Information Sciences 538 (2020) 396–414

Contents lists available at ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

On relationship between three-way concept lattices

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ARTICLE INFO

Article history:
Received 18 December 2019
Received in revised form 1 June 2020
Accepted 4 June 2020
Available online 12 June 2020

Keywords:
Formal concept
Formal concept lattice
Object-induced three-way concept
Attribute-induced three-way concept
Object-induced three-way concept lattice
Attribute-induced three-way concept lattice

ABSTRACT

By reformulating and extending the properties of three-way operators, this paper investigates the relationship between different kinds of three-way concept lattices. Three-way operators are defined through eight kinds of two-way operators which are connected by the complement operation. To examine the interrelations systematically, we study (a) the relationship between two-way operators, (b) the relationship between two-way concepts, (c) the relationship between three-way operators, and (d) the relationship between three-way concepts. The results show that the four kinds of object-induced three-way concept lattices are order isomorphic to each other and the four kinds of attribute-induced three-way concept lattices are also order isomorphic to each other.

1. Introduction

Concepts are fundamental units of conceptual knowledge presentation and processing. Proposed by Wille [38], formal concept analysis (FCA) provides an effective way to formulate and interpret concepts within a formal context. A formal context is a triple (U, V, R), where U is a set of objects, V is a set of attributes, and R is a binary relation over U and V; a concept is a pair $\langle X, A\rangle$, where $X \subseteq U$ and $A \subseteq V$. The object set X is the maximal set of objects having all properties in the attribute set A, called the extent of the concept; the attribute set A is the maximal set of attributes common to all objects in the object set X, called the intent of the concept.

Over last several decades, FCA has seen a rapid development both in theoretical foundation and in application [4,43,47,50]. Different kinds of concepts have been proposed according to different semantics. Existing results of FCA can be classified in the following way: (i) From the perspective of research technique, one relies on constructive approach or axiomatic approach or both to obtain different kinds of concepts. A constructive approach defines a concept through a pair of derivation operators [6,8,26,38,48]; an axiomatic approach characterizes concepts through a set of axioms [13,17,19,24,31,39]. (ii) From the perspective of data type, contexts are divided into formal context (which is the original context proposed by Wille [38]), incomplete context [2,5,18,44,48], **L**-context [1,10], multi-scale context [23,33], triadic context [15,35], decision context [37], etc. (iii) From the perspective of decision, one has two-way concepts (such as formal concept [8,38], property-oriented concept [6], object-oriented concept [22,40,41], cognitive concept [19,24], **L**-concept [1,3,9]) and three-way concepts (such as OE-concept [26], AE-concept [26], three-way cognitive concept [13,17,21], approximate concept [18,20], ill-known concept [5], neutrosophic concept [34]). For two-way concepts, both the intent and extent are represented by a single set; thus, the universes are divided into two disjoint parts. For three-way concepts, at least one of

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https://doi.org/10.1016/j.ins.2020.06.007
0020-0255/

the intent and extent is represented by a pair of sets or an interval set [44]. As a result, the universe is divided into three disjoint parts.

Three-way concept analysis is the outcome of the combination of FCA and three-way decision [11,12,14,28,42,45]. A semantic difference exists between three-way concepts in complete contexts and three-way concepts in incomplete contexts (in which the connections between some objects and attributes are unknown according to current information). Qi, Wei, and Yao [26] studied three-way concept analysis in complete formal contexts. They proposed two kinds of three-way concepts, namely, OE-concept and AE-concept. The intent of an OE-concept is composed by two parts: a set of attributes shared by all objects in the extent and a set of attributes not shared by any object in the extent. The extent of an AE-concept is also composed by two parts: a set of objects sharing all attributes in the intent and a set of objects not sharing any attribute in the intent. Followed by are OEP-concept and OED-concept proposed by Zhi et al. [49], and OEO-concept and AEP-concept proposed by Wei and Qian [36]. In [16], Li and Deng reviewed the development of three-way concept analysis and pointed out possible problems occurred in this area.

In an incomplete context, the actual intent and extent of a concept can not be determined precisely because of the unknown information. Burmeister and Holzer [2] generalized the standard derivation operators to a pair of modal-style derivation operators in incomplete contexts. The new operators give rise to a pair of certain and possible extents and a pair of certain and possible intents of a concept. Djouadi, Dubois, and Prade [5] represented an ill-known concept through a pair of formal concepts coming from the least and greatest completions of an incomplete context. Li, Mei, and Lv [18] introduced the notion of approximate concept with a pair of lower and upper operators. Adopting the idea of three-way concepts from Qi et al. [26,27], Li and Wang [20] constructed OE-approximate concepts and AE-approximate concepts through a pair of positive and negative operators. Based on interval interpretation of incomplete formal contexts, Yao [44] built a framework of three-way concept analysis for incomplete formal contexts. Four forms of partially-known concepts were investigated in the paper: SE-SI (i.e., set extent and set intent) concept, SE-ISI (i.e., set extent and interval set intent) concept, ISE-SI (i.e., interval set extent and set intent) concept, and ISE-ISI (i.e., interval set extent and interval set intent) concept. In this framework, the formal concept [38] is an example of Form SE-SI, the approximate concept [18] and OE-approximate concept [20] are examples of Form SE-ISI, the AE-approximate concept [20] is an example of Form ISE-SI, and the ill-known concept [5] is an example of Form ISE-SI.

The relationship between different kinds of concepts is a meaningful topic in FCA. In [32], Ren, Wei, and Yao studied the structures of and relationship between SE-ISI, ISE-SI, and ISE-ISI concepts. Qi et al. [25,27,29] investigated the relationship between three-way concept lattices and classical concept lattices. Qian, Wei, and Qi [30] studied the relationship between object (property) oriented concept lattice and three-way object (property) oriented concept lattice. In the framework of three-way granular computing, Yao [46] divided the eight kinds of two-way concepts into two groups—disjunctive group and conjunctive group—and investigated the relationship between concepts in each group. To the best of our knowledge, there is no unified framework showing the relationship between different kinds of three-way concept lattices. Inspired by the work of Yao [46], this paper mainly discusses the relationship between three-way concept lattices in complete formal contexts. Besides, the relationship between the eight kinds of two-way operators is investigated to facilitate the study of the relationship between three-way concepts.

The rest of this paper is organized as follows. Section 2 reviews basic notions and properties of formal concepts. In Section 3, we investigate first the relationship between and properties of two-way operators, and then the relationship between two-way concepts. In a similar way, Section 4 investigates the relationship between and properties of three-way operators and the relationship between three-way concepts. We prove the isomorphic relations between different kinds of three-way concept lattices. The last section serves as a conclusion part.

2. Formal concepts

A useful notion in FCA is the formal context which serves as a basic structure of the theory of FCA.

Definition 1 [8]. A formal context $K = (U, V, R)$ consists of two sets U and V and a relation R between U and V. The elements of U are called the objects of the context and the elements of V are called the attributes of the context.

Considering the maximal set of attributes shared by all objects in an object set and the maximal set of objects sharing all attributes in an attribute set, one achieves a pair of derivation operators.

Definition 2 [8]. Let $K = (U, V, R)$ be a formal context. For a set $X \subseteq U$ of objects, we define

$$X^* = \{a \in V | \forall x \in X (xRa)\}$$

the set of attributes common to the objects in X. Correspondingly, for a set $A \subseteq V$ of attributes, we define

$$A^* = \{x \in U | \forall a \in A (xRa)\}$$

the set of objects which have all attributes in A.

Applying the operator $*$ to a set of objects, one obtains a set of attributes owned by all objects in this object set. Similarly, applying the operator $*$ to a set of attributes, one obtains a set of objects having all attributes in this attribute set. An object set and an attribute set that mutually determine each other play a key role in FCA.

Definition 3 [8]. A formal concept of the context $K = (U, V, R)$ is a pair $\langle X, A\rangle$ with $X \subseteq U$ and $A \subseteq V$ such that $X^* = A$ and $A^* = X$. The set X is called the extent and A the intent of the concept $\langle X, A\rangle$.

Düntsch and Gediga [6,7] referred to the operator $*$ as a kind of modal-style operator, called sufficiency operator, and introduced another three kinds of modal-style operators: dual sufficiency operator $\#$, necessity operator $\Box$, and possibility operator $\Diamond$. Considering attributes which are not related to an object as negative attributes of the object, Qi et al. [26,27] introduced a kind of negative operator $\overline{*}$, called negative sufficiency operator. Afterwards, the negative necessity operator $\overline{\Box}$ and negative possibility operator $\overline{\Diamond}$ were given by Wei and Qian [36] and the negative dual sufficiency operator $\overline{\#}$ by Zhi et al. [49]. Each operator derives a kind of concept; each concept divides the object universe and attribute universe into two disjoint parts. To distinguish with three-way operators, we call the aforementioned eight kinds of derivation operators two-way operators [26], sometimes, object-induced two-way (short for O2W) operators and attribute-induced two-way (short for A2W) operators when considering different meanings of sets.

3. Relationship between two-way concept lattices

A formal context $K = (U, V, R)$ represents connections between objects and attributes through the binary relation R. Based on the connections, one can define eight kinds of mappings from 2^U to 2^V with existential and universal quantifiers in a natural way: for $X \in 2^U$,

$$\begin{aligned}
&f_1(X) = \{a \in V | \forall x \in X\,(xRa)\},\\
&f_2(X) = \{a \in V | \forall x \in X\,(\neg(xRa))\},\\
&f_3(X) = \{a \in V | \forall x \in X^c\,(xRa)\},\\
&f_4(X) = \{a \in V | \forall x \in X^c\,(\neg(xRa))\},\\
&f_5(X) = \{a \in V | \exists x \in X\,(xRa)\},\\
&f_6(X) = \{a \in V | \exists x \in X\,(\neg(xRa))\},\\
&f_7(X) = \{a \in V | \exists x \in X^c\,(xRa)\},\\
&f_8(X) = \{a \in V | \exists x \in X^c\,(\neg(xRa))\},
\end{aligned} \tag{1}$$

where $\neg$ is the logical negation. The eight mappings correspond to the eight O2W operators, respectively:

$$f_1(X) = X^*,\ f_2(X) = X^{\overline{*}},\ f_3(X) = X^{\overline{\Box}},\ f_4(X) = X^{\Box},\ f_5(X) = X^{\Diamond},\ f_6(X) = X^{\overline{\Diamond}},\ f_7(X) = X^{\overline{\#}},\ f_8(X) = X^{\#}. \tag{2}$$

The meanings of each obtained attribute set are listed below:

(1) X^* is the maximal set of attributes shared by all objects in X;
(2) $X^{\overline{*}}$ is the maximal set of attributes not possessed by any object in X;
(3) $X^{\overline{\Box}}$ is the maximal set of attributes shared by all objects in the complement of X;
(4) $X^{\Box}$ is the maximal set of attributes not possessed by any object in the complement of X;
(5) $X^{\Diamond}$ is the maximal set of attributes shared by at least one object in X;
(6) $X^{\overline{\Diamond}}$ is the maximal set of attributes not possessed by at least one object in X;
(7) $X^{\overline{\#}}$ is the maximal set of attributes shared by at least one object in the complement of X;
(8) $X^{\#}$ is the maximal set of attributes not possessed by at least one object in the complement of X.

Take as an example, Table 1 exhibits a formal context with $U = \{x_1, x_2, x_3, x_4\}$ and $V = \{a, b, c, d, e, f\}$. The notation 1 represents that an object has an attribute; the notation 0 means that an object does not have an attribute. For $X = \{x_1, x_2\}$, we have

Table 1
A formal context.

	a	b	c	d	e	f
x_1	0	1	0	1	1	0
x_2	1	1	1	0	1	0
x_3	0	1	0	1	0	1
x_4	1	1	1	0	0	0

$$X^{*} = \{b, e\},\ X^{\bar{*}} = \{f\},\ X^{\overline{\Box}} = \{b\},\ X^{\Box} = \{e\},$$
$$X^{\Diamond} = \{a, b, c, d, e\},\ X^{\overline{\Diamond}} = \{a, c, d, f\},\ X^{\overline{\#}} = \{a, b, c, d, f\},\ X^{\#} = \{a, c, d, e, f\}.$$

Remark 1. Above operators are revisited from the perspective of connections between objects and attributes. For simplicity, we do not change the notations of each operator which have already existed in other studies. It is, however, not difficult to find the correspondence with those notations used in [46], that is,

$$X^{*} = X^{+}_{\forall},\ X^{\Diamond} = X^{+}_{\exists},\ X^{\overline{\Box}} = X^{c+}_{\forall},\ X^{\overline{\#}} = X^{c+}_{\exists},\ X^{\overline{\Diamond}} = X^{+c}_{\forall},\ X^{\bar{*}} = X^{+c}_{\exists},\ X^{\#} = X^{c+c}_{\forall},\ X^{\Box} = X^{c+c}_{\exists}.$$

The difference is that the operators in [46] were revisited from the perspective of granule computing.

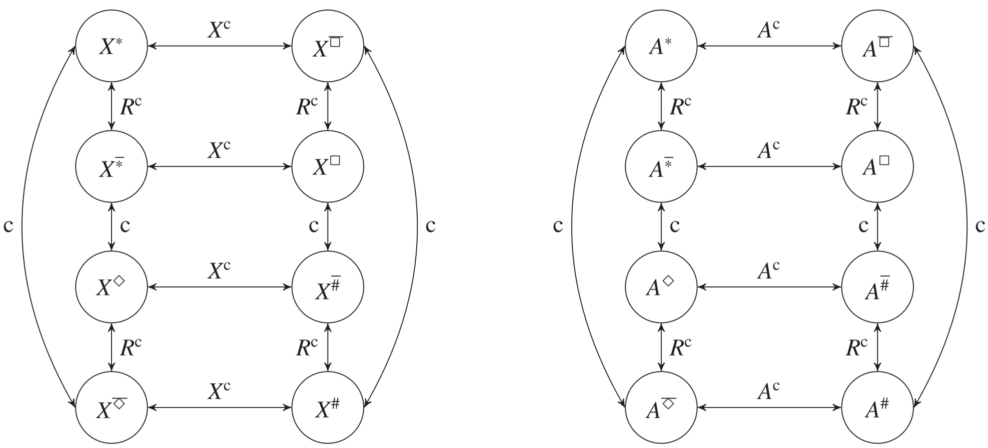
Since R is a binary relation, $\neg(xRa)$ is equivalent to $xR^{c}a$, where R^{c} is the complement of R, i.e., $R^{c} = U \times U - R$. We can thus rewrite definitions of negative sufficiency, necessity, negative possibility, and dual sufficiency operators in the following way:

$$\begin{aligned} X^{\bar{*}} &= \{a \in V \mid \forall x \in X\,(xR^{c}a)\},\\ X^{\Box} &= \{a \in V \mid \forall x \in X^{c}\,(xR^{c}a)\},\\ X^{\overline{\Diamond}} &= \{a \in V \mid \exists x \in X\,(xR^{c}a)\},\\ X^{\#} &= \{a \in V \mid \exists x \in X^{c}\,(xR^{c}a)\}. \end{aligned} \tag{3}$$

In other words, $\bar{*}$ can be regarded as $*$ (respectively, $\Box$ as $\overline{\Box}$, $\overline{\Diamond}$ as $\Diamond$, and $\#$ as $\overline{\#}$) defined in the dual formal context $K^{c} = (U, V, R^{c})$. Considering different semantics, however, each operator will be studied equally.

Inspired by Eqs. (1) and (2), the relationship between O2W operators is shown in Fig. 1a. Eight nodes represent the eight O2W operators, respectively. A double-headed arrow line connects a pair of operators from one of which the other can be obtained by taking the operation attached with the line. Totally, there are three different operations—X^{c}, R^{c}, and ^{c}—attached with lines. The notation X^{c} means the operators connected by the line can be converted into each other by replacing X with its complement X^{c}. For example, $X^{\overline{\Box}}$ can be obtained by substituting X^{c} for X in the definition of X^{*}, that is, $X^{\overline{\Box}} = (X^{c})^{*}$. The notation R^{c} means the operators connected by the line can be converted into each other by replacing R with its complement R^{c}. For example, $X^{\bar{*}}$ can be obtained by substituting X^{c} for X in the definition of X^{*}, that is, $X^{\bar{*}} = (X^{*})_{R^{c}}$. The notation ^{c} represents the complement operation. For example, $X^{\overline{\Diamond}}$ can be obtained by computing the complement of X^{*}, that is, $X^{\overline{\Diamond}} = (X^{*})^{c}$. From Fig. 1a, one can conclude that any two operators are mutually converted. For example, starting from X^{*}, one can get $X^{\#}$ in the following way: Replace X with X^{c} in X^{*} to obtain $X^{\overline{\Box}}$, and then compute the complement of $X^{\overline{\Box}}$ to obtain $X^{\#}$, namely, $X^{\#} = (X^{\overline{\Box}})^{c} = ((X^{c})^{*})^{c}$.

Table 2 summarizes the relationship between O2W operators illustrated by Fig. 1a. For notational simplicity, we omit parentheses by simply reading operations from left to right. Take as an example, $((X^{c})^{*})^{c}$ is simply denoted by X^{c*c}. The sixth line of Table 2 implies four dual pairs of operators: $(*, \#), (\Diamond, \Box), (\bar{*}, \overline{\#})$, and $(\overline{\Diamond}, \overline{\Box})$.



(a) Relationship between O2W operators. (b) Relationship between A2W operators.

Fig. 1. Relationship between two-way operators.

Table 2
Relationship between O2W operators.

$X^{*}=X_{R^c}^{\overline{*}}$	$X^{\Diamond}=X_{R^c}^{\overline{\Diamond}}$	$X^{\Box}=X_{R^c}^{\overline{\Box}}$	$X^{\#}=X_{R^c}^{\overline{\#}}$	$X^{\overline{*}}=X_{R^c}^{*}$	$X^{\overline{\Diamond}}=X_{R^c}^{\Diamond}$	$X^{\overline{\Box}}=X_{R^c}^{\Box}$	$X^{\overline{\#}}=X_{R^c}^{\#}$
$X^{*}=X^{c\overline{\Box}}$	$X^{\Diamond}=X^{c\overline{\#}}$	$X^{\Box}=X^{c\overline{*}}$	$X^{\#}=X^{c\overline{\Diamond}}$	$X^{\overline{*}}=X^{c\Box}$	$X^{\overline{\Diamond}}=X^{c\#}$	$X^{\overline{\Box}}=X^{c*}$	$X^{\overline{\#}}=X^{c\Diamond}$
$X^{*}=X^{\overline{\Diamond}c}$	$X^{\Diamond}=X^{\overline{*}c}$	$X^{\Box}=X^{\overline{\#}c}$	$X^{\#}=X^{\overline{\Box}c}$	$X^{\overline{*}}=X^{\Diamond c}$	$X^{\overline{\Diamond}}=X^{*c}$	$X^{\overline{\Box}}=X^{\#c}$	$X^{\overline{\#}}=X^{\Box c}$
$X^{*}=X_{R^c}^{c\Box}$	$X^{\Diamond}=X_{R^c}^{c\#}$	$X^{\Box}=X_{R^c}^{c*}$	$X^{\#}=X_{R^c}^{c\Diamond}$	$X^{\overline{*}}=X_{R^c}^{c\overline{\Box}}$	$X^{\overline{\Diamond}}=X_{R^c}^{c\overline{\#}}$	$X^{\overline{\Box}}=X_{R^c}^{c\overline{*}}$	$X^{\overline{\#}}=X_{R^c}^{c\overline{\Diamond}}$
$X^{*}=X_{R^c}^{\Diamond c}$	$X^{\Diamond}=X_{R^c}^{*c}$	$X^{\Box}=X_{R^c}^{\#c}$	$X^{\#}=X_{R^c}^{\Box c}$	$X^{\overline{*}}=X_{R^c}^{\overline{\Diamond}c}$	$X^{\overline{\Diamond}}=X_{R^c}^{\overline{*}c}$	$X^{\overline{\Box}}=X_{R^c}^{\overline{\#}c}$	$X^{\overline{\#}}=X_{R^c}^{\overline{\Box}c}$
$X^{*}=X^{c\#c}$	$X^{\Diamond}=X^{c\Box c}$	$X^{\Box}=X^{c\Diamond c}$	$X^{\#}=X^{c*c}$	$X^{\overline{*}}=X^{c\overline{\#}c}$	$X^{\overline{\Diamond}}=X^{c\overline{\Box}c}$	$X^{\overline{\Box}}=X^{c\overline{\Diamond}c}$	$X^{\overline{\#}}=X^{c\overline{*}c}$
$X^{*}=X_{R^c}^{c\overline{\#}c}$	$X^{\Diamond}=X_{R^c}^{c\overline{\Box}c}$	$X^{\Box}=X_{R^c}^{c\overline{\Diamond}c}$	$X^{\#}=X_{R^c}^{c\overline{*}c}$	$X^{\overline{*}}=X_{R^c}^{c\#c}$	$X^{\overline{\Diamond}}=X_{R^c}^{c\Box c}$	$X^{\overline{\Box}}=X_{R^c}^{c\Diamond c}$	$X^{\overline{\#}}=X_{R^c}^{c*c}$

The three operations, X^c, R^c, and ^{c}, are commutative with each other. For example, starting from X^*, one obtains $X^{\Box}$ by taking X^c first and R^c second, or R^c first and X^c second. But they are not commutative with any of the O2W operators. For instance, $X^{c\overline{\Box}} \neq X^{\overline{\Box}c}$, since $X^{c\overline{\Box}} = X^*$ and $X^{\overline{\Box}c} = X^{\#}$; $(X^*_{R^c})^* \neq (X^{**})_{R^c}$, since $(X^*_{R^c})^* = X^{\overline{*}*}$ and $(X^{**})_{R^c} = X^{*\overline{*}}$.

Remark 2.

(1) For an operator $X_{R^c}^{\overline{\bigstar}}$, which is obtained by replacing R with R^c in the definition of $X^{\overline{\bigstar}}$, one has two ways to interpret it: $X^{\bigstar}$ in the formal context K and $X^{\overline{\bigstar}}$ in the dual formal context K^c ($\bigstar = *, \Box, \Diamond$, and $\#$, respectively).
(2) In [46],Yao studied the connection between the eight kinds of two-way operators from the perspective of inference rules and exhibited the connection through two groups of hexagons. Each hexagon reveals the relationship between four kinds of two-way operators from positive or negative attribute view. In this study, we show the connection between the eight kinds of two-way operators from an overall view (see Fig. 1).

For comparative study, we summarize some basic properties of O2W operators.

Proposition 1. [8,36,38,41,46] *For a given formal context $K = (U, V, R)$ and $X, X_1, X_2 \subseteq U$, we have*

(1) $X_1 \subseteq X_2 \Rightarrow X_2^* \subseteq X_1^*,\ X_2^{\overline{*}} \subseteq X_1^{\overline{*}},\ X_1^{\overline{\Box}} \subseteq X_2^{\overline{\Box}},\ X_1^{\Box} \subseteq X_2^{\Box},\ X_1^{\overline{\Diamond}} \subseteq X_2^{\overline{\Diamond}},\ X_1^{\Diamond} \subseteq X_2^{\Diamond},\ X_2^{\#} \subseteq X_1^{\#},\ X_2^{\overline{\#}} \subseteq X_1^{\overline{\#}};$

(2) $X \subseteq X^{**},\ X \subseteq X^{\overline{**}},\ X^{\overline{\Box}\overline{\Diamond}} \subseteq X \subseteq X^{\overline{\Diamond}\overline{\Box}},\ X^{\Box\Diamond} \subseteq X \subseteq X^{\Diamond\Box},\ X \supseteq X^{\#\#},\ X \supseteq X^{\overline{\#\#}};$

(3) $X^* = X^{***},\ X^{\overline{*}} = X^{\overline{***}},\ X^{\overline{\Diamond\Box\Diamond}} = X^{\overline{\Diamond}},\ X^{\Diamond\Box\Diamond} = X^{\Diamond},\ X^{\overline{\Box\Diamond\Box}} = X^{\overline{\Box}},\ X^{\Box\Diamond\Box} = X^{\Box},\ X^{\#\#\#} = X^{\#},\ X^{\overline{\#\#\#}} = X^{\overline{\#}};$

(4) $(X_1 \cup X_2)^* = X_1^* \cap X_2^*,\ (X_1 \cap X_2)^{\overline{\Box}} = X_1^{\overline{\Box}} \cap X_2^{\overline{\Box}},\ (X_1 \cup X_2)^{\overline{\Diamond}} = X_1^{\overline{\Diamond}} \cup X_2^{\overline{\Diamond}},\ (X_1 \cap X_2)^{\#} = X_1^{\#} \cup X_2^{\#},$
$(X_1 \cup X_2)^{\overline{*}} = X_1^{\overline{*}} \cap X_2^{\overline{*}},\ (X_1 \cap X_2)^{\Box} = X_1^{\Box} \cap X_2^{\Box},\ (X_1 \cup X_2)^{\Diamond} = X_1^{\Diamond} \cup X_2^{\Diamond},\ (X_1 \cap X_2)^{\overline{\#}} = X_1^{\overline{\#}} \cup X_2^{\overline{\#}};$

(5) $(X_1 \cap X_2)^* \supseteq X_1^* \cup X_2^*,\ (X_1 \cup X_2)^{\overline{\Box}} \supseteq X_1^{\overline{\Box}} \cup X_2^{\overline{\Box}},\ (X_1 \cap X_2)^{\overline{\Diamond}} \subseteq X_1^{\overline{\Diamond}} \cap X_2^{\overline{\Diamond}},\ (X_1 \cup X_2)^{\#} \subseteq X_1^{\#} \cap X_2^{\#},$
$(X_1 \cap X_2)^{\overline{*}} \supseteq X_1^{\overline{*}} \cup X_2^{\overline{*}},\ (X_1 \cup X_2)^{\Box} \supseteq X_1^{\Box} \cup X_2^{\Box},\ (X_1 \cap X_2)^{\Diamond} \subseteq X_1^{\Diamond} \cap X_2^{\Diamond},\ (X_1 \cup X_2)^{\overline{\#}} \subseteq X_1^{\overline{\#}} \cap X_2^{\overline{\#}}.$

These properties can be mutually proved with the help of the information shown in Fig. 1a and Table 2. For example, suppose $X_2^* \subseteq X_1^*$ for $X_1 \subseteq X_2$, then $X_1^{\overline{\Box}} \subseteq X_2^{\overline{\Box}}$, since $X_1^{\overline{\Box}} = X_1^{c*}, X_2^{\overline{\Box}} = X_2^{c*}$, and $X_1^{c*} \subseteq X_2^{c*}$. Properties in (1) show the monotonicity of each operator. Properties in (2) show the relationship between the original set and the derived set by two applications of an operator. Properties in (3) show that the result of three applications of an operator is the same with the result of the first application of the operator. Properties in (4) and (5) show the distributivity of the eight operators with respect to set union and intersection.

Table 3 shows the connections between two derived sets by applying two operators on the same set. (Even though columns 2, 4, 6, and 8 contain the same results with columns 1, 3, 5, and 7, respectively, we keep them both to help to quickly check the properties.) Tables 2 and 3 still support the equivalent representations of three applications of an operator:

$$X^{***} = X^{\overline{\Diamond}\overline{\Box}\overline{\Diamond}c},\ X^{\Diamond\Diamond\Diamond} = X^{\overline{*}\overline{\#}\overline{*}c},\ X^{\Box\Box\Box} = X^{\overline{\#}\overline{*}\overline{\#}c},\ X^{\#\#\#} = X^{\overline{\Box}\overline{\Diamond}\overline{\Box}c},$$
$$X^{\overline{***}} = X^{\Diamond\Box\Diamond c},\ X^{\overline{\Diamond\Diamond\Diamond}} = X^{*\#*c},\ X^{\overline{\Box\Box\Box}} = X^{\#*\#c},\ X^{\overline{\#\#\#}} = X^{\Box\Diamond\Box c}. \tag{4}$$

In the very same way, for a given attribute set $A \subseteq V$, one can define eight kinds of A2W operators using existential and universal quantifiers:

Table 3
Connections between two derived sets by applying two operators.

$X^{**}=X^{\overline{\Diamond}\overline{\Box}}$	$X^{*\overline{\Box}}=X^{\overline{\Diamond}*}$	$X^{*\overline{\Diamond}}=X^{\overline{\Diamond}\#}$	$X^{*\#}=X^{\overline{\Diamond\Diamond}}$	$X^{*\overline{*}}=X^{\overline{\Diamond}\Box}$	$X^{*\Box}=X^{\overline{\Diamond}\overline{*}}$	$X^{*\Diamond}=X^{\overline{\Diamond}\overline{\#}}$	$X^{*\overline{\#}}=X^{\overline{\Diamond}\Diamond}$
$X^{\overline{\Box}*}=X^{\#\overline{\Box}}$	$X^{\overline{\Box\Box}}=X^{\#*}$	$X^{\overline{\Box\Diamond}}=X^{\#\#}$	$X^{\overline{\Box}\#}=X^{\#\overline{\Diamond}}$	$X^{\overline{\Box}\overline{*}}=X^{\#\Box}$	$X^{\overline{\Box}\Box}=X^{\#\overline{*}}$	$X^{\overline{\Box}\Diamond}=X^{\#\overline{\#}}$	$X^{\overline{\Box\#}}=X^{\#\Diamond}$
$X^{\overline{\Diamond}*}=X^{*\overline{\Box}}$	$X^{\overline{\Diamond}\overline{\Box}}=X^{**}$	$X^{\overline{\Diamond\Diamond}}=X^{*\#}$	$X^{\overline{\Diamond}\#}=X^{*\overline{\Diamond}}$	$X^{\overline{\Diamond}\overline{*}}=X^{*\Box}$	$X^{\overline{\Diamond}\Box}=X^{*\overline{*}}$	$X^{\overline{\Diamond}\Diamond}=X^{*\overline{\#}}$	$X^{\overline{\Diamond}\overline{\#}}=X^{*\Diamond}$
$X^{\#*}=X^{\overline{\Box\Box}}$	$X^{\#\overline{\Box}}=X^{\overline{\Box}*}$	$X^{\#\overline{\Diamond}}=X^{\overline{\Box}\#}$	$X^{\#\#}=X^{\overline{\Box}\overline{\Diamond}}$	$X^{\#\overline{*}}=X^{\overline{\Box}\Box}$	$X^{\#\Box}=X^{\overline{\Box}\overline{*}}$	$X^{\#\Diamond}=X^{\overline{\Box\#}}$	$X^{\#\overline{\#}}=X^{\overline{\Box}\Diamond}$
$X^{\overline{*}*}=X^{\Diamond\overline{\Box}}$	$X^{\overline{*}\overline{\Box}}=X^{\Diamond*}$	$X^{\overline{*}\overline{\Diamond}}=X^{\Diamond\#}$	$X^{\overline{*}\#}=X^{\Diamond\overline{\Diamond}}$	$X^{\overline{**}}=X^{\Diamond\Box}$	$X^{\overline{*}\Box}=X^{\Diamond\overline{*}}$	$X^{\overline{*}\Diamond}=X^{\Diamond\overline{\#}}$	$X^{\overline{*}\overline{\#}}=X^{\Diamond\Diamond}$
$X^{\Box*}=X^{\overline{\#\Box}}$	$X^{\Box\overline{\Box}}=X^{\overline{\#}*}$	$X^{\Box\overline{\Diamond}}=X^{\overline{\#}\#}$	$X^{\Box\#}=X^{\overline{\#}\overline{\Diamond}}$	$X^{\Box\overline{*}}=X^{\overline{\#}\Box}$	$X^{\Box\Box}=X^{\overline{\#}\overline{*}}$	$X^{\Box\Diamond}=X^{\overline{\#\#}}$	$X^{\Box\overline{\#}}=X^{\overline{\#}\Diamond}$
$X^{\Diamond*}=X^{\overline{*}\overline{\Box}}$	$X^{\Diamond\overline{\Box}}=X^{\overline{*}*}$	$X^{\Diamond\overline{\Diamond}}=X^{\overline{*}\#}$	$X^{\Diamond\#}=X^{\overline{*}\overline{\Diamond}}$	$X^{\Diamond\ \overline{*}}=X^{\overline{*}\Box}$	$X^{\Diamond\Box}=X^{\overline{**}}$	$X^{\Diamond\Diamond}=X^{\overline{*}\overline{\#}}$	$X^{\Diamond\overline{\#}}=X^{\overline{*}\Diamond}$
$X^{\overline{\#}*}=X^{\Box\overline{\Box}}$	$X^{\overline{\#\Box}}=X^{\Box*}$	$X^{\overline{\#\Diamond}}=X^{\Box\#}$	$X^{\overline{\#}\#}=X^{\Box\overline{\Diamond}}$	$X^{\overline{\#}\overline{*}}=X^{\Box\Box}$	$X^{\overline{\#}\Box}=X^{\Box\overline{*}}$	$X^{\overline{\#}\Diamond}=X^{\Box\overline{\#}}$	$X^{\overline{\#\#}}=X^{\Box\Diamond}$

$$
\begin{aligned}
A^{*} &= \{x \in U \mid \forall a \in A\,(xRa)\},\\
A^{\overline{*}} &= \{x \in U \mid \forall a \in A\,(xR^{c}a)\},\\
A^{\overline{\Box}} &= \{x \in U \mid \forall a \in A^{c}\,(xRa)\},\\
A^{\Box} &= \{x \in U \mid \forall a \in A^{c}\,(xR^{c}a)\},\\
A^{\Diamond} &= \{x \in U \mid \exists a \in A\,(xRa)\},\\
A^{\overline{\Diamond}} &= \{x \in U \mid \exists a \in A\,(xR^{c}a)\},\\
A^{\overline{\#}} &= \{x \in U \mid \exists a \in A^{c}\,(xRa)\},\\
A^{\#} &= \{x \in U \mid \exists a \in A^{c}\,(xR^{c}a)\}.
\end{aligned}
$$

The meanings of each obtained object set are listed below:

(1) A^{*} is the maximal set of objects sharing all attributes in A;
(2) $A^{\overline{*}}$ is the maximal set of objects not possessing any attribute in A;
(3) $A^{\overline{\Box}}$ is the maximal set of objects sharing all attributes in the complement of A;
(4) $A^{\Box}$ is the maximal set of objects not possessing any attribute in the complement of A;
(5) $A^{\Diamond}$ is the maximal set of objects sharing at least one attribute in A;
(6) $A^{\overline{\Diamond}}$ is the maximal set of objects not possessing at least one attribute in A;
(7) $A^{\overline{\#}}$ is the maximal set of objects sharing at least one attribute in the complement of A;
(8) $A^{\#}$ is the maximal set of objects not possessing at least one attribute in the complement of A.

Remark 3. If we do not consider the concrete meanings of sets, each A2W operator just corresponds to one O2W operator. Therefore, the relationship between the eight kinds of A2W operators is the same with that between O2W operators (shown in Fig. 1b). Properties of A2W operators are consequently the same with those of O2W operators. One actually obtains the properties of A2W operators by simply replacing the object set X with an attribute set A in Proposition 1 and Tables 2 and 3.

The connections between O2W operators and A2W operators are listed as follows. They can be mutually proved as the properties in Proposition 1.

Proposition 2. [36,38,41,46]*Let $K=(U,V,R)$ be a formal context, $X\subseteq U$, and $A\subseteq V$. Then,*

(1) $X\subseteq A^{*} \iff A\subseteq X^{*}$;
(2) $X\subseteq A^{\overline{*}} \iff A\subseteq X^{\overline{*}}$;
(3) $X\subseteq A^{\overline{\Box}} \iff X^{\overline{\Diamond}}\subseteq A$;
(4) $X\subseteq A^{\Box} \iff X^{\Diamond}\subseteq A$;
(5) $X\supseteq A^{\Diamond} \iff X^{\Box}\supseteq A$;
(6) $X\supseteq A^{\overline{\Diamond}} \iff X^{\overline{\Box}}\supseteq A$;
(7) $X\supseteq A^{\overline{\#}} \iff X^{\overline{\#}}\supseteq A$;
(8) $X\supseteq A^{\#} \iff X^{\#}\supseteq A$.

Based on different O2W operators and A2W operators, we obtain different kinds of two-way concepts.

Definition 4. Let $K=(U,V,R)$ be a formal context, $X\subseteq U$, and $A\subseteq V$. Then,

(1) $\langle X,A\rangle$ is a $*$-concept (i.e., formal concept in [8] or positive formal concept in [46]) if $X^{*}=A$ and $A^{*}=X$;
(2) $\langle X,A\rangle$ is a $\Box\Diamond$-concept (i.e., object-oriented formal concept in [41] or positive object-oriented formal concept in [46]) if $X^{\Box}=A$ and $A^{\Diamond}=X$;
(3) $\langle X,A\rangle$ is a $\Diamond\Box$-concept (i.e., property-oriented formal concept in [6,41] or positive attribute-oriented formal concept in [46]) if $X^{\Diamond}=A$ and $A^{\Box}=X$;
(4) $\langle X,A\rangle$ is a $\#$-concept (i.e., dual positive formal concept in [46]) if $X^{\#}=A$ and $A^{\#}=X$;
(5) $\langle X,A\rangle$ is a $\overline{*}$-concept (i.e., formal concept in [40] or negative formal concept in [46]) if $X^{\overline{*}}=A$ and $A^{\overline{*}}=X$;
(6) $\langle X,A\rangle$ is a $\overline{\Box}\overline{\Diamond}$-concept (i.e., negative object-induced formal concept in [46]) if $X^{\overline{\Box}}=A$ and $A^{\overline{\Diamond}}=X$;
(7) $\langle X,A\rangle$ is a $\overline{\Diamond}\overline{\Box}$-concept (i.e., negative attribute-induced formal concept in [46]) if $X^{\overline{\Diamond}}=A$ and $A^{\overline{\Box}}=X$;
(8) $\langle X,A\rangle$ is a $\overline{\#}$-concept (i.e., dual negative formal concept in [46]) if $X^{\overline{\#}}=A$ and $A^{\overline{\#}}=X$.

A $\overline{*}$-concept in K is a $*$-concept in K^{c}, and a $*$-concept in K is a $\overline{*}$-concept in K^{c}. The same correspondences exist between $\overline{\Box}\overline{\Diamond}$-concepts and $\Box\Diamond$-concepts, $\overline{\Diamond}\overline{\Box}$-concepts and $\Diamond\Box$-concepts, $\overline{\#}$-concepts and $\#$-concepts, respectively. Besides, we have the following equivalences between $*$-concept, $\overline{\Box}\overline{\Diamond}$-concept, $\#$-concept, and $\overline{\Diamond}\overline{\Box}$-concept.

Theorem 1. [[46] *Let (U,V,R) be a formal context, $X\subseteq U$, and $A\subseteq V$. Then, the following statements are equivalent:*

(1) $\langle X,A\rangle$ *is a* $*$*-concept;*
(2) $\langle X^{c},A\rangle$ *is* $\overline{\Box}\overline{\Diamond}$*-concept;*
(3) $\langle X,A^{c}\rangle$ *is a* $\overline{\Diamond}\overline{\Box}$*-concept;*
(4) $\langle X^{c},A^{c}\rangle$ *is a* $\#$*-concept.*

Proof. We first prove that Item (1) is equivalent to Item (2). According to Definition 4, Table 2, and Remark 3, we have

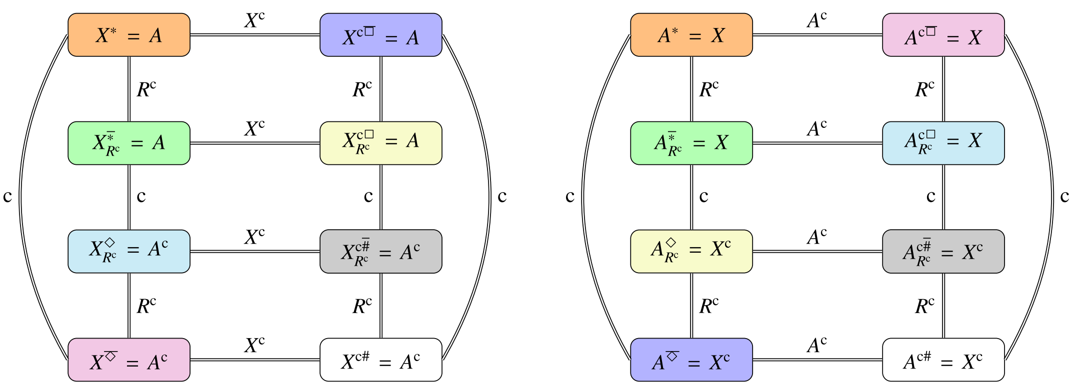
$$\langle X,A\rangle \text{ is a } *-\text{concept} \iff X^{*}=A,\ A^{*}=X \iff X^{c\overline{\Box}}=A,\ A^{\overline{\Diamond}c}=X \iff X^{c\overline{\Box}}=A,\ A^{\overline{\Diamond}}=X^{c} \iff \langle X^{c},A\rangle \text{ is a } \overline{\Box}\overline{\Diamond}-\text{concept}.$$

The others can be proved similarly based on the information shown in Fig. 2. (A double line connects two equivalent statements, and the equivalence can be proved by taking the operation attached with the line. Two nodes of the same color in Fig. 2 correspond to a kind of two-way concept.)

Expressing one kind of two-way concept in terms of another, although theoretically possible, is somewhat inconvenient in formulation and interpretation. From a practical point of view, it is useful to keep the eight kinds of two-way concepts. A subset of objects generates a two-way concept; also, a subset of attributes generates a two-way concept.

Theorem 2. [8,36,38,41,46]*Let $K=(U,V,R)$ be a formal context, $X\subseteq U$, and $A\subseteq V$. Then,*

(1) $\langle X^{**},X^{*}\rangle$ and $\langle A^{*},A^{**}\rangle$ are $*$-concepts;
(2) $\langle X^{\overline{**}},X^{\overline{*}}\rangle$ and $\langle A^{\overline{*}},A^{\overline{**}}\rangle$ are $\overline{*}$-concepts;



(a) Equivalences between O2W operators. (b) Equivalences between A2W operators.

Fig. 2. Equivalences between two-way concepts.

(3) $\left\langle X^{\overline{\Diamond}\overline{\Box}}, X^{\overline{\Diamond}} \right\rangle$ and $\left\langle A^{\overline{\Box}}, A^{\overline{\Box}\overline{\Diamond}} \right\rangle$ are $\overline{\Diamond}\overline{\Box}$-concepts;

(4) $\left\langle X^{\Diamond\Box}, X^{\Diamond} \right\rangle$ and $\left\langle A^{\Box}, A^{\Box\Diamond} \right\rangle$ are $\Diamond\Box$-concepts;

(5) $\left\langle X^{\overline{\Box}\overline{\Diamond}}, X^{\overline{\Box}} \right\rangle$ and $\left\langle A^{\overline{\Diamond}}, A^{\overline{\Diamond}\overline{\Box}} \right\rangle$ are $\overline{\Box}\overline{\Diamond}$-concepts;

(6) $\left\langle X^{\Box\Diamond}, X^{\Box} \right\rangle$ and $\left\langle A^{\Diamond}, A^{\Diamond\Box} \right\rangle$ are $\Box\Diamond$-concepts;

(7) $\langle X^{\#\#}, X^{\#} \rangle$ and $\left\langle A^{\#}, A^{\#\#} \right\rangle$ are $\#$-concepts;

(8) $\left\langle X^{\overline{\#\#}}, X^{\overline{\#}} \right\rangle$ and $\left\langle A^{\overline{\#}}, A^{\overline{\#\#}} \right\rangle$ are $\overline{\#}$-concepts.

Theorem 2 provides us a way to construct different kinds of two-way concepts by using object sets and attribute sets. The order, supremum, and infimum of two-way concepts are defined as follows.

Definition 5. Let $K = (U, V, R)$ be a formal context, $X_1, X_2 \subseteq U$, and $A_1, A_2 \subseteq V$.

(1) [8] For two $*$-concepts $\langle X_1, A_1 \rangle$ and $\langle X_2, A_2 \rangle$, $\langle X_1, A_1 \rangle \leq_* \langle X_2, A_2 \rangle$ iff $X_1 \subseteq X_2$ (equivalently, $A_2 \subseteq A_1$), and

$$\begin{aligned} \langle X_1, A_1 \rangle \wedge_* \langle X_2, A_2 \rangle &= \langle X_1 \cap X_2, (A_1 \cup A_2)^{**} \rangle = \langle X_1 \cap X_2, (X_1 \cap X_2)^{*} \rangle, \\ \langle X_1, A_1 \rangle \vee_* \langle X_2, A_2 \rangle &= \langle (X_1 \cup X_2)^{**}, A_1 \cap A_2 \rangle = \langle (A_1 \cap A_2)^{*}, A_1 \cap A_2 \rangle. \end{aligned} \tag{5}$$

(2) [46] For two $\overline{*}$-concepts $\langle X_1, A_1 \rangle$ and $\langle X_2, A_2 \rangle$, $\langle X_1, A_1 \rangle \leq_{\overline{*}} \langle X_2, A_2 \rangle$ iff $X_1 \subseteq X_2$ (equivalently, $A_2 \subseteq A_1$), and

$$\begin{aligned} \langle X_1, A_1 \rangle \wedge_{\overline{*}} \langle X_2, A_2 \rangle &= \langle X_1 \cap X_2, (A_1 \cup A_2)^{\overline{**}} \rangle = \langle X_1 \cap X_2, (X_1 \cap X_2)^{\overline{*}} \rangle, \\ \langle X_1, A_1 \rangle \vee_{\overline{*}} \langle X_2, A_2 \rangle &= \langle (X_1 \cup X_2)^{\overline{**}}, A_1 \cap A_2 \rangle = \langle (A_1 \cap A_2)^{\overline{*}}, A_1 \cap A_2 \rangle. \end{aligned} \tag{6}$$

(3) For two $\overline{\Box}\overline{\Diamond}$-concepts $\langle X_1, A_1 \rangle$ and $\langle X_2, A_2 \rangle$, $\langle X_1, A_1 \rangle \leq_{\overline{\Box}\overline{\Diamond}} \langle X_2, A_2 \rangle$ iff $X_2 \subseteq X_1$ (equivalently, $A_2 \subseteq A_1$), and

$$\begin{aligned} \langle X_1, A_1 \rangle \wedge_{\overline{\Box}\overline{\Diamond}} \langle X_2, A_2 \rangle &= \left\langle X_1 \cup X_2, (A_1 \cup A_2)^{\overline{\Diamond}\overline{\Box}} \right\rangle = \left\langle X_1 \cup X_2, (X_1 \cup X_2)^{\overline{\Box}} \right\rangle, \\ \langle X_1, A_1 \rangle \vee_{\overline{\Box}\overline{\Diamond}} \langle X_2, A_2 \rangle &= \left\langle (X_1 \cap X_2)^{\overline{\Box}\overline{\Diamond}}, A_1 \cap A_2 \right\rangle = \left\langle (A_1 \cap A_2)^{\overline{\Diamond}}, A_1 \cap A_2 \right\rangle. \end{aligned} \tag{7}$$

(4) For two $\Box\Diamond$-concepts $\langle X_1, A_1 \rangle$ and $\langle X_2, A_2 \rangle$, $\langle X_1, A_1 \rangle \leq_{\Box\Diamond} \langle X_2, A_2 \rangle$ iff $X_2 \subseteq X_1$ (equivalently, $A_2 \subseteq A_1$), and

$$\begin{aligned} \langle X_1, A_1 \rangle \wedge_{\Box\Diamond} \langle X_2, A_2 \rangle &= \left\langle X_1 \cup X_2, (A_1 \cup A_2)^{\Diamond\Box} \right\rangle = \left\langle X_1 \cup X_2, (X_1 \cup X_2)^{\Box} \right\rangle, \\ \langle X_1, A_1 \rangle \vee_{\Box\Diamond} \langle X_2, A_2 \rangle &= \left\langle (X_1 \cap X_2)^{\Box\Diamond}, A_1 \cap A_2 \right\rangle = \left\langle (A_1 \cap A_2)^{\Diamond}, A_1 \cap A_2 \right\rangle. \end{aligned} \tag{8}$$

(5) [46] For two $\overline{\Diamond}\overline{\Box}$-concepts $\langle X_1, A_1 \rangle$ and $\langle X_2, A_2 \rangle$, $\langle X_1, A_1 \rangle \leq_{\overline{\Diamond}\overline{\Box}} \langle X_2, A_2 \rangle$ iff $X_1 \subseteq X_2$ (equivalently, $A_1 \subseteq A_2$), and

$$\begin{aligned} \langle X_1, A_1 \rangle \wedge_{\overline{\Diamond}\overline{\Box}} \langle X_2, A_2 \rangle &= \left\langle X_1 \cap X_2, (A_1 \cap A_2)^{\overline{\Box}\overline{\Diamond}} \right\rangle = \left\langle X_1 \cap X_2, (X_1 \cap X_2)^{\overline{\Diamond}} \right\rangle, \\ \langle X_1, A_1 \rangle \vee_{\overline{\Diamond}\overline{\Box}} \langle X_2, A_2 \rangle &= \left\langle (X_1 \cup X_2)^{\overline{\Diamond}\overline{\Box}}, A_1 \cup A_2 \right\rangle = \left\langle (A_1 \cup A_2)^{\overline{\Box}}, A_1 \cup A_2 \right\rangle. \end{aligned} \tag{9}$$

(6) [40,46] For two $\Diamond\Box$-concepts $\langle X_1, A_1 \rangle$ and $\langle X_2, A_2 \rangle$, $\langle X_1, A_1 \rangle \leq_{\Diamond\Box} \langle X_2, A_2 \rangle$ iff $X_1 \subseteq X_2$ (equivalently, $A_1 \subseteq A_2$), and

$$\begin{aligned} \langle X_1, A_1 \rangle \wedge_{\Diamond\Box} \langle X_2, A_2 \rangle &= \left\langle X_1 \cap X_2, (A_1 \cap A_2)^{\Box\Diamond} \right\rangle = \left\langle X_1 \cap X_2, (X_1 \cap X_2)^{\Diamond} \right\rangle, \\ \langle X_1, A_1 \rangle \vee_{\Diamond\Box} \langle X_2, A_2 \rangle &= \left\langle (X_1 \cup X_2)^{\Diamond\Box}, A_1 \cup A_2 \right\rangle = \left\langle (A_1 \cup A_2)^{\Box}, A_1 \cup A_2 \right\rangle. \end{aligned} \tag{10}$$

(7) For two $\#$-concepts $\langle X_1, A_1 \rangle$ and $\langle X_2, A_2 \rangle$, $\langle X_1, A_1 \rangle \leq_{\#} \langle X_2, A_2 \rangle$ iff $X_2 \subseteq X_1$ (equivalently, $A_1 \subseteq A_2$), and

$$\begin{aligned} \langle X_1, A_1 \rangle \wedge_{\#} \langle X_2, A_2 \rangle &= \left\langle X_1 \cup X_2, (A_1 \cap A_2)^{\#\#} \right\rangle = \left\langle X_1 \cup X_2, (X_1 \cup X_2)^{\#} \right\rangle, \\ \langle X_1, A_1 \rangle \vee_{\#} \langle X_2, A_2 \rangle &= \left\langle (X_1 \cap X_2)^{\#\#}, A_1 \cup A_2 \right\rangle = \left\langle (A_1 \cup A_2)^{\#}, A_1 \cup A_2 \right\rangle. \end{aligned} \tag{11}$$

(8) For two $\overline{\#}$-concepts $\langle X_1, A_1 \rangle$ and $\langle X_2, A_2 \rangle$, $\langle X_1, A_1 \rangle \leq_{\overline{\#}} \langle X_2, A_2 \rangle$ iff $X_2 \subseteq X_1$ (equivalently, $A_1 \subseteq A_2$), and

$$\begin{aligned} \langle X_1, A_1 \rangle \wedge_{\overline{\#}} \langle X_2, A_2 \rangle &= \left\langle X_1 \cup X_2, (A_1 \cap A_2)^{\overline{\#\#}} \right\rangle = \left\langle X_1 \cup X_2, (X_1 \cup X_2)^{\overline{\#}} \right\rangle, \\ \langle X_1, A_1 \rangle \vee_{\overline{\#}} \langle X_2, A_2 \rangle &= \left\langle (X_1 \cap X_2)^{\overline{\#\#}}, A_1 \cup A_2 \right\rangle = \left\langle (A_1 \cup A_2)^{\overline{\#}}, A_1 \cup A_2 \right\rangle. \end{aligned} \tag{12}$$

Denote by $C^{\star}(K)$ the set of all $\star$-concepts in the formal context $K = (U, V, R)$, where $\star = *, \overline{*}, \overline{\Box}\overline{\Diamond}, \Box\Diamond, \overline{\Diamond}\overline{\Box}, \Diamond\Box, \#$, and $\overline{\#}$, respectively. According to the equivalences between two-way concepts shown in Theorem 1 and Fig. 2, we have changed the order of $\overline{\Box}\overline{\Diamond}$-concepts, $\Box\Diamond$-concepts, $\#$-concepts, and $\overline{\#}$-concepts, respectively, compared with those in [46]; consequently, the infimums and supremums defined in Eqs. (7), (8), (11), and (12) are different from those in [46]. The collection of two-way concepts of the same kind forms a complete lattice with the corresponding infimum and supremum defined in Definition 5.

Theorem 3 [46]. *Let $K=(U,V,R)$ be a formal context. Then, $\left(C^{\bigstar}(K),\wedge_{\bigstar},\vee_{\bigstar}\right)$ is a complete lattice, where $\bigstar=*,\overline{*},\overline{\Box\Diamond},\Box\Diamond,\overline{\Diamond\Box},\Diamond\Box,\#$, and $\overline{\#}$, respectively.*

Proof. (We only prove that $(C^{\#}(K),\wedge_{\#},\vee_{\#})$ is a complete lattice, for the others can be similarly proved.) Let $\langle X_1,A_1\rangle,\langle X_2,A_2\rangle\in C^{\#}(K)$. Using Proposition 1, we have

$$(A_1\cap A_2)^{\#\#\#}=(A_1\cap A_2)^{\#}=A_1^{\#}\cup A_2^{\#}=X_1\cup X_2,$$
$$(X_1\cup X_2)^{\#}=\left((A_1\cap A_2)^{\#\#\#}\right)^{\#}=\left((A_1\cap A_2)^{\#}\right)^{\#}=(A_1\cap A_2)^{\#\#},$$

which means $\left\langle X_1\cup X_2,(A_1\cap A_2)^{\#\#}\right\rangle\in C^{\#}(K)$.

By Definition 5, Proposition 1, and Remark 3, it is straightforward that

$$\langle X_1\cup X_2,(A_1\cap A_2)^{\#\#}\rangle\le_{\#}\langle X_1,A_1\rangle,$$
$$\langle X_1\cup X_2,(A_1\cap A_2)^{\#\#}\rangle\le_{\#}\langle X_2,A_2\rangle,$$

which support that $\left\langle X_1\cup X_2,(A_1\cap A_2)^{\#\#}\right\rangle$ is a lower bound of $\langle X_1,A_1\rangle$ and $\langle X_2,A_2\rangle$.

Next, we prove that $\left\langle X_1\cup X_2,(A_1\cap A_2)^{\#\#}\right\rangle$ is the infimum of $\langle X_1,A_1\rangle$ and $\langle X_2,A_2\rangle$. If not, suppose $\langle X,A\rangle\le_{\#}\langle X_1,A_1\rangle,\langle X,A\rangle\le_{\#}\langle X_2,A_2\rangle$, and $\left\langle X_1\cup X_2,(A_1\cap A_2)^{\#\#}\right\rangle\le_{\#}\langle X,A\rangle$. Using Definition 5, we have $X_1\subseteq X$ and $X_2\subseteq X$ (which follow the result that $X_1\cup X_2\subseteq X$), and at the same time $X\subseteq X_1\cup X_2$. Finally, we obtain $X=X_1\cup X_2$; besides, $A=X^{\#}=(X_1\cup X_2)^{\#}=(A_1\cap A_2)^{\#\#}$. Equivalently saying, $\left\langle X_1\cup X_2,(A_1\cap A_2)^{\#\#}\right\rangle$ is the infimum of $\langle X_1,A_1\rangle$ and $\langle X_2,A_2\rangle$.

In a similar way, one can prove that $\left\langle (X_1\cap X_2)^{\#\#},A_1\cup A_2\right\rangle$ is a $\#$-concept and also the supremum of $\langle X_1,A_1\rangle$ and $\langle X_2,A_2\rangle$. Consequently, $\left(C^{\#}(K),\wedge_{\#},\vee_{\#}\right)$ is a complete lattice.

The results in Fig. 2 imply two groups of isomorphism relation among the eight kinds of concept lattices.

Theorem 4. *Let $K=(U,V,R)$ be a formal context. Then,*

(1) $C^{*}(K)\cong C^{\overline{\Box\Diamond}}(K)\cong C^{\overline{\Diamond\Box}}(K)\cong C^{\#}(K)$;
(2) $C^{\overline{*}}(K)\cong C^{\Box\Diamond}(K)\cong C^{\Diamond\Box}(K)\cong C^{\overline{\#}}(K)$.

The notation $\cong$ means isomorphic relation.

Proof.

(1) Given $\langle X,A\rangle\in C^{*}(K)$, let $f(\langle X,A\rangle)=\langle X^{c},A\rangle$. Then, $\langle X^{c},A\rangle\in C^{\overline{\Box\Diamond}}(K)$, and f is a bijection between $C^{*}(K)$ and $C^{\overline{\Box\Diamond}}(K)$ by Theorem 1.
Suppose $\langle X_1,A_1\rangle,\langle X_2,A_2\rangle\in C^{*}(K)$, then according to Eqs. (5) and (7), we have

$$f(\langle X_1,A_1\rangle\wedge_{*}\langle X_2,A_2\rangle)=f(\langle X_1\cap X_2,(A_1\cup A_2)^{**}\rangle)=\langle (X_1\cap X_2)^{c},(A_1\cup A_2)^{**}\rangle,$$
$$f(\langle X_1,A_1\rangle)\wedge_{\overline{\Box\Diamond}}f(\langle X_2,A_2\rangle)=\langle X_1^{c},A_1\rangle\wedge_{\overline{\Box\Diamond}}\langle X_2^{c},A_2\rangle=\left\langle X_1^{c}\cup X_2^{c},(A_1\cup A_2)^{\overline{\Diamond\Box}}\right\rangle.$$

It is obvious that $(X_1\cap X_2)^{c}=X_1^{c}\cup X_2^{c}$, and at the same time it follows from Table 3 that $(A_1\cup A_2)^{**}=(A_1\cup A_2)^{\overline{\Diamond\Box}}$. Equivalently saying, we have

$$f(\langle X_1,A_1\rangle\wedge_{*}\langle X_2,A_2\rangle)=f(\langle X_1,A_1\rangle)\wedge_{\overline{\Box\Diamond}}f(\langle X_2,A_2\rangle),$$

which demonstrates that f is $\wedge$-preserving.
On the other hand, it follows from Eqs. (5) and (7) that

$$f(\langle X_1,A_1\rangle\vee_{*}\langle X_2,A_2\rangle)=f(\langle (X_1\cup X_2)^{**},A_1\cap A_2\rangle)=\langle (X_1\cup X_2)^{**c},A_1\cap A_2\rangle,$$
$$f(\langle X_1,A_1\rangle)\vee_{\overline{\Box\Diamond}}f(\langle X_2,A_2\rangle)=\langle X_1^{c},A_1\rangle\vee_{\overline{\Box\Diamond}}\langle X_2^{c},A_2\rangle=\langle (X_1^{c}\cap X_2^{c})^{\overline{\Box\Diamond}},A_1\cap A_2\rangle,$$

and from Table 2 that

$$\begin{aligned}(X_1 \cup X_2)^{**c} &= (X_1 \cup X_2)^{*\overline{\Diamond}},\\ \left(X_1^c \cap X_2^c\right)^{\overline{\Box\Diamond}} = \left((X_1 \cup X_2)^c\right)^{\overline{\Box\Diamond}} &= (X_1 \cup X_2)^{c\overline{\Box\Diamond}} = (X_1 \cup X_2)^{*\overline{\Diamond}}.\end{aligned}$$

Equivalently saying, we have

$$f(\langle X_1, A_1\rangle \vee_* \langle X_2, A_2\rangle) = f(\langle X_1, A_1\rangle) \vee_{\overline{\Box\Diamond}} f(\langle X_2, A_2\rangle),$$

which supports that f is $\vee$-preserving.

Therefore, $C^*(K) \cong C^{\overline{\Box\Diamond}}(K)$. Similarly, by setting $f : C^{\overline{\Box\Diamond}}(K) \to C^{\#}(K), f(\langle X, A\rangle) = \langle X, A^c\rangle$ (respectively, $f : C^{\#}(K) \to C^{\overline{\Diamond\Box}}(K), f(\langle X, A\rangle) = \langle X^c, A\rangle$, and $f : C^{\overline{\Diamond\Box}}(K) \to C^*(K), f(\langle X, A\rangle) = \langle X, A^c\rangle$), one can prove that $C^{\overline{\Box\Diamond}}(K) \cong C^{\#}(K)$ (respectively, $C^{\#}(K) \cong C^{\overline{\Diamond\Box}}(K)$ and $C^{\overline{\Diamond\Box}}(K) \cong C^*(K)$).

(2) The proof is similar to that of (1).

On the basis of Theorem 4, the eight kinds of concept lattices are reasonably grouped into two classes, namely, $\left\{C^*(K), C^{\overline{\Box\Diamond}}(K), C^{\overline{\Diamond\Box}}(K), C^{\#}(K)\right\}$ and $\left\{C^{\bar{*}}(K), C^{\Box\Diamond}(K), C^{\Diamond\Box}(K), C^{\overline{\#}}(K)\right\}$. The concept lattices in each group are order isomorphic to each other. A similar result is given in [46], however, there is a difference in the way of the definition of infimum and supremum of $\Box\Diamond$-concepts, $\overline{\Box\Diamond}$-concepts, $\#$-concepts, and $\overline{\#}$-concepts. Theoretically, one kind of concept lattice is enough to generate other three kinds within the same class. To maintain the semantic interpretations of different concepts, however, each kind of concept lattice is equally important.

4. Relationship between three-way concept lattices

Three-way concepts are defined by three-way operators. Table 4 summarizes different kinds of three-way operators appearing in literatures [26,27,36,49], where X and Y represent object sets, and A and B represent attribute sets. To avoid confusion, the eight kinds of object-induced three-way (short for O3W) operators are classified into two groups: Type-I O3W operators and Type-II O3W operators; each contains four kinds of operators. In the same way, we classified the eight kinds of attribute-induced three-way (short for A3W) operators. A Type-I O3W operator and a Type-II A3W operator determine a kind of O3W concept; a Type-II O3W operator and a Type-I A3W operator determine a kind of A3W concept. Note that definitions of $X^{\triangledown}$ and $X^{\blacktriangledown}$ in Table 4 are slightly different from those in [36,49]. We change the order of $X^{\Box}$ and $X^{\overline{\Box}}$ in $X^{\triangledown}$ and the order of $X^{\Diamond}$ and $X^{\overline{\Diamond}}$ in $X^{\blacktriangledown}$ for the convenience of operator relation discussion (see Table 5); it's the same case for $A^{\triangledown}$ and $A^{\blacktriangledown}$. Regardless of the meanings of sets, there are totally eight kinds of three-way operators. In other word, O3W operators and A3W operators are correspondingly same.

For two pairs of sets (P, Q) and (Z, W) of the same meaning, define

$$\begin{aligned}&(P, Q) \cap (Z, W) = (P \cap Z, Q \cap W),\\ &(P, Q) \cup (Z, W) = (P \cup Z, Q \cup W),\\ &(P, Q)^c = (P^c, Q^c).\end{aligned} \tag{13}$$

The pairs of sets are ordered in the following way:

$$(P, Q) \subseteq (Z, W) \Longleftrightarrow P \subseteq Z,\ Q \subseteq W.$$

4.1. Relationship between object-induced three-way concept lattices

Some basic properties of Type-I and Type-II O3W operators are listed in Propositions 3–6. We only prove the properties in Proposition 5, for the proof of others can be found in corresponding references.

Proposition 3 [27]. *For a given formal context $K = (U, V, R)$ and $X, X_1, X_2 \subseteq U$, we have*

$$\begin{aligned}&(1)\quad X_1 \subseteq X_2 \Rightarrow X_2^{\lessdot} \subseteq X_1^{\lessdot}; (2)\quad (X_1, Y_1) \subseteq (X_2, Y_2) \Rightarrow (X_2, Y_2)^{\gtrdot} \subseteq (X_1, Y_1)^{\gtrdot};\\ &(3)\quad X \subseteq X^{\lessdot\gtrdot}; (4)\quad (X, Y) \subseteq (X, Y)^{\gtrdot\lessdot};\\ &(5)\quad X^{\lessdot} = X^{\lessdot\gtrdot\lessdot}; (6)\quad (X, Y)^{\gtrdot} = (X, Y)^{\gtrdot\lessdot\gtrdot};\\ &(7)\quad (X_1 \cup X_2)^{\lessdot} = X_1^{\lessdot} \cap X_2^{\lessdot}; (8)\quad ((X_1, Y_1) \cup (X_2, Y_2))^{\gtrdot} = (X_1, Y_1)^{\gtrdot} \cap (X_2, Y_2)^{\gtrdot};\\ &(9)\quad (X_1 \cap X_2)^{\lessdot} \supseteq X_1^{\lessdot} \cup X_2^{\lessdot}; (10)\quad ((X_1, Y_1) \cap (X_2, Y_2))^{\gtrdot} \supseteq (X_1, Y_1)^{\gtrdot} \cup (X_2, Y_2)^{\gtrdot}.\end{aligned}$$

Proposition 4. [36] *For a given formal context $K = (U, V, R)$ and $X, X_1, X_2 \subseteq U$, we have*

$$(1)\quad X_1 \subseteq X_2 \Rightarrow X_1^{\triangledown} \subseteq X_2^{\triangledown};\quad (2)\quad (X_1,Y_1) \subseteq (X_2,Y_2) \Rightarrow (X_1,Y_1)^{\vartriangle} \subseteq (X_2,Y_2)^{\vartriangle};$$
$$(3)\quad X^{\triangledown\vartriangle} \subseteq X;\quad (4)\quad (X,Y) \subseteq (X,Y)^{\vartriangle\triangledown};$$
$$(5)\quad X^{\triangledown} = X^{\triangledown\vartriangle\triangledown};\quad (6)\quad (X,Y)^{\vartriangle} = (X,Y)^{\vartriangle\triangledown\vartriangle};$$
$$(7)\quad (X_1 \cap X_2)^{\triangledown} = X_1^{\triangledown} \cap X_2^{\triangledown};\quad (8)\quad ((X_1,Y_1) \cup (X_2,Y_2))^{\vartriangle} = (X_1,Y_1)^{\vartriangle} \cup (X_2,Y_2)^{\vartriangle};$$
$$(9)\quad (X_1 \cup X_2)^{\triangledown} \supseteq X_1^{\triangledown} \cup X_2^{\triangledown};\quad (10)\quad ((X_1,Y_1) \cap (X_2,Y_2))^{\vartriangle} \subseteq (X_1,Y_1)^{\vartriangle} \cap (X_2,Y_2)^{\vartriangle}.$$

Proposition 5. *For a given formal context $K = (U,V,R)$ and $X, X_1, X_2 \subseteq U$, we have*

$$(1)\quad X_1 \subseteq X_2 \Rightarrow X_1^{\blacktriangledown} \subseteq X_2^{\blacktriangledown};\quad (2)\quad (X_1,Y_1) \subseteq (X_2,Y_2) \Rightarrow (X_1,Y_1)^{\blacktriangle} \subseteq (X_2,Y_2)^{\blacktriangle};$$
$$(3)\quad X \subseteq X^{\blacktriangledown\blacktriangle};\quad (4)\quad (X,Y)^{\blacktriangle\blacktriangledown} \subseteq (X,Y);$$
$$(5)\quad X^{\blacktriangledown} = X^{\blacktriangledown\blacktriangle\blacktriangledown}\quad (6)\quad (X,Y)^{\blacktriangle} = (X,Y)^{\blacktriangle\blacktriangledown\blacktriangle};$$
$$(7)\quad (X_1 \cup X_2)^{\blacktriangledown} = X_1^{\blacktriangledown} \cup X_2^{\blacktriangledown};\quad (8)\quad ((X_1,Y_1) \cap (X_2,Y_2))^{\blacktriangle} = (X_1,Y_1)^{\blacktriangle} \cap (X_2,Y_2)^{\blacktriangle};$$
$$(9)\quad (X_1 \cap X_2)^{\blacktriangledown} \subseteq X_1^{\blacktriangledown} \cup X_2^{\blacktriangledown};\quad (10)\quad ((X_1,Y_1) \cup (X_2,Y_2))^{\blacktriangle} \supseteq (X_1,Y_1)^{\blacktriangle} \cup (X_2,Y_2)^{\blacktriangle}.$$

Proof.

(1) Since $X_1 \subseteq X_2$, then $X_1^{\blacktriangledown} = \left(X_1^{\overline{\Diamond}}, X_1^{\Diamond}\right) \subseteq \left(X_2^{\overline{\Diamond}}, X_2^{\Diamond}\right) = X_2^{\blacktriangledown}$ using Proposition 1.

(2) Suppose $(X_1,Y_1) \subseteq (X_2,Y_2)$, then $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ by Eq. (14). Accordingly, we have $(X_1,Y_1)^{\blacktriangle} = X_1^{\overline{\Box}} \cap Y_1^{\Box} \subseteq X_2^{\overline{\Box}} \cap Y_2^{\Box} = (X_2,Y_2)^{\blacktriangle}$.

(3) It follows from Proposition 1 that $X^{\blacktriangledown\blacktriangle} = \left(X^{\overline{\Diamond}}, X^{\Diamond}\right)^{\blacktriangle} = X^{\overline{\overline{\Diamond}\Box}} \cap X^{\Diamond\Box} \supseteq X \cap X = X$.

(4) It follows from Proposition 1 that $(X,Y)^{\blacktriangle\blacktriangledown} = \left(X^{\overline{\Box}} \cap Y^{\Box}\right)^{\blacktriangledown} = \left(\left(X^{\overline{\Box}} \cap Y^{\Box}\right)^{\overline{\Diamond}}, \left(X^{\overline{\Box}} \cap Y^{\Box}\right)^{\Diamond}\right) \subseteq \left(X^{\overline{\Box\Diamond}} \cap Y^{\Box\overline{\Diamond}}, X^{\overline{\Box}\Diamond} \cap Y^{\Box\Diamond}\right) \subseteq \left(X^{\overline{\Box\Diamond}}, Y^{\Box\Diamond}\right) \subseteq (X,Y)$.

(5) According to Items (1) and (3) in Proposition 5, it follows $X^{\blacktriangledown} \subseteq X^{\blacktriangledown\blacktriangle\blacktriangledown}$. On the other hand, $X^{\blacktriangledown\blacktriangle\blacktriangledown} = \left(X^{\overline{\Diamond}}, X^{\Diamond}\right)^{\blacktriangle\blacktriangledown} = \left(X^{\overline{\overline{\Diamond}\Box}} \cap X^{\Diamond\Box}\right)^{\blacktriangledown} = \left(\left(X^{\overline{\overline{\Diamond}\Box}} \cap X^{\Diamond\Box}\right)^{\overline{\Diamond}}, \left(X^{\overline{\overline{\Diamond}\Box}} \cap X^{\Diamond\Box}\right)^{\Diamond}\right) \subseteq \left(X^{\overline{\overline{\Diamond}\Box\Diamond}} \cap X^{\Diamond\Box\overline{\Diamond}}, X^{\overline{\overline{\Diamond}\Box}\Diamond} \cap X^{\Diamond\Box\Diamond}\right) \subseteq \left(X^{\overline{\overline{\Diamond}\Box\Diamond}}, X^{\Diamond\Box\Diamond}\right) = \left(X^{\overline{\Diamond}}, X^{\Diamond}\right) = X^{\blacktriangledown}$. Finally, $X^{\blacktriangledown} = X^{\blacktriangledown\blacktriangle\blacktriangledown}$.

(6) According to Items (2) and (4) in Proposition 5, it follows $(X,Y)^{\blacktriangle\blacktriangledown\blacktriangle} \subseteq (X,Y)^{\blacktriangle}$. On the other hand, $(X,Y)^{\blacktriangle\blacktriangledown\blacktriangle} = \left((X,Y)^{\blacktriangle\overline{\Diamond}}, (X,Y)^{\blacktriangle\Diamond}\right)^{\blacktriangle} = (X,Y)^{\blacktriangle\overline{\overline{\Diamond}\Box}} \cap (X,Y)^{\blacktriangle\Diamond\Box} \supseteq (X,Y)^{\blacktriangle} \cap (X,Y)^{\blacktriangle} = (X,Y)^{\blacktriangle}$.

(7) It follows from Item (4) in Proposition 1 that $(X_1 \cup X_2)^{\blacktriangledown} = \left((X_1 \cup X_2)^{\overline{\Diamond}}, (X_1 \cup X_2)^{\Diamond}\right) = \left(X_1^{\overline{\Diamond}} \cup X_2^{\overline{\Diamond}}, X_1^{\Diamond} \cup X_2^{\Diamond}\right) = \left(X_1^{\overline{\Diamond}}, X_1^{\Diamond}\right) \cup \left(X_2^{\overline{\Diamond}}, X_2^{\Diamond}\right) = X_1^{\blacktriangledown} \cup X_2^{\blacktriangledown}$.

(8) It follows from Item (4) in Proposition 1 that $((X_1,Y_1) \cap (X_2,Y_2))^{\blacktriangle} = (X_1 \cap X_2, Y_1 \cap Y_2)^{\blacktriangle} = (X_1 \cap X_2)^{\overline{\Box}} \cap (Y_1 \cap Y_2)^{\Box} = \left(X_1^{\overline{\Box}} \cap X_2^{\overline{\Box}}\right) \cap \left(Y_1^{\Box} \cap Y_2^{\Box}\right) = \left(X_1^{\overline{\Box}} \cap Y_1^{\Box}\right) \cap \left(X_2^{\overline{\Box}} \cap Y_2^{\Box}\right) = (X_1,Y_1)^{\blacktriangle} \cap (X_2,Y_2)^{\blacktriangle}$.

(9) It follows from Item (5) in Proposition 1 that $(X_1 \cap X_2)^{\blacktriangledown} = \left((X_1 \cap X_2)^{\overline{\Diamond}}, (X_1 \cap X_2)^{\Diamond}\right) \subseteq \left(X_1^{\overline{\Diamond}} \cap X_2^{\overline{\Diamond}}, X_1^{\Diamond} \cap X_2^{\Diamond}\right) = \left(X_1^{\overline{\Diamond}}, X_1^{\Diamond}\right) \cap \left(X_2^{\overline{\Diamond}}, X_2^{\Diamond}\right) = X_1^{\blacktriangledown} \cap X_2^{\blacktriangledown}$.

(10) It follows from Item (5) in Proposition 1 that $((X_1,Y_1) \cup (X_2,Y_2))^{\blacktriangle} = (X_1 \cup X_2, Y_1 \cup Y_2)^{\blacktriangle} = (X_1 \cup X_2)^{\overline{\Box}} \cap (Y_1 \cup Y_2)^{\Box} \supseteq \left(X_1^{\overline{\Box}} \cup X_2^{\overline{\Box}}\right) \cap \left(Y_1^{\Box} \cup Y_2^{\Box}\right) \supseteq \left(X_1^{\overline{\Box}} \cap Y_1^{\Box}\right) \cup \left(X_2^{\overline{\Box}} \cap Y_2^{\Box}\right) = (X_1,Y_1)^{\blacktriangle} \cup (X_2,Y_2)^{\blacktriangle}$.

Proposition 6 [49]. *For a given formal context $K = (U,V,R)$ and $X, X_1, X_2 \subseteq U$, we have*

$$(1)\quad X_1 \subseteq X_2 \Rightarrow X_2^{\triangleright} \subseteq X_1^{\triangleright};\quad (2)\quad (X_1,Y_1) \subseteq (X_2,Y_2) \Rightarrow (X_2,Y_2)^{\triangleleft} \subseteq (X_1,Y_1)^{\triangleleft};$$
$$(3)\quad X \supseteq X^{\triangleright\triangleleft};\quad (4)\quad (X,Y) \supseteq (X,Y)^{\triangleleft\triangleright};$$
$$(5)\quad X^{\triangleright} = X^{\triangleright\triangleleft\triangleright};\quad (6)\quad (X,Y)^{\triangleleft} = (X,Y)^{\triangleleft\triangleright\triangleleft};$$
$$(7)\quad (X_1 \cap X_2)^{\triangleright} = X_1^{\triangleright} \cup X_2^{\triangleright};\quad (8)\quad ((X_1,Y_1) \cap (X_2,Y_2))^{\triangleleft} = (X_1,Y_1)^{\triangleleft} \cup (X_2,Y_2)^{\triangleleft};$$
$$(9)\quad (X_1 \cup X_2)^{\triangleright} \subseteq X_1^{\triangleright} \cap X_2^{\triangleright};\quad (10)\quad ((X_1,Y_1) \cup (X_2,Y_2))^{\triangleleft} \subseteq (X_1,Y_1)^{\triangleleft} \cup (X_2,Y_2)^{\triangleleft}.$$

The relationship between different kinds of Type-I O3W operators are exhibited in Fig. 3a. Four nodes represent the four kinds of Type-I O3W operators, respectively. A double-headed arrow line connects a pair of operators from one of which the other can be obtained by taking the operation (namely, X^{c} or ^{c}) attached with the line. The notation X^{c} means the operators connected by the line can be converted into each other by replacing X with its complement X^{c}. For example, $X^{\lessdot}$ can be obtained by replacing X with X^{c} in $X^{\triangledown}$, that is, $X^{\lessdot} = (X^{c})^{\triangledown}$. The notation ^{c} represents the complement operation. For example, $X^{\lessdot}$ can be obtained by computing the complement of $X^{\blacktriangledown}$, that is, $X^{\lessdot} = \left(X^{\blacktriangledown}\right)^{c}$. (In the following discussion, we will omit the parentheses for simplicity.) Similar interpretations are for notations in Fig. 3b. The difference is that the notation $(X, Y)^{c}$ means that the operators connected by the line can be converted into each other by replacing X and Y both with their complements X^{c} and Y^{c}. We list the properties in detail in Table 5 and only present the proofs of equations in the first and fifth columns, for the others are similarly proved. According to Tables 2–4, the following hold:

$$X^{c\triangledown} = \left(X^{c\overline{\square}}, X^{c\square}\right) = \left(X^{*}, X^{\overline{*}}\right) = X^{\lessdot};$$

$$X^{c\triangleright c} = (X^{c\triangleright})^{c} = \left(X^{c\#}, X^{c\overline{\#}}\right)^{c} = \left(X^{*c}, X^{\overline{*}c}\right)^{c} = \left(X^{*}, X^{\overline{*}}\right) = X^{\lessdot};$$

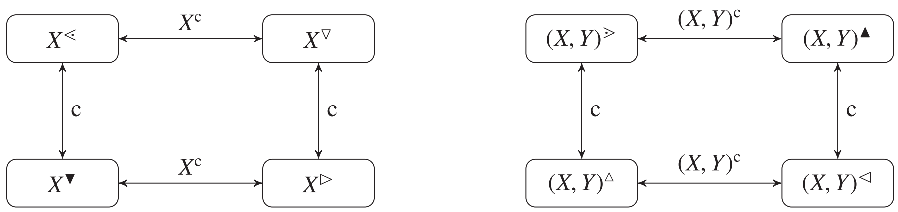
$$X^{\blacktriangledown c} = \left(X^{\overline{\diamond}}, X^{\diamond}\right)^{c} = \left(X^{\overline{\diamond}c}, X^{\diamond c}\right) = \left(X^{*}, X^{\overline{*}}\right) = X^{\lessdot};$$

$$(X, Y)^{c\blacktriangle} = (X^{c}, Y^{c})^{\blacktriangle} = X^{c\overline{\square}} \cap Y^{c\square} = X^{*} \cap Y^{\overline{*}} = (X, Y)^{\gtrdot};$$

$$(X, Y)^{c\triangleleft c} = (X^{c}, Y^{c})^{\triangleleft c} = \left(X^{c\#} \cup Y^{c\overline{\#}}\right)^{c} = X^{c\#c} \cap Y^{c\overline{\#}c} = X^{*} \cap Y^{\overline{*}} = (X, Y)^{\gtrdot};$$

$$(X, Y)^{\triangle c} = \left(X^{\overline{\diamond}} \cup Y^{\diamond}\right)^{c} = X^{\overline{\diamond}c} \cap Y^{\diamond c} = X^{*} \cap Y^{\overline{*}} = (X, Y)^{\gtrdot}.$$

Table 5 implies four pairs of dual operators, that is, $(\lessdot, \triangleright)$, $(\triangledown, \blacktriangledown)$, $(\gtrdot, \triangleleft)$, and $(\triangle, \blacktriangle)$. Properties of O3W operators shown in Propositions 3–6 can be mutually proved based on the results shown in Fig. 3 and Table 5. For example, suppose $X_1^{\blacktriangledown} \subseteq X_2^{\blacktriangledown}$ for $X_1 \subseteq X_2$, then it follows $X_2^{\lessdot} \subseteq X_1^{\lessdot}$, since $X_1^{\lessdot} = X_1^{\blacktriangledown c}$ and $X_2^{\lessdot} = X_2^{\blacktriangledown c}$.



(a) Relationship between Type-I O3W operators. (b) Relationship between Type-II O3W operators.

Fig. 3. Relationship between O3W operators.

Table 4
Three-way operators.

Type-I O3W operator	Type-II O3W operator	Type-I A3W operator	Type-II A3W operator
$X^{\lessdot} = \left(X^{*}, X^{\overline{*}}\right)$	$(X, Y)^{\gtrdot} = X^{*} \cap Y^{\overline{*}}$	$A^{\lessdot} = \left(A^{*}, A^{\overline{*}}\right)$	$(A, B)^{\gtrdot} = A^{*} \cap B^{\overline{*}}$
$X^{\triangledown} = \left(X^{\overline{\square}}, X^{\square}\right)$	$(X, Y)^{\triangle} = X^{\overline{\diamond}} \cup Y^{\diamond}$	$A^{\triangledown} = \left(A^{\overline{\square}}, A^{\square}\right)$	$(A, B)^{\triangle} = A^{\overline{\diamond}} \cup B^{\diamond}$
$X^{\blacktriangledown} = \left(X^{\overline{\diamond}}, X^{\diamond}\right)$	$(X, Y)^{\blacktriangle} = X^{\overline{\square}} \cap Y^{\square}$	$A^{\blacktriangledown} = \left(A^{\overline{\diamond}}, A^{\diamond}\right)$	$(A, B)^{\blacktriangle} = A^{\overline{\square}} \cap B^{\square}$
$X^{\triangleright} = \left(X^{\#}, X^{\overline{\#}}\right)$	$(X, Y)^{\triangleleft} = X^{\#} \cup Y^{\overline{\#}}$	$A^{\triangleright} = \left(A^{\#}, A^{\overline{\#}}\right)$	$(A, B)^{\triangleleft} = A^{\#} \cup A^{\overline{\#}}$

Table 5
Relationship between O3W operators.

$X^{\lessdot} = X^{\lessdot}$	$X^{\triangledown} = X^{c\lessdot}$	$X^{\triangleright} = X^{c\lessdot c}$	$X^{\blacktriangledown} = X^{\lessdot c}$	$(X, Y)^{\gtrdot} = (X, Y)^{\gtrdot}$	$(X, Y)^{\blacktriangle} = (X, Y)^{c\gtrdot}$	$(X, Y)^{\triangleleft} = (X, Y)^{c\gtrdot c}$	$(X, Y)^{\triangle} = (X, Y)^{\gtrdot c}$
$X^{\lessdot} = X^{c\triangledown}$	$X^{\triangledown} = X^{\triangledown}$	$X^{\triangleright} = X^{\triangledown c}$	$X^{\blacktriangledown} = X^{c\triangledown c}$	$(X, Y)^{\gtrdot} = (X, Y)^{c\blacktriangle}$	$(X, Y)^{\blacktriangle} = (X, Y)^{\blacktriangle}$	$(X, Y)^{\triangleleft} = (X, Y)^{\blacktriangle c}$	$(X, Y)^{\triangle} = (X, Y)^{c\blacktriangle c}$
$X^{\lessdot} = X^{c\triangleright c}$	$X^{\triangledown} = X^{\triangleright c}$	$X^{\triangleright} = X^{\triangleright}$	$X^{\blacktriangledown} = X^{c\triangleright}$	$(X, Y)^{\gtrdot} = (X, Y)^{c\triangleleft c}$	$(X, Y)^{\blacktriangle} = (X, Y)^{\triangleleft c}$	$(X, Y)^{\triangleleft} = (X, Y)^{\triangleleft}$	$(X, Y)^{\triangle} = (X, Y)^{c\triangleleft}$
$X^{\lessdot} = X^{\blacktriangledown c}$	$X^{\triangledown} = X^{c\blacktriangledown c}$	$X^{\triangleright} = X^{c\blacktriangledown}$	$X^{\blacktriangledown} = X^{\blacktriangledown}$	$(X, Y)^{\gtrdot} = (X, Y)^{\triangle c}$	$(X, Y)^{\blacktriangle} = (X, Y)^{c\triangle c}$	$(X, Y)^{\triangleleft} = (X, Y)^{c\triangle}$	$(X, Y)^{\triangle} = (X, Y)^{\triangle}$

Proposition 7. *Let $K=(U,V,R)$ be a formal context and $X,Y\subseteq U$. Then,*

$$(1)\quad X^{\lessdot\gtrdot}=X^{\blacktriangledown\blacktriangle},\ X^{\lessdot\triangle}=X^{\blacktriangledown\triangleleft},\ X^{\lessdot\blacktriangle}=X^{\blacktriangledown\gtrdot},\ X^{\lessdot\triangleleft}=X^{\blacktriangledown\triangle},$$
$$X^{\triangledown\gtrdot}=X^{\triangleright\blacktriangle},\ X^{\triangledown\triangle}=X^{\triangleright\triangleleft},\ X^{\triangledown\blacktriangle}=X^{\triangleright\gtrdot},\ X^{\triangledown\triangleleft}=X^{\triangleright\triangle};$$
$$(2)\quad (X,Y)^{\gtrdot\lessdot}=(X,Y)^{\triangle\triangledown},\ (X,Y)^{\gtrdot\triangledown}=(X,Y)^{\triangle\lessdot},\ (X,Y)^{\gtrdot\triangleright}=(X,Y)^{\triangle\blacktriangledown},\ (X,Y)^{\gtrdot\blacktriangledown}=(X,Y)^{\triangle\triangleright},$$
$$(X,Y)^{\blacktriangle\lessdot}=(X,Y)^{\triangleleft\triangledown},\ (X,Y)^{\blacktriangle\triangledown}=(X,Y)^{\triangleleft\lessdot},\ (X,Y)^{\blacktriangle\triangleright}=(X,Y)^{\triangleleft\blacktriangledown},\ (X,Y)^{\blacktriangle\blacktriangledown}=(X,Y)^{\triangleleft\triangleright}.$$

Proof. For $X\subseteq U$, by repeatedly using properties in Table 5, we have $X^{\lessdot\gtrdot}=\left(X^{\blacktriangledown c}\right)^{\gtrdot}=\left(X^{\blacktriangledown c}\right)^{c\blacktriangle}=X^{\blacktriangledown\blacktriangle}$. All other items can be proved in the same way.

Item (1) shows the connection between two derived sets by applying a Type-I and a Type-II O3W operators successively. Item (2) shows the connection between two derived sets by applying a Type-II and a Type-I O3W operators successively.

Remark 4. Since A3W operators and O3W operators are correspondingly same, the properties of and relationship between O3W operators discussed above also hold for A3W operators. In case of redundancy, discussions of the relationship between and properties of A3W operators are omitted.

An O3W concept is defined by a pair of Type-I O3W and Type-II A3W operators.

Definition 6. [26,36,49] Let $K=(U,V,R)$ be a formal context, $X\subseteq U$, and $A,B\subseteq V$. Then,

(1) $\langle X,(A,B)\rangle$ is a $\lessdot$-object-induced three-way concept (short for $\lessdot$-O3W concept) if $X^{\lessdot}=(A,B)$ and $(A,B)^{\gtrdot}=X$;
(2) $\langle X,(A,B)\rangle$ is a $\triangledown$-object-induced three-way concept (short for $\triangledown$-O3W concept) if $X^{\triangledown}=(A,B)$ and $(A,B)^{\triangle}=X$;
(3) $\langle X,(A,B)\rangle$ is a $\blacktriangledown$-object-induced three-way concept (short for $\blacktriangledown$-O3W concept) if $X^{\blacktriangledown}=(A,B)$ and $(A,B)^{\blacktriangle}=X$;
(4) $\langle X,(A,B)\rangle$ is a $\triangleright$-object-induced three-way concept (short for $\triangleright$-O3W concept) if $X^{\triangleright}=(A,B)$ and $(A,B)^{\triangleleft}=X$.

Denote by $OC_3^{\star}(K)$ the set of all $\star$-O3W concepts of the formal context $K=(U,V,R)$, where $\star=\lessdot,\triangledown,\blacktriangledown$, and $\triangleright$. The relationship between different O3W concepts is shown in the following theorem.

Theorem 5. [49] *Let $K=(U,V,R)$ be a formal context, $X\subseteq U$, and $A,B\subseteq V$. Then, the following statements are equivalent:*

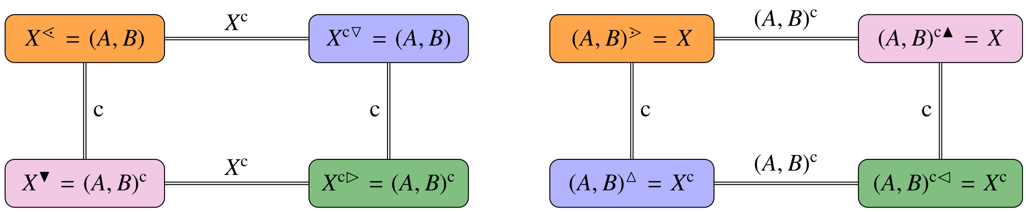
(1) *$\langle X,(A,B)\rangle$ is a $\lessdot$-O3W concept;*
(2) *$\langle X^c,(A,B)\rangle$ is a $\triangledown$-O3W concept;*
(3) *$\langle X^c,(A,B)^c\rangle$ is a $\triangleright$-O3W concept;*
(4) *$\langle X,(A,B)^c\rangle$ is a $\blacktriangledown$-O3W concept.*

Proof. Suppose that $\langle X,(A,B)\rangle$ is a $\lessdot$-O3W concept, then according to Table 5, we have

$$\begin{aligned}\langle X,(A,B)\rangle \text{ is a } \lessdot-\text{O3W concept} &\iff X^{\lessdot}=(A,B),\ (A,B)^{\gtrdot}=X\\ &\iff X^{c\triangledown}=(A,B),\ (A,B)^{\triangle c}=X\\ &\iff (X^c)^{\triangledown}=(A,B),\ (A,B)^{\triangle}=X^c\\ &\iff \langle X^c,(A,B)\rangle \text{ is a } \triangledown-\text{O3W concept.}\end{aligned}$$

The other equivalences can be similarly proved.

Theorem 5 provides a theoretical support of expressing one kind of O3W concept in terms of another. Fig. 4 helps us to explain and prove Theorem 5. Two nodes of the same color in Figs. 4a and 4b represent a kind of O3W concept. The equivalence between two kinds of O3W concepts is depicted by two double lines in Figs. 4a and 4b, respectively. In consideration



(a) Equivalences between Type-I O3W operators. (b) Equivalences between Type-II A3W operators.

Fig. 4. Equivalences between O3W concepts.

of formulation and interpretation, even though the four kinds of O3W concepts are equivalent to each other, it is meaningful and more convenient to use separate of them. Based on Eqs. (13) and (14), the order, infimum, and supremum of O3W concepts are defined as follows.

Definition 7. Let $K=(U,V,R)$ be a formal context, $X_1,X_2\subseteq U$, and $A_1,A_2,B_1,B_2\subseteq V$.

(1) [26] For $\langle X_1,(A_1,B_1)\rangle,\langle X_2,(A_2,B_2)\rangle\in OC_3^{\lessdot}(K)$, $\langle X_1,(A_1,B_1)\rangle\leq_{\lessdot}\langle X_2,(A_2,B_2)\rangle$ iff $X_1\subseteq X_2$ (equivalently, $(A_2,B_2)\subseteq(A_1,B_1)$), and

$$\begin{aligned}\langle X_1,(A_1,B_1)\rangle\wedge_{\lessdot}\langle X_2,(A_2,B_2)\rangle &= \left\langle X_1\cap X_2,((A_1,B_1)\cup(A_2,B_2))^{\gtrdot\lessdot}\right\rangle\\ &= \left\langle X_1\cap X_2,(X_1\cap X_2)^{\lessdot}\right\rangle,\\ \langle X_1,(A_1,B_1)\rangle\vee_{\lessdot}\langle X_2,(A_2,B_2)\rangle &= \left\langle (X_1\cup X_2)^{\lessdot\gtrdot},(A_1,B_1)\cap(A_2,B_2)\right\rangle\\ &= \left\langle ((A_1,B_1)\cap(A_2,B_2))^{\gtrdot},(A_1,B_1)\cap(A_2,B_2)\right\rangle.\end{aligned}\tag{15}$$

(2) For $\langle X_1,(A_1,B_1)\rangle,\langle X_2,(A_2,B_2)\rangle\in OC_3^{\triangledown}(K)$, $\langle X_1,(A_1,B_1)\rangle\leq_{\triangledown}\langle X_2,(A_2,B_2)\rangle$ iff $X_2\subseteq X_1$ (equivalently, $(A_2,B_2)\subseteq(A_1,B_1)$), and

$$\begin{aligned}\langle X_1,(A_1,B_1)\rangle\wedge_{\triangledown}\langle X_2,(A_2,B_2)\rangle &= \left\langle X_1\cup X_2,((A_1,B_1)\cup(A_2,B_2))^{\vartriangle\triangledown}\right\rangle\\ &= \langle X_1\cup X_2,(X_1\cup X_2)^{\triangledown}\rangle,\\ \langle X_1,(A_1,B_1)\rangle\vee_{\triangledown}\langle X_2,(A_2,B_2)\rangle &= \left\langle (X_1\cap X_2)^{\triangledown\vartriangle},(A_1,B_1)\cap(A_2,B_2)\right\rangle\\ &= \left\langle ((A_1,B_1)\cap(A_2,B_2))^{\vartriangle},(A_1,B_1)\cap(A_2,B_2)\right\rangle.\end{aligned}\tag{16}$$

(3) For $\langle X_1,(A_1,B_1)\rangle,\langle X_2,(A_2,B_2)\rangle\in OC_3^{\triangleright}(K)$, $\langle X_1,(A_1,B_1)\rangle\leq_{\triangleright}\langle X_2,(A_2,B_2)\rangle$ iff $X_2\subseteq X_1$ (equivalently, $(A_1,B_1)\subseteq(A_2,B_2)$), and

$$\begin{aligned}\langle X_1,(A_1,B_1)\rangle\wedge_{\triangleright}\langle X_2,(A_2,B_2)\rangle &= \langle X_1\cup X_2,((A_1,B_1)\cap(A_2,B_2))^{\triangleleft\triangleright}\rangle\\ &= \langle X_1\cup X_2,(X_1\cup X_2)^{\triangleright}\rangle,\\ \langle X_1,(A_1,B_1)\rangle\vee_{\triangleright}\langle X_2,(A_2,B_2)\rangle &= \langle (X_1\cap X_2)^{\triangleright\triangleleft},(A_1,B_1)\cup(A_2,B_2)\rangle\\ &= \langle ((A_1,B_1)\cup(A_2,B_2))^{\triangleleft},(A_1,B_1)\cup(A_2,B_2)\rangle.\end{aligned}\tag{17}$$

(4) [29] For $\langle X_1,(A_1,B_1)\rangle,\langle X_2,(A_2,B_2)\rangle\in OC_3^{\blacktriangledown}(K)$, $\langle X_1,(A_1,B_1)\rangle\leq_{\blacktriangledown}\langle X_2,(A_2,B_2)\rangle$ iff $X_1\subseteq X_2$ (equivalently, $(A_1,B_1)\subseteq(A_2,B_2)$), and

$$\begin{aligned}\langle X_1,(A_1,B_1)\rangle\wedge_{\blacktriangledown}\langle X_2,(A_2,B_2)\rangle &= \left\langle X_1\cap X_2,((A_1,B_1)\cap(A_2,B_2))^{\blacktriangle\blacktriangledown}\right\rangle\\ &= \left\langle X_1\cap X_2,(X_1\cap X_2)^{\blacktriangledown}\right\rangle,\\ \langle X_1,(A_1,B_1)\rangle\vee_{\blacktriangledown}\langle X_2,(A_2,B_2)\rangle &= \left\langle (X_1\cup X_2)^{\blacktriangledown\blacktriangle},(A_1,B_1)\cup(A_2,B_2)\right\rangle\\ &= \left\langle ((A_1,B_1)\cup(A_2,B_2))^{\blacktriangle},(A_1,B_1)\cup(A_2,B_2)\right\rangle.\end{aligned}\tag{18}$$

To be consistent with the equivalences between O3W concepts, we have changed the order, infimum, and supremum of $\triangledown$-O3W concepts and $\triangleright$-O3W concepts in [29]. The collection of O3W concepts of the same kind forms a complete lattice with the corresponding infimum and supremum defined in Definition 7.

Theorem 6. *Let $K=(U,V,R)$ be a formal context. Then, $\left(OC_3^{\star}(K),\wedge_{\star},\vee_{\star}\right)$ is a complete lattice, where $\star=\lessdot,\triangledown,\blacktriangledown$, and $\triangleright$, respectively.*

Proof. (Given is the proof of $(OC_3^{\triangledown}(K),\wedge_{\triangledown},\vee_{\triangledown})$ being a complete lattice. The others can be proved similarly.) Suppose $\langle X_1,(A_1,B_1)\rangle,\langle X_2,(A_2,B_2)\rangle\in OC_3^{\triangledown}(K)$, then $X_i^{\triangledown}=(A_i,B_i)$ and $(A_i,B_i)^{\vartriangle}=X_i, i=1,2$. According to Proposition 1 and 4, we have

$$\begin{aligned}\left(((A_1,B_1)\cup(A_2,B_2))^{\vartriangle\triangledown}\right)^{\vartriangle} &= ((A_1,B_1)\cup(A_2,B_2))^{\vartriangle\triangledown\vartriangle}=((A_1,B_1)\cup(A_2,B_2))^{\vartriangle}\\ &= (A_1,B_1)^{\vartriangle}\cup(A_2,B_2)^{\vartriangle}=X_1\cup X_2.\end{aligned}$$

Still, applying Proposition 4, it follows that

$$(X_1\cup X_2)^{\triangledown}=\left(((A_1,B_1)\cup(A_2,B_2))^{\vartriangle\triangledown\vartriangle}\right)^{\triangledown}=((A_1,B_1)\cup(A_2,B_2))^{\vartriangle\triangledown}.$$

Therefore, $\left\langle X_1\cup X_2,((A_1,B_1)\cup(A_2,B_2))^{\vartriangle\triangledown}\right\rangle\in OC_3^{\triangledown}(K)$.

It is straightforward from Definition 7 and Proposition 4 that

$$\left\langle X_1 \cup X_2, ((A_1,B_1)\cup(A_2,B_2))^{\triangle\triangledown}\right\rangle \leq_{\triangledown} \langle X_1,(A_1,B_1)\rangle,$$
$$\left\langle X_1 \cup X_2, ((A_1,B_1)\cup(A_2,B_2))^{\triangle\triangledown}\right\rangle \leq_{\triangledown} \langle X_2,(A_2,B_2)\rangle.$$

Next, we prove that $\left\langle X_1 \cup X_2, ((A_1,B_1)\cup(A_2,B_2))^{\triangle\triangledown}\right\rangle$ is the infimum of $\langle X_1,(A_1,B_1)\rangle$ and $\langle X_2,(A_2,B_2)\rangle$. If else, suppose that $\langle X,(A,B)\rangle$ is another lower bound of $\langle X_1,(A_1,B_1)\rangle$ and $\langle X_2,(A_2,B_2)\rangle$, and $\left\langle X_1 \cup X_2, ((A_1,B_1)\cup(A_2,B_2))^{\triangle\triangledown}\right\rangle \leq_{\triangledown} \langle X,(A,B)\rangle$. Then, $X_1 \subseteq X$ and $X_2 \subseteq X$ on one hand; on the other $X \subseteq X_1 \cup X_2$. This leads to $X = X_1 \cup X_2$ and $(A,B) = X^{\triangledown} = (X_1\cup X_2)^{\triangledown} = ((A_1,B_1)\cup(A_2,B_2))^{\triangle\triangledown}$. Equivalently saying, $\left\langle X_1 \cup X_2, ((A_1,B_1)\cup(A_2,B_2))^{\triangle\triangledown}\right\rangle$ is the infimum of $\langle X_1,(A_1,B_1)\rangle$ and $\langle X_2,(A_2,B_2)\rangle$.

In a similar process, one can prove that $\left\langle (X_1\cap X_2)^{\triangledown\triangle}, (A_1,B_1)\cap(A_2,B_2)\right\rangle$ is a $\triangledown$-O3W concept and also the supremum of $\langle X_1,(A_1,B_1)\rangle$ and $\langle X_2,(A_2,B_2)\rangle$. Consequently, $(OC_3^{\triangledown}(K), \wedge_{\triangledown}, \vee_{\triangledown})$ is a complete lattice.

The equivalences between different kinds of O3W concepts in Theorem 5 provide a hint on the relationship between the four kinds of O3W concept lattices stated in Theorem 6.

Theorem 7. *Let $K = (U,V,R)$ be a formal context. Then, $OC_3^{\lessdot}(K) \cong OC_3^{\triangledown}(K) \cong OC_3^{\gtrdot}(K) \cong OC_3^{\blacktriangledown}(K)$.*

Proof. Given $\langle X,(A,B)\rangle \in OC^{\lessdot}(K)$, let $f(\langle X,(A,B)\rangle) = \langle X^c,(A,B)\rangle$. Then, using Theorem 5, we have $\langle X^c,(A,B)\rangle \in OC^{\triangledown}(K)$, and f is a bijection between $OC^{\lessdot}(K)$ and $OC^{\triangledown}(K)$.

Suppose $\langle X_1,(A_1,B_1)\rangle, \langle X_2,(A_2,B_2)\rangle \in OC^{\lessdot}(K)$, then according to Eqs. (15) and (16), we have

$$f(\langle X_1,(A_1,B_1)\rangle \wedge_{\lessdot} \langle X_2,(A_2,B_2)\rangle) = f\left(\left\langle X_1\cap X_2, ((A_1,B_1)\cup(A_2,B_2))^{\gtrdot\lessdot}\right\rangle\right) = \left\langle (X_1\cap X_2)^c, ((A_1,B_1)\cup(A_2,B_2))^{\gtrdot\lessdot}\right\rangle,$$
$$f(\langle X_1,(A_1,B_1)\rangle) \wedge_{\triangledown} f(\langle X_2,(A_2,B_2)\rangle) = \langle X_1^c,(A_1,B_1)\rangle \wedge_{\triangledown} \langle X_2^c,(A_2,B_2)\rangle = \left\langle X_1^c\cup X_2^c, ((A_1,B_1)\cup(A_2,B_2))^{\triangle\triangledown}\right\rangle.$$

It is obvious that $(X_1\cap X_2)^c = X_1^c \cup X_2^c$, and at the same time it follows from Proposition 7 that $((A_1,B_1)\cup(A_2,B_2))^{\gtrdot\lessdot} = ((A_1,B_1)\cup(A_2,B_2))^{\triangle\triangledown}$. That is to say,

$$f(\langle X_1,(A_1,B_1)\rangle \wedge_{\lessdot} \langle X_2,(A_2,B_2)\rangle) = f(\langle X_1,(A_1,B_1)\rangle) \wedge_{\triangledown} f(\langle X_2,(A_2,B_2)\rangle),$$

which means that f is $\wedge$-preserving.

On the other hand, it follows from Eqs. (15) and (16) that

$$f(\langle X_1,(A_1,B_1)\rangle \vee_{\lessdot} \langle X_2,(A_2,B_2)\rangle) = f\left(\left\langle (X_1\cup X_2)^{\lessdot\gtrdot}, (A_1,B_1)\cap(A_2,B_2)\right\rangle\right) = \left\langle (X_1\cup X_2)^{\lessdot\gtrdot c}, (A_1,B_1)\cap(A_2,B_2)\right\rangle,$$
$$f(\langle X_1,(A_1,B_1)\rangle) \vee_{\triangledown} f(\langle X_2,(A_2,B_2)\rangle) = \langle X_1^c,(A_1,B_1)\rangle \vee_{\triangledown} \langle X_2^c,(A_2,B_2)\rangle = \left\langle (X_1^c\cap X_2^c)^{\triangledown\triangle}, (A_1,B_1)\cap(A_2,B_2)\right\rangle.$$

According to the properties shown in Table 5, we have

$$(X_1\cup X_2)^{\lessdot\gtrdot c} = (X_1\cup X_2)^{\lessdot\triangle},$$
$$(X_1^c\cap X_2^c)^{\triangledown\triangle} = ((X_1\cup X_2)^c)^{\triangledown\triangle} = (X_1\cup X_2)^{c\triangledown\triangle} = (X_1\cup X_2)^{\lessdot\triangle},$$

which means

$$f(\langle X_1,(A_1,B_1)\rangle \vee_{\lessdot} \langle X_2,(A_2,B_2)\rangle) = f(\langle X_1,(A_1,B_1)\rangle) \vee_{\triangledown} f(\langle X_2,(A_2,B_2)\rangle).$$

That is to say, f is $\vee$-preserving.

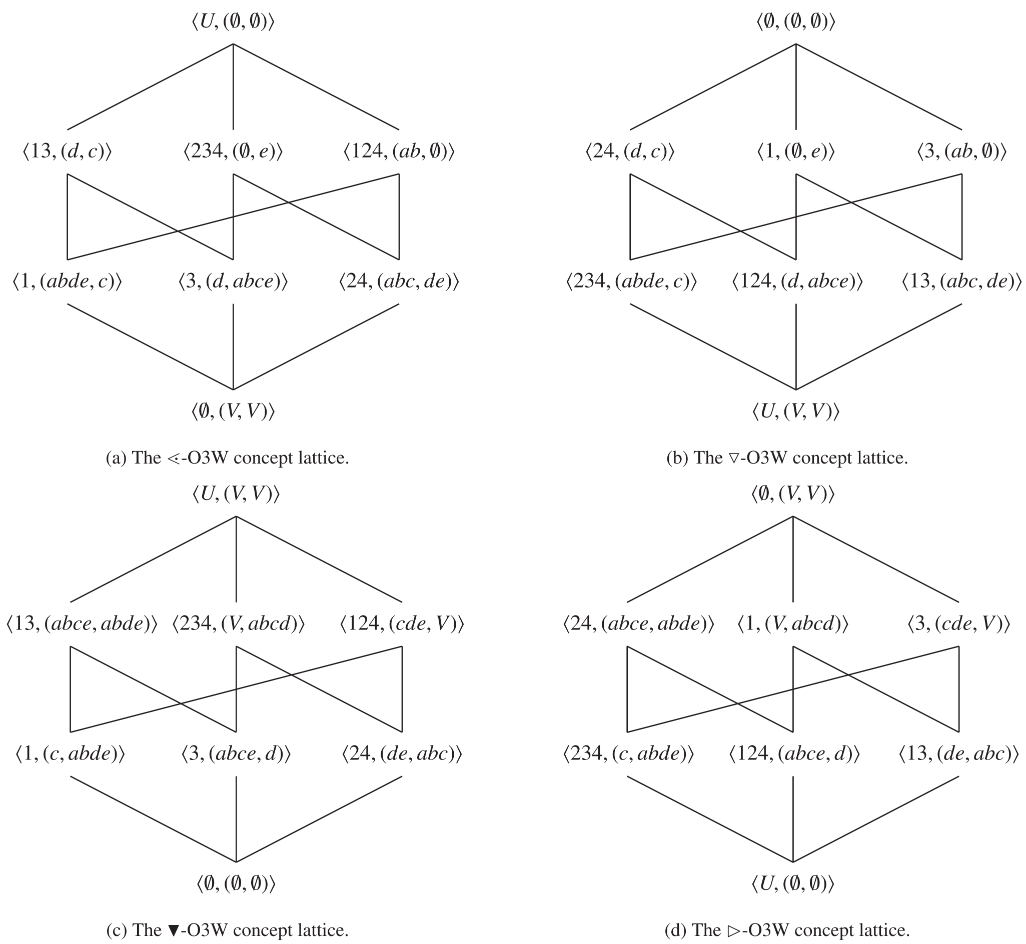
Therefore, $OC_3^{\lessdot}(K) \cong OC_3^{\triangledown}(K)$. Similarly, by setting $f: OC_3^{\triangledown}(K) \to OC_3^{\gtrdot}(K), f(\langle X,(A,B)\rangle) = \langle X,(A,B)^c\rangle$ (respectively, $f: OC_3^{\gtrdot}(K) \to OC_3^{\blacktriangledown}(K), f(\langle X,(A,B)\rangle) = \langle X^c,(A,B)\rangle$, and $f: OC_3^{\blacktriangledown}(K) \to OC_3^{\lessdot}(K), f(\langle X,(A,B)\rangle) = \langle X,(A,B)^c\rangle$), one can prove that $C^{\triangledown}(K) \cong OC_3^{\blacktriangledown}(K)$ (respectively, $OC_3^{\gtrdot}(K) \cong OC_3^{\blacktriangledown}(K)$ and $OC_3^{\blacktriangledown}(K) \cong OC_3^{\lessdot}(K)$).

The order isomorphic relation in Theorem 7 supports that any of the four kinds of O3W concept lattices can produce another three. The following is an example of the application of Theorem 7.

Example 1. Table 6 is a formal context with $U = \{x_1,x_2,x_3,x_4\}$ and $V = \{a,b,c,d,e\}$ (cited from [26]). The $\lessdot$-O3W concept lattice is shown in Fig. 5a, where 13 represents $\{x_1,x_3\}$ and (d,c) represents $(\{d\},\{c\})$; similar interpretations are for other notations of different nodes. Applying Theorems 5 and 7, the $\triangledown$-O3W concept lattice (shown in Fig. 5b) is obtained by replacing all extents in Fig. 5a with their complements. The $\blacktriangledown$-O3W concept lattice (shown in Fig. 5c) is obtained by replacing all intents in Fig. 5a with their complements. The $\gtrdot$-O3W concept lattice (shown in Fig. 5d) is obtained by replacing all extents and intents in Fig. 5a with their complements, respectively. A line in these figures connects two O3W concepts one of which from the lower level is the subconcept of the one in the upper level.

Table 6
A formal context (cited from [26]).

	a	b	c	d	e
x_1	1	1	0	1	1
x_2	1	1	1	0	0
x_3	0	0	0	1	0
x_4	1	1	1	0	0



(a) The $\lessdot$-O3W concept lattice. (b) The $\triangledown$-O3W concept lattice. (c) The $\blacktriangledown$-O3W concept lattice. (d) The $\triangleright$-O3W concept lattice.

Fig. 5. The O3W concept lattices.

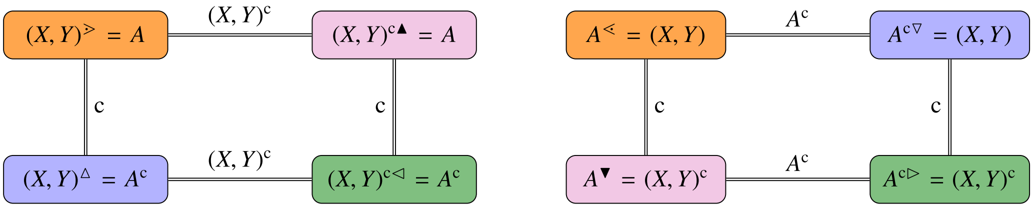
4.2. Relationship between attribute-induced three-way concept lattices

This section mainly discusses the relationship between different kinds of A3W concept lattices. Since the properties of A3W operators are similar to those of O3W operators, we only present some critical results of A3W concepts.

Definition 8. [26,36] Let $K = (U, V, R)$ be a formal context, $X, Y \subseteq U$, and $A \subseteq V$. Then,

(1) $\langle (X, Y), A\rangle$ is a $\lessdot$-attribute-induced three-way concept (short for $\lessdot$-A3W concept) if $A^{\lessdot} = (X, Y)$ and $(X, Y)^{\gtrdot} = A$;
(2) $\langle (X, Y), A\rangle$ is a $\triangledown$-attribute-induced three-way concept (short for $\triangledown$-A3W concept) if $A^{\triangledown} = (X, Y)$ and $(X, Y)^{\vartriangle} = A$;
(3) $\langle (X, Y), A\rangle$ is a $\blacktriangledown$-attribute-induced three-way concept (short for $\blacktriangledown$-A3W concept) if $A^{\blacktriangledown} = (X, Y)$ and $(X, Y)^{\blacktriangle} = A$;
(4) $\langle (X, Y), A\rangle$ is a $\triangleright$-attribute-induced three-way concept (short for $\triangleright$-A3W concept) if $A^{\triangleright} = (X, Y)$ and $(X, Y)^{\triangleleft} = A$.

Denote $AC_3^{\star}(K)$ the family of all $\star$-A3W concepts of the formal context $K = (U, V, R)$, where $\star = \lessdot, \triangledown, \blacktriangledown$, and $\triangleright$. The relationship between the four kinds of A3W concepts is summarized in Theorem 8 and explained through Fig. 6.



(a) Equivalences between Type-II O3W operators. (b) Equivalences between Type-I A3W operators.

Fig. 6. Equivalences between A3W concepts.

Theorem 8. *Let $K=(U,V,R)$ be a formal context, $X,Y\subseteq U$, and $A\subseteq V$. Then, the following statements are equivalent:*

(1) $\langle (X,Y),A\rangle$ *is a* $\lessdot$*-A3W concept in K;*
(2) $\left\langle (X,Y),A^{c}\right\rangle$ *is a* $\triangledown$*-A3W concept in K;*
(3) $\left\langle (X,Y)^{c},A^{c}\right\rangle$ *is a* $\triangleright$*-A3W concept in K;*
(4) $\left\langle (X,Y)^{c},A\right\rangle$ *is a* $\blacktriangledown$*-A3W concept in K.*

The order, infimum, and supremum of A3W concepts are defined as follows.

Definition 9. Let $K=(U,V,R)$ be a formal context, $X_1,X_2,Y_1,Y_2\subseteq U$, and $A_1,A_2\subseteq V$.

(1) [26] For $\langle (X_1,Y_1),A_1\rangle,\langle (X_2,Y_2),A_2\rangle\in AC_3^{\lessdot}(K)$, $\langle (X_1,Y_1),A_1\rangle\leq_{\lessdot}\langle (X_2,Y_2),A_2\rangle$ iff $(X_1,Y_1)\subseteq (X_2,Y_2)$ (equivalently, $A_2\subseteq A_1$), and

$$\begin{aligned}\langle (X_1,Y_1),A_1\rangle\wedge_{\lessdot}\langle (X_2,Y_2),A_2\rangle &= \left\langle (X_1,Y_1)\cap (X_2,Y_2),(A_1\cup A_2)^{\lessdot\gtrdot}\right\rangle\\ &= \left\langle (X_1,Y_1)\cap (X_2,Y_2),((X_1,Y_1)\cap (X_2,Y_2))^{\gtrdot}\right\rangle,\\ \langle (X_1,Y_1),A_1\rangle\vee_{\lessdot}\langle (X_2,Y_2),A_2\rangle &= \left\langle ((X_1,Y_1)\cup (X_2,Y_2))^{\gtrdot\lessdot},A_1\cap A_2\right\rangle\\ &= \left\langle (A_1\cap A_2)^{\lessdot},A_1\cap A_2\right\rangle.\end{aligned}\tag{19}$$

(2) For $\langle (X_1,Y_1),A_1\rangle,\langle (X_2,Y_2),A_2\rangle\in AC_3^{\triangledown}(K)$, $\langle (X_1,Y_1),A_1\rangle\leq_{\triangledown}\langle (X_2,Y_2),A_2\rangle$ iff $(X_1,Y_1)\subseteq (X_2,Y_2)$ (equivalently, $A_1\subseteq A_2$), and

$$\begin{aligned}\langle (X_1,Y_1),A_1\rangle\wedge_{\triangledown}\langle (X_2,Y_2),A_2\rangle &= \left\langle (X_1,Y_1)\cap (X_2,Y_2),(A_1\cap A_2)^{\triangledown\vartriangle}\right\rangle\\ &= \left\langle (X_1,Y_1)\cap (X_2,Y_2),((X_1,Y_1)\cap (X_2,Y_2))^{\vartriangle}\right\rangle,\\ \langle (X_1,Y_1),A_1\rangle\vee_{\triangledown}\langle (X_2,Y_2),A_2\rangle &= \left\langle ((X_1,Y_1)\cup (X_2,Y_2))^{\vartriangle\triangledown},A_1\cup A_2\right\rangle\\ &= \langle (A_1\cup A_2)^{\triangledown},A_1\cup A_2\rangle.\end{aligned}\tag{20}$$

(3) For $\langle (X_1,Y_1),A_1\rangle,\langle (X_2,Y_2),A_2\rangle\in AC_3^{\triangleright}(K)$, $\langle (X_1,Y_1),A_1\rangle\leq_{\triangleright}\langle (X_2,Y_2),A_2\rangle$ iff $(X_2,Y_2)\subseteq (X_1,Y_1)$ (equivalently, $A_1\subseteq A_2$), and

$$\begin{aligned}\langle (X_1,Y_1),A_1\rangle\wedge_{\triangleright}\langle (X_2,Y_2),A_2\rangle &= \langle (X_1,Y_1)\cup (X_2,Y_2),(A_1\cap A_2)^{\triangleright\triangleleft}\rangle\\ &= \langle (X_1,Y_1)\cup (X_2,Y_2),((X_1,Y_1)\cup (X_2,Y_2))^{\triangleleft}\rangle,\\ \langle (X_1,Y_1),A_1\rangle\vee_{\triangleright}\langle (X_2,Y_2),A_2\rangle &= \langle ((X_1,Y_1)\cap (X_2,Y_2))^{\triangleleft\triangleright},A_1\cup A_2\rangle\\ &= \langle (A_1\cup A_2)^{\triangleright},A_1\cup A_2\rangle.\end{aligned}\tag{21}$$

(4) For $\langle (X_1,Y_1),A_1\rangle,\langle (X_2,Y_2),A_2\rangle\in AC_3^{\blacktriangledown}(K)$, $\langle (X_1,Y_1),A_1\rangle\leq_{\blacktriangledown}\langle (X_2,Y_2),A_2\rangle$ iff $(X_2,Y_2)\subseteq (X_1,Y_1)$ (equivalently, $A_2\subseteq A_1$), and

$$\begin{aligned}\langle (X_1,Y_1),A_1\rangle\wedge_{\blacktriangledown}\langle (X_2,Y_2),A_2\rangle &= \left\langle (X_1,Y_1)\cup (X_2,Y_2),(A_1\cup A_2)^{\blacktriangledown\blacktriangle}\right\rangle\\ &= \left\langle (X_1,Y_1)\cup (X_2,Y_2),((X_1,Y_1)\cup (X_2,Y_2))^{\blacktriangle}\right\rangle,\\ \langle (X_1,Y_1),A_1\rangle\vee_{\blacktriangledown}\langle (X_2,Y_2),A_2\rangle &= \left\langle ((X_1,Y_1)\cap (X_2,Y_2))^{\blacktriangle\blacktriangledown},A_1\cap A_2\right\rangle\\ &= \left\langle (A_1\cap A_2)^{\blacktriangledown},A_1\cap A_2\right\rangle.\end{aligned}\tag{22}$$

The collection of A3W concepts of the same kind forms a complete lattice with the corresponding infimum and supremum defined in Definition 9.

Theorem 9. *Let $K=(U,V,R)$ be a formal context. Then, $\left(AC_3^{\bigstar}(K),\wedge_{\bigstar},\vee_{\bigstar}\right)$ is a complete lattice, where $\bigstar=\lessdot,\triangledown,\blacktriangledown$, and $\triangleright$, respectively.*

Proof. Similar to that of Theorem 6.

The four kinds of A3W concept lattices are order isomorphic to each other.

Theorem 10. *Let* $K = (U, V, R)$ *be a formal context. Then,* $AC_3^{\lessdot}(K) \cong AC_3^{\triangledown}(K) \cong AC_3^{\triangleright}(K) \cong AC_3^{\blacktriangledown}(K)$.

Proof. Similar to that of Theorem 7.

5. Conclusion

Three-way concept analysis is a new theory which investigates concepts in the background of three-way decision. In this paper, we examined the relationship between different kinds of three-way concept lattices in complete formal contexts by revisiting the connections between different kinds of two-way operators. A unified framework exhibiting the relationship between the eight kinds of two-way operators was given to gain an overall understanding of two-way operators and two-way concepts. On the basis of the connections between two-way operators, three-way operators are classified into four groups: Type-I O3W operators, Type-II O3W operators, Type-I A3W operators, and Type-II A3W operators. A Type-I O3W operator and a Type-II A3W operator determine a kind of O3W concept; a Type-II O3W operator and a Type-I A3W operator determine a kind of A3W concept. Based on the equivalences between different kinds of three-way concepts, we proved that the four kinds of O3W concept lattices are order isomorphic to each other and the four kinds of A3W concept lattices are also order isomorphic to each other.

The results in this paper provide a systematic understanding of three-way concepts including how to construct three-way concept lattices through existing ones. One can define a three-way concept in incomplete formal contexts through a Type-I O3W operator and a Type-II A3W operator, or through a Type-II O3W operator and a Type-I A3W operator [20]. Therefore, the relationship between three-way concepts in complete formal contexts can be applied into three-way concepts in incomplete formal contexts. On the other hand, by generalizing formal contexts to **L**-contexts, sets to **L**-sets, and binary relations to **L**-relations, one can use a similar method proposed in this paper to characterize the relationship between **L**-two-way concepts [1,3,9] and between **L**-three-way concepts [10]. These two subjects are our future work.

CRediT authorship contribution statement

Xuerong Zhao: Conceptualization, Methodology, Writing - original draft. **Duoqian Miao:** Conceptualization, Writing - review & editing, Funding acquisition. **Bao Qing Hu:** Conceptualization, Writing - review & editing, Funding acquisition.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The authors thank the editor and reviewers for their constructive comments and valuable suggestions which lead to a significant improvement on the manuscript. This research was supported in part by the National Nature Science Foundation of China (Grant Nos. 11971365, 61179038, 61976158, and 61673301).

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