sets.

6 Summary and outlook

As we known, with the research scope continues to expand, and the research content becomes more and more complex, it is always accompanied by increasingly severe inconsistency between the uncertainty of reality and the accuracy of classical mathematics. In the case, the research on uncertainty has become more and more significantly. As a special type of binary relations, uncertain order relation is universal, and which is also the focus of this study. In light of inclusion degree is a type of powerful tool for measuring uncertain order relations, while concept lattice is essentially a type of data analysis tool using binary relations as research objects, including order relations, the paper tries to build connections among concept lattice, inclusion degree, order relations, and further offers a new feasible way for analyzing and processing ordered information systems. The mainly contributions are listed as follows:

(1) For properly widening the research scope of order relations, the paper offers a kind of new way from the perspective of the integration theory of concept lattice and inclusion degree, which not only has good robustness and generalization ability, but also can meet actual needs flexibly.

(2) To solve problems such as algebraic structure, reduction, core, dependency in ordered information systems, the paper provides some new and simply ways within the framework of integration theory. Especially, for eliminating redundant dependencies, a new idea on the basis of inclusion-inference is proposed, by which a smaller dependency set referred to as a dependency generation set can be obtain. In short, the paper mainly focuses on the integration of concept lattice and inclusion degree to solve the problems in ordered information systems. Both theories and examples demonstrate the validity and rationality. Although some theoretical findings are achieved, which should be further supplemented and improved. Issues of how to autonomously determine parameter values, and how to extend concept lattice based on inclusion degree in more complex data sets, et al. will remain focuses of our future research.

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