Three-Way Group Conflict Analysis Based on Pythagorean Fuzzy Set Theory

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Abstract—In some real-world situations, Pythagorean fuzzy sets are more powerful and effective than intuitionistic fuzzy sets to describe vague and uncertain information, and there are many Pythagorean fuzzy information systems for conflicts in which attitudes of agents on issues are depicted by Pythagorean fuzzy numbers. In this paper, we first provide the concepts of positive, neutral, and negative alliances with two thresholds and employ examples to illustrate how to compute positive, neutral, and negative alliances in Pythagorean fuzzy information systems for conflicts. Then, we focus on three-way conflict analysis based on the Bayesian minimum risk theory and explore examples to show how to compute the positive, neutral, and negative alliances with a Pythagorean fuzzy loss function given by an expert. Finally, we study how to calculate positive, neutral, and negative alliances with group decision theory and take examples to demonstrate how to construct the positive, neutral, and negative alliances with a group of Pythagorean fuzzy loss functions given by more experts.

Index Terms—Bayesian minimum risk theory, conflict analysis, Pythagorean fuzzy information system, Pythagorean fuzzy loss function, Pythagorean fuzzy sets (PFSs).

I. INTRODUCTION

C ONFLICTS are undoubtedly one of the most essential characteristics of human society, and the study of which is of utmost significance both theoretically and practically. Especially, conflict analysis [1]–[21], which plays an important role in many fields, such as business, political, and legal disputes, investigates conflict structures with conflict, neutrality, and alliance relations and gives some guidance to conflict resolution. For example, Pawlak [1] initially considered the auxiliary functions and distance functions and offered deeper insight into the structure of conflicts. Cholvy *et al.* [4] proposed a method for estimating the relative reliability of information sources. Deja

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[7] transformed conflict analysis problems and conflict resolving problems into Boolean reasoning problems with the rough sets and Boolean reasoning methods. Jabbour et al. [9] provided the notion of conflicting variable and investigated quantifying conflicts in propositional logic through prime implicates. Ramanna et al. [13] studied how to model a combination of complex situations among agents where there are disagreements leading to a conflict situation. Silva and Almeida-Filho [14] presented a multicriteria approach for analysis of conflicts in evidence theory. Skowron and Deja [15] explained the nature of conflict and defined the conflict situation model in a way to encapsulate the conflict components in a clear manner. Sun *et al.* [18] proposed a conflict analysis decision model and developed a matrix approach for conflict analysis based on rough set theory over two universes. Yang et al. [19] investigated evidence conflict and belief convergence based on the analysis of the degree of coherence between two sources of evidence and illustrated the stochastic interpretation for basic probability assignments. Yu et al. [20] provided the supporting probability distance to characterize the differences among bodies of evidence and gave a new combination rule for the combination of the conflicting evidence. Zhu and Wang [21] studied the problems of conflicts of interest in database access security using granular computing based on covering rough set theory.

Three-way decision theory, proposed by Yao [22] for decision making with less risks, promotes thinking and problem solving in threes, such as using three regions, three elements, three views, three levels, and three stages. Many scholars [23]-[38] have developed three-way decision theory in theoretical and practical aspects, which has become a new mathematical tool to deal with uncertain information and problems. For instance, Chen et al. [23] focused on three-way decision support for diagnosis on focal liver lesions. Feng et al. [24] studied uncertainty and reduction of variable precision multigranulation fuzzy rough sets based on three-way decisions. Hu et al. [25] provided two types of three-way decisions in three-way decision spaces and discussed properties of the three-way decisions. Khan et al. [26] introduced a three-way approach for learning rules in automatic knowledge-based topic models. Li et al. [30] presented costsensitive sequential three-way decision modeling using a deep neural network. Qian et al. [31] investigated attribute reduction for sequential three-way decisions under dynamic granulation. Sun et al. [32] studied three-way group decision making based on multigranulation fuzzy decision-theoretic rough sets over two universes. Xu et al. [33] provided a three-way decision model with probabilistic rough sets for stream computing. Yang *et al.*

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[37] proposed a unified model of sequential three-way decisions and multilevel incremental processing.

Pythagorean fuzzy sets (PFSs), introduced by Yager and Abbasov [39] for describing uncertainty, are considered as a generalization of intuitionistic fuzzy sets (IFSs) and characterized by a membership degree and a nonmembership degree satisfying the condition that the square sum of its membership degree and nonmembership degree is equal to or less than one. Many investigations [39]–[57] have focused on PFSs, which have more powerful ability than IFSs to model the uncertain information in decision making problems. For example, Beliakov and James [41] provided the averaging aggregation functions for preferences expressed as Pythagorean membership grades and fuzzy orthopairs. Bustince et al. [42] investigated a historical account of types of fuzzy sets and discussed their relationships. Peng and Yang [45] developed a Pythagorean fuzzy superiority and inferiority ranking method to solve uncertainty multiple attribute group decision making problem. Peng and Selvachandran [46] presented an overview on PFSs with aim of offering a clear perspective on the different concepts, tools, and trends related to their extension and provided two novel algorithms in decision making problems under Pythagorean fuzzy environment. Reformat and Yager [50] proposed a novel collaborative-based recommender system that provides a user with the ability to control a process of constructing a list of suggested items using PFSs. Ren et al. [51] extended the TODIM approach to solve the multi-criteria decision making (MCDM) problems with Pythagorean fuzzy information and analyzed how the risk attitudes of the decision makers exert the influence on the results of MCDM under uncertainty. Wu and Liu [52] proposed a knowledge-augmented logical analysis framework for policy conflicts in order to make services collaboration possible and smooth. Zhang [55] presented a hierarchical QUALIFLEX approach with the closeness index-based ranking methods for multicriteria Pythagorean fuzzy decision analysis. Zhang et al. [57] introduced the models of Pythagorean fuzzy rough sets over two universes and Pythagorean fuzzy multigranulation rough sets over two universes.

In conflict analysis, we are mainly interested in finding the relationship among agents taking part in the dispute and study what measures can be taken for solving the conflict. In this paper, we investigate how to compute positive, neutral, and negative alliances based on PFS theory. The motivations and innovations of this paper are given by answering the following three questions.

1) Why study conflicts based on PFS theory? In practice, PFSs are more suitable than IFSs for describing attitudes of agents in conflicts. For instance, when a person expresses his preference about the degree of an issue, he gives the degree to support this issue as $\frac{\sqrt{3}}{2}$, and the degree to against this issue as $\frac{1}{2}$, and we have $(\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 = 1$ and $\frac{\sqrt{3}}{2} + \frac{1}{2} > 1$, and IFSs cannot work in this situation. Furthermore, Pythagorean fuzzy loss functions are more accurate than intuitionistic fuzzy loss functions for measuring losses and risks in decision making problems, which helps people make decisions with less losses and risks.

- 2) Why investigate conflicts with three-way decision theory and group decision theory? In conflict situations, we ask the agents to specify their views from disagreement, neutral, and agreement and classify all agents into conflict set, neutral set, and alliance set of an agent with conflict, neutral, and alliance relations, respectively. We also find that three-way decision theory partitions all agents into positive, boundary, and negative regions based on the Bayesian minimum risk theory, which is consistent with the thought of conflict analysis. Moreover, we see that Pythagorean fuzzy loss functions are given by experts, and different experts have different opinions for the same problem and give different Pythagorean fuzzy loss functions. We employ a group of Pythagorean fuzzy loss functions given by many famous experts to calculate positive, neutral, and negative alliances in conflict analysis so as to make decisions with less losses and risks.
- 3) What are innovations of this paper? We have not observed studies on Pythagorean fuzzy information systems for conflicts, where attitudes of agents are Pythagorean fuzzy numbers (PFNs). The innovations of this paper mainly include the following.
 - Construct positive, neutral, and negative alliances with three-way decision theory.
 - Classify all agents into positive, neutral, and negative alliances based on Bayesian minimum risk theory.
 - Employ a group of Pythagorean fuzzy loss functions to compute positive, neutral, and negative alliances with the minimum risk.

The contributions of this paper are shown as follows. First, we provide the concept of Pythagorean fuzzy information system and employ an example to illustrate the difference between Pawlak information systems and Pythagorean fuzzy information systems. We provide the concepts of positive, neutral, and negative alliances with two thresholds and employ several examples to illustrate how to compute the positive, neutral, and negative alliances in Pythagorean fuzzy information systems for conflicts. Second, we provide the concept of Pythagorean fuzzy loss function for conflict analysis of Pythagorean fuzzy information systems, and illustrate mechanisms of computing the positive, neutral, and negative alliances based on Bayesian minimum risk theory. We also employ several examples to illustrate how to compute the positive, neutral, and negative alliances with a Pythagorean fuzzy loss function given by an expert. Third, we demonstrate mechanisms of calculating the positive, neutral, and negative alliances for conflict analysis with group decision theory, and employ several examples to illustrate how to construct the positive, neutral, and negative alliances with a group of Pythagorean fuzzy loss functions given by more experts.

The rest of this paper is organized as follows. Section II reviews the basic concepts of PFSs and conflict analysis. Section III proposes the concepts of positive, neutral, and negative alliances with two thresholds. Section IV focuses on threeway conflict analysis based on Bayesian minimum risk theory. Section V provides three-way group conflict analysis based on group decision theory. The conclusion is given in Section VI.



Fig. 1. Relationship between a PFN and an intuitionistic fuzzy number.

II. PRELIMINARIES

In this section, we review the related concepts of PFSs and conflict analysis.

A. Pythagorean Fuzzy Sets

Definition 2.1 (see [53]): Let U be an arbitrary nonempty set, and a PFS P is a mathematical object of the form as follows:

$$P = \{ \langle x, P(\mu_P(x), \nu_P(x)) \rangle | x \in U \}$$

where $\mu_P(x), \nu_P(x) : U \to [0, 1]$ such as $\mu_P^2(x) + \nu_P^2(x) \le 1$, for every $x \in U$, $\mu_P(x)$ and $\nu_P(x)$ denote the membership degree and the nonmembership degree of the element $x \in U$ in P, respectively.

For convenience, we denote the PFN and the hesitant degree as $\gamma = P(\mu_{\gamma}, \nu_{\gamma})$ and $\pi_{\gamma} = \sqrt{1 - \mu_{\gamma}^2 - \nu_{\gamma}^2}$, respectively. Moreover, if $\gamma = P(\mu_{\gamma}, \nu_{\gamma})$ satisfying $\mu_{\gamma} + \nu_{\gamma} \leq 1$, then γ is an intuitionistic fuzzy number, and the relationship between a PFN and an intuitionistic fuzzy number is illustrated by Fig. 1. Therefore, PFSs, as a generalization of IFSs, are powerful for describing imprecise information.

Definition 2.2 (see [53]): Let $\gamma_1 = P(\mu_{\gamma_1}, \nu_{\gamma_1})$ and $\gamma_2 = P(\mu_{\gamma_2}, \nu_{\gamma_2})$ be PFNs. Then, a nature quasi-ordering on PFNs is defined as follows:

$$\gamma_1 \geq \gamma_2$$
 if and only if $\mu_{\gamma_1} \geq \mu_{\gamma_2}$ and $\nu_{\gamma_1} \leq \nu_{\gamma_2}$.

For convenience, we define the function $I(\gamma) = \gamma$ for any PFN γ . So $\gamma_1 \ge \gamma_2 \Leftrightarrow I(\gamma_1) \ge I(\gamma_2)$. Moreover, Yager provided multiplication and summation operations for PFNs as follows.

1)
$$k\gamma = P(\sqrt{1 - (1 - \mu_{\gamma}^2)^k}, \nu_{\gamma}^k).$$

2) $\gamma_1 \oplus \gamma_2 = P(\sqrt{\mu_{\gamma_1}^2 + \mu_{\gamma_2}^2 - \mu_{\gamma_1}^2 * \mu_{\gamma_2}^2}, \nu_{\gamma_1} * \nu_{\gamma_2})$
Definition 2.3 (see (561): Let $\gamma = P(\mu_1, \mu_2)$ be a

Definition 2.3 (see [56]): Let $\gamma = P(\mu_{\gamma}, \nu_{\gamma})$ be a PFN. Then, the score function S for γ is defined as follows:

$$S(\gamma) = \mu_{\gamma}^2 - \nu_{\gamma}^2.$$

We have that $-1 \le S(\gamma) \le 1$ for the PFN γ . Especially, the score function is effective to discern PFNs.

Definition 2.4 (see [56]): Let $\gamma_1 = P(\mu_{\gamma_1}, \nu_{\gamma_1})$ and $\gamma_2 = P(\mu_{\gamma_2}, \nu_{\gamma_2})$ be PFNs. Then, the Euclidean distance d between γ_1 and γ_2 is defined as follows:

$$d(\gamma_1, \gamma_2) = \frac{1}{2} (|\mu_{\gamma_1}^2 - \mu_{\gamma_2}^2| + |\nu_{\gamma_1}^2 - \nu_{\gamma_2}^2| + |\pi_{\gamma_1}^2 - \pi_{\gamma_2}^2|).$$

Especially, we obtain the Euclidean distance between the PFN $P(\mu_{\gamma}, \nu_{\gamma})$ and the positive ideal PFN $\gamma^+ = P(1, 0)$ as follows:

$$d(\gamma, \gamma^{+}) = \frac{1}{2}(1 - \mu_{\gamma}^{2} + \nu_{\gamma}^{2} + \pi_{\gamma}^{2}) = 1 - \mu_{\gamma}^{2}$$

and the Euclidean distance between the PFN $P(\mu_{\gamma}, \nu_{\gamma})$ and the negative ideal PFN $\gamma^{-} = P(0, 1)$ as follows:

$$d(\gamma, \gamma^{-}) = \frac{1}{2}(1 - \nu_{\gamma}^{2} + \mu_{\gamma}^{2} + \pi_{\gamma}^{2}) = 1 - \nu_{\gamma}^{2}.$$

Definition 2.5 (see [55]): Let $\gamma = P(\mu_{\gamma}, \nu_{\gamma})$ be a PFN, $\gamma^+ = P(1, 0)$ and $\gamma^- = P(0, 1)$. Then, the closeness index \mathscr{P} for γ is defined as follows:

$$\mathscr{P}(\gamma) = \frac{d(\gamma, \gamma^-)}{d(\gamma, \gamma^+) + d(\gamma, \gamma^-)} = \frac{1 - \nu_\gamma^2}{2 - \mu_\gamma^2 - \nu_\gamma^2}$$

We see that the closeness index $\mathscr{P}(\gamma)$ of γ is constructed based on the Euclidean distance between the PFN $P(\mu_{\gamma}, \nu_{\gamma})$ and the positive ideal PFN γ^+ and the Euclidean distance between the PFN $P(\mu_{\gamma}, \nu_{\gamma})$ and the negative ideal PFN γ^- . Especially, we have $0 \leq \mathscr{P}(\gamma) \leq 1$ for the PFN γ .

Definition 2.6 (see [39]): Let $\gamma = P(\mu_{\gamma}, \nu_{\gamma})$ be a PFN. Then, the function F for γ is defined as follows:

$$F(\gamma) = \frac{1}{2} + \sqrt{\mu_{\gamma}^2 + \nu_{\gamma}^2} * \left(\frac{1}{2} - \frac{2 \arccos(\frac{\mu_{\gamma}}{\sqrt{\mu_{\gamma}^2 + \nu_{\gamma}^2}})}{\pi}\right).$$

The function F provides an effective approach to comparing PFNs. Moreover, by Definitions 2.2–2.6, we provide the comparison law for discerning PFNs as follows.

Definition 2.7: Let $\gamma_1 = P(\mu_{\gamma_1}, \nu_{\gamma_1})$ and $\gamma_2 = P(\mu_{\gamma_2}, \nu_{\gamma_2})$ be PFNs, and $\bullet = I, S, \mathscr{P}, F$. Then, the following statements can be concluded:

- 1) if $\bullet(\gamma_1) > \bullet(\gamma_2)$, then γ_1 is bigger than γ_2 , denoted by $\gamma_1 \succ \bullet \gamma_2$;
- if •(γ₁) < •(γ₂), then γ₁ is smaller than γ₂, denoted by γ₁ ≺ • γ₂; and
- if •(γ₁) = •(γ₂), then γ₁ is equal to γ₂, denoted by γ₁ ~.
 γ₂.

We employ the following example to illustrate how to discern PFNs with Definition 2.7.

Example 2.8:

1) Taking $\gamma_1 = P(\frac{\sqrt{5}}{3}, \frac{1}{3})$ and $\gamma_2 = P(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3})$, we have $I(\gamma_1) = P(\frac{\sqrt{5}}{3}, \frac{1}{3})$ and $I(\gamma_2) = P(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3})$. Since $\frac{\sqrt{5}}{3} > \frac{\sqrt{2}}{3}$ and $\frac{1}{3} < \frac{\sqrt{2}}{3}$, then we have $\gamma_1 \succ_I \gamma_2$. Sometimes, we cannot discern PFNs with Definition 2.2.

For example, it does not work for $\gamma_1 = P(\frac{\sqrt{5}}{3}, \frac{\sqrt{2}}{3})$ and $\gamma_2 = P(\frac{\sqrt{2}}{3}, \frac{1}{3})$.

2) Taking $\gamma_1 = P(\frac{\sqrt{5}}{3}, \frac{\sqrt{2}}{3})$ and $\gamma_2 = P(\frac{\sqrt{2}}{3}, \frac{1}{3})$, by Definition 2.3, we have $S(\gamma_1) = S(P(\frac{\sqrt{5}}{3}, \frac{\sqrt{2}}{3})) = \frac{3}{9}$ and $S(\gamma_2)=S(P(\frac{\sqrt{2}}{3},\frac{1}{3}))=\frac{1}{9}.$ Therefore, we have $\gamma_1\succ_S$

We see that the score function fails to discern some PFNs. For example, for $\gamma_1 = P(\frac{\sqrt{5}}{3}, \frac{2}{3})$ and $\gamma_2 = P(\frac{2}{3}, \frac{\sqrt{3}}{3})$, we have

$$S(\gamma_1) = \frac{1}{9}$$
 and $S(\gamma_2) = \frac{1}{9}$

3) Taking $\gamma_1 = P(\frac{\sqrt{5}}{3}, \frac{2}{3})$ and $\gamma_2 = P(\frac{2}{3}, \frac{\sqrt{3}}{3})$, by Definition 2.5, we have

$$\mathscr{P}(\gamma_1) = \frac{5}{9}$$
 and $\mathscr{P}(\gamma_2) = \frac{6}{11}$.

Therefore, we get $\gamma_1 \succ \mathscr{P} \gamma_2$. 4) Taking $\gamma_1 = P(\frac{\sqrt{5}}{3}, \frac{2}{3})$ and $\gamma_2 = P(\frac{2}{3}, \frac{\sqrt{3}}{3})$, by Definition 2.6, we have

$$F(\gamma_1) \approx 0.5355$$
 and $F(\gamma_2) \approx 0.5402$.

Therefore, we get $\gamma_2 \succ_F \gamma_1$.

We observe that there are four types of functions for comparing PFNs. If we choose one of them to discern PFNs, and it does not work, then the other functions can be applied in practical situations.

B. Conflict Analysis

Definition 2.9 (see [1]): An information system is a 4-tuple S = (U, A, V, f), where $U = \{x_1, x_2, \dots, x_n\}$ is a finite set of agents, A is a finite set of issues, $V = \{V_c \mid c \in A\}$, where V_c is the set of values of issue c, and $card(V_c) > 1$, f is a function from $U \times A$ into V.

The classical information system given by Definition 2.9 is called Pawlak information system, and information systems mentioned in this section are Pawlak information systems.

Definition 2.10 (see [2]): Let S = (U, A, V, f) be an information system. Then, the auxiliary function $\phi_c(x, y)$ for any $c \in A$ is defined as follows:

$$\phi_c(x,y) = \begin{cases} 1, & \text{if } c(x) \cdot c(y) = 1 \lor x = y \\ 0, & \text{if } c(x) \cdot c(y) = 0 \land x \neq y \\ -1, & \text{if } c(x) \cdot c(y) = -1 \end{cases}$$

where c(x) and c(y) denote issue values of x and y on c, respectively.

If $\phi_c(x, y) = 1$, then x and y have the same opinion about issue c; if $\phi_c(x, y) = 0$, then it means that x or y has a neutral opinion about issue c; and if $\phi_c(x, y) = -1$, then x and y have different opinions about issue c.

Example 2.11 (see [2]): Table I shows the information system for the Middle East conflict, where x_1, x_2, x_3, x_4, x_5 , and x_6 denote six countries and c_1, c_2, c_3, c_4, c_5 , and c_6 denote six issues. For example, $c_1(x_1) = -1$ denotes the agent x_1 is against the issue c_1 , and $c_1(x_2) = +1$ denotes the agent x_2 supports the issue c_1 , and $c_1(x_4) = 0$ denotes the agent x_4 is neutral to the issue c_1 .

TABLE I INFORMATION SYSTEM FOR THE MIDDLE EAST CONFLICT

U	c_1	<i>c</i> ₂	С3	С4	c_5
x_1	-1	+1	+1	+1	+1
<i>x</i> ₂	+1	0	-1	-1	-1
<i>x</i> ₃	+1	-1	-1	-1	0
<i>x</i> ₄	0	-1	-1	0	-1
<i>x</i> ₅	+1	-1	-1	-1	-1
<i>x</i> ₆	0	+1	-1	0	+1

Remarks: x_1, x_2, x_3, x_4, x_5 , and x_6 denote Israel, Egypt, Palestine, Jordan, Syria, and Saudi Arabia, respectively. Moreover, c_1 means autonomous Palestinian state on the West Bank and Gaza; c2 denotes Israeli military outpost along the Jordan River; c_3 stands for Israel retains East Jerusalem; c_4 is Israeli military outposts on the Golan Heights; and c_5 denotes Arab countries grant citizenship to Palestinians who choose to remain within their borders.

Definition 2.12 (see [2]): Let S = (U, A, V, f) be an information system. Then, the distance function $\rho_A(x, y)$ for $x, y \in$ U is defined as follows:

$$\rho_A(x,y) = \frac{\sum_{c \in A} \phi_c^*(x,y)}{|A|}$$

where

$$\phi_c^*(x,y) = \frac{1 - \phi_c(x,y)}{2} = \begin{cases} 0, & \text{if } c(x) \cdot c(y) = 1 \lor x = y\\ 0.5, & \text{if } c(x) \cdot c(y) = 0 \land x \neq y\\ 1, & \text{if } c(x) \cdot c(y) = -1 \end{cases}$$

After that, Pawlak provided the conflict, neutral, and allied relations for conflict analysis with Definition 2.12 as follows.

Definition 2.13 (see [2]): Let S = (U, A, V, f) be an information system, and the distance function $\rho_A(x, y)$ for $x, y \in U$. Then, a pair x and y is said to be

- 1) conflict if $\rho_A(x, y) > 0.5$;
- 2) neutral if $\rho_A(x, y) = 0.5$; and
- 3) allied if $\rho_A(x, y) < 0.5$.

Pawlak also proposed the allied, conflict, and neutral sets as follows.

Definition 2.14 (see [2]): Let S = (U, A, V, f) be an information system. Then, the conflict, neutral, and allied sets of $x \in U$ are defined as follows.

- 1) $CO(x) = \{y \in U \mid \rho_A(x, y) > 0.5\}.$
- 2) $NE(x) = \{y \in U \mid \rho_A(x, y) = 0.5\}.$
- 3) $AL(x) = \{y \in U \mid \rho_A(x, y) < 0.5\}.$

We classify all agents with respect to x into three parts: M(x), M(x), and AL(x). Since decision-theoretic rough set theory is a powerful mathematical tool for depicting ambiguous information, Lang et al. [27] investigated conflict analysis using decision-theoretic rough set theory, which actually provides constructive advice for decision making with less loss.

Definition 2.15 (see [27]): Let S = (U, A, V, f) be an information system, and $0 \le \beta \le \alpha \le 1$. For any $x \in U$, the probabilistic conflict, neutral, and allied sets $CO^{\alpha}_{\beta}(x)$, $NE^{\alpha}_{\beta}(x)$, and $AL^{\alpha}_{\beta}(x)$ of x are defined as follows.

1)
$$CO^{\alpha}_{\beta}(x) = \{y \in U \mid \rho_A(x, y) > \alpha\}.$$

2) $NE^{\alpha}_{\beta}(x) = \{y \in U \mid \alpha \ge \rho_A(x, y) \ge \beta\}.$

TABLE II Pythagorean Fuzzy Information System for the Middle East Conflict

U	c_1	<i>c</i> ₂	<i>c</i> ₃	С4	c_5
x_1	P(1.0, 0.0)	P(0.9, 0.3)	P(0.8, 0.2)	P(0.9, 0.1)	P(0.9, 0.2)
x_2	P(0.9, 0.1)	P(0.5, 0.5)	P(0.1, 0.9)	P(0.3, 0.8)	P(0.1, 0.9)
<i>x</i> ₃	P(0.1, 0.9)	P(0.1, 0.9)	P(0.2, 0.8)	P(0.1, 0.9)	P(0.5, 0.5)
<i>x</i> 4	P(0.5, 0.5)	P(0.1, 0.9)	P(0.3, 0.7)	P(0.5, 0.5)	P(0.1, 0.9)
x_5	P(0.9, 0.2)	P(0.4, 0.6)	P(0.1, 0.9)	P(0.1, 0.9)	P(0.3, 0.9)
x_6	P(0.0, 1.0)	P(0.9, 0.1)	P(0.2, 0.9)	P(0.5, 0.5)	P(0.8, 0.4)

3)
$$AL^{\alpha}_{\beta}(x) = \{y \in U \mid \rho_A(x, y) < \beta\}$$

In some practical situations, PFSs are effective for describing uncertain information, and there are some Pythagorean fuzzy information systems for conflicts, where all issue values are PFNs, and there has been relatively little progress in developing effective methods for studying Pythagorean fuzzy information systems for conflicts.

III. CONFLICT ANALYSIS OF PYTHAGOREAN FUZZY INFORMATION SYSTEMS

In this section, we investigate Pythagorean fuzzy information systems for conflicts.

Definition 3.1: A Pythagorean fuzzy information system is a 4-tuple S = (U, A, V, f), where $U = \{x_1, x_2, \ldots, x_n\}$ is a finite set of agents, $A = \{c_1, c_2, \ldots, c_l\}$ is a finite set of issues, $V = \{V_c \mid c \in A\}$, where V_c is the set of issue values on c, all issue values are PFNs, and f is a function from $U \times A$ into V.

We see that Pythagorean fuzzy information systems, as a generalization of Pawlak information systems, represent all available information and knowledge, where agents are measured by using a finite number of issues and issue values are PFNs, which provides more information than intuitionistic fuzzy information systems. Furthermore, we provide matrix representation M(S)of the Pythagorean fuzzy information system S for conflict analysis as follows:

where *n* and *l* are the numbers of agents and issues, respectively. *Example 3.2:*

1) We employ a Pythagorean fuzzy information system depicted by Table II to show the Middle East conflict, where x_1, x_2, x_3, x_4, x_5 , and x_6 denote six agents and c_1, c_2, c_3, c_4, c_5 , and c_6 denote six issues. For example, we have $c_1(x_1) = P(\mu_P(x_1), \nu_P(x_1)) = P(1.0, 0.0)$, where $\mu_P(x_1) = 1.0$ denotes the support degree of the agent x_1 to the issue c_1 , and $\nu_P(x_1) = 0.0$ denotes the against degree of the agent x_1 to the issue c_1 ; we have $c_5(x_6) = P(\mu_P(x_6), \nu_P(x_6)) = P(0.8, 0.4)$, where $\mu_P(x_6) = 0.8$ denotes the support degree of the agent x_6 to the issue c_5 .

 From Table II, we have the Pythagorean matrix M(S) of the Pythagorean fuzzy information system S in Example 3.2(1) as in unnumbered equation shown at the bottom of this page:

Remark: We denote Israel, Egypt, Palestinians, Jordan, Syria, and Saudi Arabia as x_1, x_2, x_3, x_4, x_5 , and x_6 , respectively; c_1 means autonomous Palestinian state on the West Bank and Gaza; c_2 denotes Israeli military outpost along the Jordan River; c_3 stands for Israeli retains East Jerusalem; c_4 is Israeli military outposts on the Golan Heights; c_5 notes Arab countries grant citizenship to Palestinians who choose to remain with their borders. Furthermore, we employ Tables I and II to depict the Middle East conflict, and there is no relationship among issue values of agents. We also employ Pythagorean matrix M(S) to represent the Pythagorean fuzzy information system S, which provides an effective tool for studying Pythagorean fuzzy information systems for conflicts.

Definition 3.3 (see [53]): Let $\mathscr{P} = \{\gamma_i | \gamma_i = P(\mu_{\gamma_i}, \nu_{\gamma_i}), i = 1, 2, \ldots, l\}$ be a collection of PFNs, and $\mathscr{K} = \{k_1, k_2, \ldots, k_l\}$ be the weight vector of γ_i $(i = 1, 2, \ldots, l)$, where k_i indicates the importance degree of γ_i , and satisfies $k_i \ge 0$ $(i = 1, 2, \ldots, l)$ and $\Sigma_{i=1}^l k_i = 1$. Then, the Pythagorean fuzzy weighted averaging operator $\mathscr{R}: \Theta^l \to \Theta$ is defined as follows: $\mathscr{R}(\gamma_1, \gamma_2, \ldots, \gamma_l) = P(\Sigma_{i=1}^l k_i \mu_{\gamma_i}, \Sigma_{i=1}^l k_i \nu_{\gamma_i})$.

By Definition 3.3, we aggregate a collection of PFNs $\{\gamma_i | \gamma_i = P(\mu_{\gamma_i}, \nu_{\gamma_i}), i = 1, 2, ..., l\}$ into a PFN $\mathscr{R}(\gamma_1, \gamma_2, ..., \gamma_l)$ with the weight vector. For simplicity, we denote $\mathscr{R}(c_1(x), c_2(x), ..., c_l(x))$ as $\mathscr{R}(x)$ in the following discussion. Moreover, we provide the positive, neutral, and negative alliances with two thresholds as follows.

Definition 3.4: Let S = (U, A, V, f) be a Pythagorean fuzzy information system, α and β are two thresholds, and \bullet denotes a function for PFNs. Then, the positive, neutral, and negative alliances are defined as follows:

$$POA_{(\bullet,\alpha,\beta)}(U) = \{x \in U \mid \bullet(\mathscr{R}(x)) \ge \alpha\}$$
$$CTA_{(\bullet,\alpha,\beta)}(U) = \{x \in U \mid \beta < \bullet(\mathscr{R}(x)) < \alpha\}$$
$$NEA_{(\bullet,\alpha,\beta)}(U) = \{x \in U \mid \bullet(\mathscr{R}(x)) \le \beta\}.$$

	P(1.0, 0.0)	P(0.9, 0.3)	P(0.8, 0.2)	P(0.9, 0.1)	P(0.9, 0.2)
	P(0.9, 0.1)	P(0.5, 0.5)	P(0.1, 0.9)	P(0.3, 0.8)	P(0.1, 0.9)
M(S) =	P(0.1, 0.9)	P(0.1, 0.9)	P(0.2, 0.8)	P(0.1, 0.9)	P(0.5, 0.5)
M(S) =	P(0.5, 0.5)	P(0.1, 0.9)	P(0.3, 0.7)	P(0.5, 0.5)	P(0.1, 0.9)
	P(0.9, 0.2)	P(0.4, 0.6)	P(0.1, 0.9)	P(0.1, 0.9)	P(0.3, 0.9)
	P(0.0, 1.0)	P(0.9, 0.1)	P(0.2, 0.9)	P(0.5, 0.5)	P(0.8, 0.4)

By Definition 3.4, we get the positive, neutral, and negative alliances $POA_{(\bullet,\alpha,\beta)}(U)$, $CTA_{(\bullet,\alpha,\beta)}(U)$, and $NEA_{(\bullet,\alpha,\beta)}(U)$ with two thresholds α and β , and we have $POA_{(\bullet,\alpha,\beta)}(U) \cup$ $CTA_{(\bullet,\alpha,\beta)}(U) \cup NEA_{(\bullet,\alpha,\beta)}(U) \subseteq U$. Furthermore, we provide four types of positive, neutral, and negative alliances when $\bullet = I, S, \mathscr{P}, F$ in Definition 3.4 as follows.

Definition 3.5: Let S = (U, A, V, f) be a Pythagorean fuzzy information system.

1) If α and β are PFNs, and $P(0,1) \le \beta \le \alpha \le P(1,0)$, then we define the first positive, neutral, and negative alliances as follows:

$$POA_{(I,\alpha,\beta)}(U) = \{x \in U \mid I(\mathscr{R}(x)) \ge \alpha\}$$

$$CTA_{(I,\alpha,\beta)}(U) = \{x \in U \mid \beta < I(\mathscr{R}(x)) < \alpha\}$$

$$NEA_{(I,\alpha,\beta)}(U) = \{x \in U \mid I(\mathscr{R}(x)) \le \beta\}.$$

 If −1 ≤ β ≤ α ≤ 1, then we define the second positive, neutral, and negative alliances as follows:

$$\begin{aligned} &POA_{(S,\alpha,\beta)}(U) = \ \{x \in U \mid S(\mathscr{R}(x)) \geq \alpha\} \\ &CTA_{(S,\alpha,\beta)}(U) = \ \{x \in U \mid \beta < S(\mathscr{R}(x)) < \alpha\} \\ &NEA_{(S,\alpha,\beta)}(U) = \ \{x \in U \mid S(\mathscr{R}(x)) \leq \beta\}. \end{aligned}$$

3) If $0 \le \beta \le \alpha \le 1$, then we define the third positive, neutral, and negative alliances as follows:

$$POA_{(\mathscr{P},\alpha,\beta)}(U) = \{x \in U \mid \mathscr{P}(\mathscr{R}(x)) \ge \alpha\}$$
$$CTA_{(\mathscr{P},\alpha,\beta)}(U) = \{x \in U \mid \beta < \mathscr{P}(\mathscr{R}(x)) < \alpha\}$$
$$NEA_{(\mathscr{P},\alpha,\beta)}(U) = \{x \in U \mid \mathscr{P}(\mathscr{R}(x)) \le \beta\}.$$

4) If $0 \le \beta \le \alpha \le 1$, then we define the fourth positive, neutral, and negative alliances as follows:

$$\begin{aligned} &POA_{(F,\alpha,\beta)}(U) = \ \{x \in U \mid F(\mathscr{R}(x)) \geq \alpha\} \\ &CTA_{(F,\alpha,\beta)}(U) = \ \{x \in U \mid \beta < F(\mathscr{R}(x)) < \alpha\} \\ &NEA_{(F,\alpha,\beta)}(U) = \ \{x \in U \mid F(\mathscr{R}(x)) \leq \beta\} \end{aligned}$$

 $\begin{array}{ll} \text{where} \quad \mathscr{R}(x) = (\mu(x), \nu(x)) \quad \text{and} \quad F(\mathscr{R}(x)) = \frac{1}{2} + \\ \sqrt{\mu(x)^2 + \nu(x)^2} * (\frac{1}{2} - \frac{2 \arccos(\frac{\mu(x)}{\sqrt{\mu(x)^2 + \nu(x)^2}})}{\pi}) \text{ for } x \in U. \\ \text{Example 3.6 (Continuation from Example 3.2):} \end{array}$

1) Taking $k_1 = k_2 = k_3 = k_4 = k_5 = \frac{1}{5}$, $\alpha = P(0.7, 0.4)$ and $\beta = P(0.25, 0.85)$. By Definition 3.3, we have the Pythagorean fuzzy weighted averaging closeness index of x_1, x_2, x_3, x_4, x_5 , and x_6 on A as follows:

$$I(\mathscr{R}(x_1)) = P(0.90, 0.16), I(\mathscr{R}(x_2)) = P(0.38, 0.64),$$

$$I(\mathscr{R}(x_3)) = P(0.20, 0.80), I(\mathscr{R}(x_4)) = P(0.30, 0.70),$$

$$I(\mathscr{R}(x_5)) = P(0.36, 0.70), \text{ and } I(\mathscr{R}(x_6)) = P(0.48, 0.58).$$

By Definition 3.5(1), we have $POA_{(I,\alpha,\beta)}(U) = \{x_1\}$, $CTA_{(I,\alpha,\beta)}(U) = \{x_2, x_4, x_5, x_6\}$, and $NEA_{(I,\alpha,\beta)}(U)$ $= \emptyset$. So, we classify $\{x_1, x_2, x_4, x_5, x_6\}$ into $POA_{(I,\alpha,\beta)}(U), CTA_{(I,\alpha,\beta)}(U)$, and $NEA_{(I,\alpha,\beta)}(U)$. But it does not work for x_3 . Furthermore, we see that $\{x_1\}$ and $\{x_2, x_4, x_5, x_6\}$ are difference alliances, but x_3 does not belong to any alliance. 2) Taking $\alpha = 0.5$ and $\beta = -0.5$, we have

$$\begin{split} S(\mathscr{R}(x_1)) &= +0.7844, S(\mathscr{R}(x_2)) = -0.2652, \\ S(\mathscr{R}(x_3)) &= -0.6000, S(\mathscr{R}(x_4)) = -0.4000, \\ S(\mathscr{R}(x_5)) &= -0.3604, \text{ and } S(\mathscr{R}(x_6)) = -0.1060. \end{split}$$

By Definition 3.5(2), we have $POA_{(S,\alpha,\beta)}(U) = \{x_1\}, CTA_{(S,\alpha,\beta)}(U) = \{x_2, x_4, x_5, x_6\}, \text{ and } NEA_{(S,\alpha,\beta)}(U) = \{x_3\}.$ Furthermore, we see that $\{x_1\}, \{x_3\}, \text{ and } \{x_2, x_4, x_5, x_6\}$ are difference alliances. Especially, $\{x_1\}$ and $\{x_3\}$ have different opinions on issues, and $\{x_2, x_4, x_5, x_6\}$ is a neutral alliance.

3) Taking $\alpha = 0.75$ and $\beta = 0.3$, we have

$$\begin{split} \mathscr{P}(\mathscr{R}(x_1)) &= \ 0.8368, \, \mathscr{P}(\mathscr{R}(x_2)) = 0.4083, \\ \mathscr{P}(\mathscr{R}(x_3)) &= \ 0.2727, \, \mathscr{P}(\mathscr{R}(x_4)) = 0.3592, \\ \mathscr{P}(\mathscr{R}(x_5)) &= \ 0.3695, \text{ and } \, \mathscr{P}(\mathscr{R}(x_6)) = 0.4584. \end{split}$$

By Definition 3.5(3), we have $POA_{(\mathscr{P},\alpha,\beta)}(U) = \{x_1\}$, $CTA_{(\mathscr{P},\alpha,\beta)}(U) = \{x_2, x_4, x_5, x_6\}$, and $NEA_{(\mathscr{P},\alpha,\beta)}(U) = \{x_3\}$. Moreover, we observe that $\{x_1\}$, $\{x_3\}$, and $\{x_2, x_4, x_5, x_6\}$ are difference alliances. Especially, $\{x_2, x_4, x_5, x_6\}$ is a neutral alliance, and $\{x_1\}$ and $\{x_3\}$ are opposite alliances.

4) Taking $\alpha = 0.75$ and $\beta = 0.3$, we have

$$\begin{split} F(\mathscr{R}(x_1)) &= 0.8547, F(\mathscr{R}(x_2)) = 0.3817, \\ F(\mathscr{R}(x_3)) &= 0.2163, F(\mathscr{R}(x_4)) = 0.3155, \\ F(\mathscr{R}(x_5)) &= 0.3445, \text{ and} F(\mathscr{R}(x_6)) = 0.4549. \end{split}$$

By Definition 3.5(4), we have $POA_{(F,\alpha,\beta)}(U) = \{x_1\}$, $CTA_{(F,\alpha,\beta)}(U) = \{x_2, x_4, x_5, x_6\}$, and $NEA_{(F,\alpha,\beta)}(U) = \{x_3\}$. Moreover, we find that $\{x_1\}$, $\{x_3\}$, and $\{x_2, x_4, x_5, x_6\}$ are difference alliances. Especially, $\{x_2, x_4, x_5, x_6\}$ is a neutral alliance, and $\{x_1\}$ and $\{x_3\}$ are opposite alliances.

By Definition 3.5, we partition the universe into three regions: positive, neutral, and negative alliances with different operators, and denote the positive, neutral, and negative alliances of U as POA(U), CTA(U), and NEA(U) for simplicity. Furthermore, we classify all agents into three regions by Definition 3.5(1) when they are depicted by PFNs. If Definition 3.5(1) does not work, then we choose Definition 3.5(2) to partition these agents. Especially, if Definitions 3.5(1) and 3.5(2) do not work, we apply Definition 3.5(3) to classify these agents.

IV. THREE-WAY CONFLICT ANALYSIS OF PYTHAGOREAN FUZZY INFORMATION SYSTEMS

In this section, we study Pythagorean fuzzy information systems for conflicts based on three-way decision theory and Bayesian minimum risk theory.

Definition 4.1: A Pythagorean fuzzy loss function given by an expert is a 3-tuple $\lambda = (\Omega, \mathscr{A}, \mathscr{L})$ shown as Table III, where $\Omega = \{X, \neg X\}$ is a set of two states, $\mathscr{A} = \{a_P, a_B, a_N\}$ is a set of three actions for each state, $\mathscr{L} = \{\lambda_{PP}, \lambda_{BP}, \lambda_{NP}, \lambda_{PN}, \lambda_{BN}, \lambda_{NN}\}$, $X(\subseteq U)$ and $\neg X(\subseteq U)$ indicate that an agent is

 TABLE III

 Pythagorean Fuzzy Loss Function Given by an Expert

Action	X	$\neg X$
a_P	$\lambda_{PP} = P(\mu_{\lambda_{PP}}, \nu_{\lambda_{PP}})$	$\lambda_{PN} = P(\mu_{\lambda_{PN}}, \nu_{\lambda_{PN}})$
a_B	$\lambda_{BP} = P(\mu_{\lambda_{BP}}, \nu_{\lambda_{BP}})$	$\lambda_{BN} = P(\mu_{\lambda_{BN}}, \nu_{\lambda_{BN}})$
a_N	$\lambda_{NP} = P(\mu_{\lambda_{NP}}, \nu_{\lambda_{NP}})$	$\lambda_{NN} = P(\mu_{\lambda_{NN}}, \nu_{\lambda_{NN}})$
	TABLE IV	
	PYTHAGOREAN FUZZY LOS	s Function

Action	X	$\neg X$
a_P	$\lambda_{PP} = P(0.1, 0.8)$	$\lambda_{PN} = P(0.9, 0.2)$
a_B	$\lambda_{BP} = P(0.6, 0.5)$	$\lambda_{BN} = P(0.5, 0.6)$
a_N	$\lambda_{NP}=P(0.9,0.3)$	$\lambda_{NN} = P(0.2, 0.8)$

in X and not in X, respectively; a_P, a_B , and a_N denote three actions in classifying an agent x into POA(U), CTA(U), and NEA(U), respectively; $\lambda_{PP}, \lambda_{BP}$, and λ_{NP} stand for the losses of taking actions a_P, a_B , and a_N , respectively, when an agent belongs to X; $\lambda_{PN}, \lambda_{BN}$, and λ_{NN} mean the losses of taking actions a_P, a_B , and a_N , respectively, when an agent belongs to $\neg X$, where $\lambda_{PP}, \lambda_{BP}, \lambda_{NP}, \lambda_{PN}, \lambda_{BN}$, and λ_{NN} are PFNs.

For simplicity, we employ the same symbol to denote both the set C and the corresponding state; we also denote both the set $\neg C$ and the corresponding state as the same symbol. Furthermore, we assume that the loss of assigning an agent into the boundary region is between an incorrect classification and a correct classification. That is, the loss of right decision is less than that of deferred decision, and the loss of deferred decision is less than that of the wrong decision in practice. So, λ_{PN} , λ_{BN} , $\lambda_{NN} \lambda_{PN}$, λ_{BN} , and λ_{NN} should satisfy the above-mentioned relations. Furthermore, there are four types of Pythagorean fuzzy loss functions as follows.

- 1) The Pythagorean fuzzy loss function satisfying $\lambda_{PP} \leq \lambda_{BP} \leq \lambda_{NP}$ and $\lambda_{NN} \leq \lambda_{BN} \leq \lambda_{PN}$.
- 2) The Pythagorean fuzzy loss function satisfying $S(\lambda_{PP}) \leq S(\lambda_{BP}) \leq S(\lambda_{NP})$ and $S(\lambda_{NN}) \leq S(\lambda_{BN}) \leq S(\lambda_{PN})$.
- 3) The Pythagorean fuzzy loss function satisfying $\mathscr{P}(\lambda_{PP}) \leq \mathscr{P}(\lambda_{BP}) \leq \mathscr{P}(\lambda_{NP})$ and $\mathscr{P}(\lambda_{NN}) \leq \mathscr{P}(\lambda_{BN}) \leq \mathscr{P}(\lambda_{PN})$.
- 4) The Pythagorean fuzzy loss function satisfying $F(\lambda_{PP}) \leq F(\lambda_{BP}) \leq F(\lambda_{NP})$ and $F(\lambda_{NN}) \leq F(\lambda_{BN}) \leq F(\lambda_{PN})$.

In practice, loss functions are very important for conflict analysis of Pythagorean fuzzy information systems, there are many methods of deriving loss functions, such as practical experience, and given by famous experts. Although there are plenty of loss functions besides the aforementioned four types, we only discuss the Pythagorean fuzzy loss functions given by experts satisfying $\lambda_{PP} \leq \lambda_{BP} \leq \lambda_{NP}$ and $\lambda_{NN} \leq \lambda_{BN} \leq \lambda_{PN}$ in this section.

Example 4.2: Table IV depicts a Pythagorean fuzzy loss function given by an expert, and $\lambda_{PP}, \lambda_{BP}, \lambda_{NP}, \lambda_{NN}, \lambda_{BN}$, and λ_{PN} are PFNs. Especially, we have $\lambda_{PP} \leq \lambda_{BP} \leq \lambda_{NP}$ and $\lambda_{NN} \leq \lambda_{BN} \leq \lambda_{PN}$.

Suppose $\lambda_{PP}, \lambda_{BP}, \lambda_{NP}, \lambda_{PN}, \lambda_{BN}$, and λ_{NN} are PFNs, which satisfy $\lambda_{PP} \leq \lambda_{BP} \leq \lambda_{NP}$ and $\lambda_{NN} \leq \lambda_{BN} \leq \lambda_{PN}$. For the agent $x \in U$, the expected losses $R(a_P|x), R(a_B|x)$, and $R(a_N|x)$ under the actions a_P, a_B , and a_N , respectively, are shown as follows:

$$R(a_P|x) = \mathscr{P}(\mathscr{R}(x)) * \lambda_{PP} \oplus [1 - \mathscr{P}(\mathscr{R}(x))] * \lambda_{PN}$$

$$R(a_B|x) = \mathscr{P}(\mathscr{R}(x)) * \lambda_{BP} \oplus [1 - \mathscr{P}(\mathscr{R}(x))] * \lambda_{BN}$$

$$R(a_N|x) = \mathscr{P}(\mathscr{R}(x)) * \lambda_{NP} \oplus [1 - \mathscr{P}(\mathscr{R}(x))] * \lambda_{NN}$$

We see that the expected loss functions $R(a_P|x)$, $R(a_B|x)$, and $R(a_N|x)$ are constructed on the closeness index function $\mathscr{P}(\mathscr{R}(x))$, which are different from the expected loss functions of reference [27]. According to Definition 4.1, we have the expected losses $R(a_P|x)$, $R(a_B|x)$, and $R(a_N|x)$ as follows:

$$\begin{split} R(a_P|x) &= P\left(\sqrt{1 - (1 - \mu_{\lambda_{PP}}^2)^{\mathscr{P}(\mathscr{R}(x))}}, (\nu_{\lambda_{PP}})^{\mathscr{P}(\mathscr{R}(x))}\right) \\ \oplus P\left(\sqrt{1 - (1 - \mu_{\lambda_{PN}}^2)^{1 - \mathscr{P}(\mathscr{R}(x))}}, (\nu_{\lambda_{PN}})^{1 - \mathscr{P}(\mathscr{R}(x))}\right) \\ R(a_B|x) &= P\left(\sqrt{1 - (1 - \mu_{\lambda_{BP}}^2)^{\mathscr{P}(\mathscr{R}(x))}}, (\nu_{\lambda_{BP}})^{\mathscr{P}(\mathscr{R}(x))}\right) \\ \oplus P\left(\sqrt{1 - (1 - \mu_{\lambda_{BN}}^2)^{1 - \mathscr{P}(\mathscr{R}(x))}}, (\nu_{\lambda_{BN}})^{1 - \mathscr{P}(\mathscr{R}(x))}\right) \\ R(a_N|x) &= P\left(\sqrt{1 - (1 - \mu_{\lambda_{NN}}^2)^{\mathscr{P}(\mathscr{R}(x))}}, (\nu_{\lambda_{NN}})^{1 - \mathscr{P}(\mathscr{R}(x))}\right) \\ \oplus P\left(\sqrt{1 - (1 - \mu_{\lambda_{NN}}^2)^{1 - \mathscr{P}(\mathscr{R}(x))}}, (\nu_{\lambda_{NN}})^{1 - \mathscr{P}(\mathscr{R}(x))}\right). \end{split}$$

Theorem 4.3: Let $R(a_{\bullet}|x)$ be the expected loss under the action a_{\bullet} for the agent $x \in U$, where $\bullet = P, B, N$. Then

$$R(a_P|x)$$

$$= P\left(\sqrt{1 - (1 - \mu_{\lambda_{PP}}^2)^{\mathscr{P}(\mathscr{R}(x))} * (1 - \mu_{\lambda_{PN}}^2)^{1 - \mathscr{P}(\mathscr{R}(x))}}, (\nu_{\lambda_{PP}})^{\mathscr{P}(\mathscr{R}(x))} * (\nu_{\lambda_{PN}})^{1 - \mathscr{P}(\mathscr{R}(x))}\right)$$

 $R(a_B|x)$

$$= P\left(\sqrt{1 - (1 - \mu_{\lambda_{BP}}^2)^{\mathscr{P}(\mathscr{R}(x))} * (1 - \mu_{\lambda_{BN}}^2)^{1 - \mathscr{P}(\mathscr{R}(x))}}, (\nu_{\lambda_{BP}})^{\mathscr{P}(\mathscr{R}(x))} * (\nu_{\lambda_{BN}})^{1 - \mathscr{P}(\mathscr{R}(x))}\right)$$

 $R(a_N|x)$

$$= P\left(\sqrt{1 - (1 - \mu_{\lambda_{NP}}^2)^{\mathscr{P}(\mathscr{R}(x))} * (1 - \mu_{\lambda_{NN}}^2)^{1 - \mathscr{P}(\mathscr{R}(x))}}, (\nu_{\lambda_{NP}})^{\mathscr{P}(\mathscr{R}(x))} * (\nu_{\lambda_{NN}})^{1 - \mathscr{P}(\mathscr{R}(x))}\right).$$

Proof: We assume $t_1 = (1 - \mu_{\lambda_{PP}}^2)^{\mathscr{P}(\mathscr{R}(x))}, t_2 = (1 - \mu_{\lambda_{PN}}^2)^{1 - \mathscr{P}(\mathscr{R}(x))}, y_1 = (\nu_{\lambda_{PP}})^{\mathscr{P}(\mathscr{R}(x))}, \text{ and } y_2 = (\nu_{\lambda_{PN}})^{1 - \mathscr{P}(\mathscr{R}(x))}.$ By Definition 2.1, we have

$$\begin{aligned} R(a_P|x) &= P\left(\sqrt{1 - (1 - \mu_{\lambda_{PP}}^2)^{\mathscr{P}(\mathscr{R}(x))}}, (\nu_{\lambda_{PP}})^{\mathscr{P}(\mathscr{R}(x))}\right) \\ \oplus P\left(\sqrt{1 - (1 - \mu_{\lambda_{PN}}^2)^{1 - \mathscr{P}(\mathscr{R}(x))}}, (\nu_{\lambda_{PN}})^{1 - \mathscr{P}(\mathscr{R}(x))}\right) \\ &= P\left(\sqrt{1 - t_1}, y_1\right) \oplus P(\sqrt{1 - t_2}, y_2) \end{aligned}$$

TABLE V EXPECTED LOSS TABLE IR(S)

Action	Р	В	Ν
x_1	$I(R(a_P x_1))$	$I(R(a_B x_1))$	$I(R(a_N x_1))$
<i>x</i> ₂	$I(R(a_P x_2))$	$I(R(a_B x_2))$	$I(R(a_N x_2))$
•			•
x_n	$I(R(a_P x_n))$	$I(R(a_B x_n))$	$I(R(a_N x_n))$

TABLE VI SCORE TABLE SR(S)

Action	Р	В	Ν
<i>x</i> ₁	$S(R(a_P x_1))$	$S(R(a_B x_1))$	$S(R(a_N x_1))$
<i>x</i> ₂	$S(R(a_P x_2))$	$S(R(a_B x_2))$	$S(R(a_N x_2))$
		•	•
•	•	•	
Xn	$S(R(a_P x_n))$	$S(R(a_B x_n))$	$S(R(a_N x_n))$

$$= P\left(\sqrt{1 - t_1 + 1 - t_2 - (1 - t_1)(1 - t_2)}, y_1 y_2\right)$$

= $P\left(\sqrt{1 - t_1 t_2}, y_1 y_2\right)$
= $P\left(\sqrt{1 - (1 - \mu_{\lambda_{PP}}^2)^{\mathscr{P}(\mathscr{R}(x))} * (1 - \mu_{\lambda_{PN}}^2)^{1 - \mathscr{P}(\mathscr{R}(x))}}}{(\nu_{\lambda_{PP}})^{\mathscr{P}(\mathscr{R}(x))} * (\nu_{\lambda_{PN}})^{1 - \mathscr{P}(\mathscr{R}(x))}}\right).$

Furthermore, we also prove

$$\begin{aligned} R(a_B|x) \\ &= P\left(\sqrt{1 - (1 - \mu_{\lambda_{BP}}^2)^{\mathscr{P}(\mathscr{R}(x))} * (1 - \mu_{\lambda_{BN}}^2)^{1 - \mathscr{P}(\mathscr{R}(x))}}, \\ (\nu_{\lambda_{BP}})^{\mathscr{P}(\mathscr{R}(x))} * (\nu_{\lambda_{BN}})^{1 - \mathscr{P}(\mathscr{R}(x))}\right) \end{aligned}$$

$$R(a_N|x)$$

$$= P\left(\sqrt{1 - (1 - \mu_{\lambda_{NP}}^2)^{\mathscr{P}(\mathscr{R}(x))} * (1 - \mu_{\lambda_{NN}}^2)^{1 - \mathscr{P}(\mathscr{R}(x))}}, \\ (\nu_{\lambda_{NP}})^{\mathscr{P}(\mathscr{R}(x))} * (\nu_{\lambda_{NN}})^{1 - \mathscr{P}(\mathscr{R}(x))}\right).$$

We observe that Theorem 4.3 illustrates that the expected losses $R(a_P|x)$, $R(a_B|x)$, and $R(a_N|x)$ are PFNs. Especially, it implies that how to compute the expected losses $R(a_P|x)$, $R(a_B|x)$, and $R(a_N|x)$ with the closeness index function $\mathscr{P}(\mathscr{R}(x))$.

Definition 4.4: Let S = (U, A, V, f) be a Pythagorean fuzzy information system, $R(a_P|x)$, $R(a_B|x)$, and $R(a_N|x)$ are the expected losses under the actions a_P, a_B , and a_N , respectively, for the agent $x \in U$. Then, the expected loss table IR(S), score table SR(S), closeness table PR(S), and preferred table FR(S) are defined as Tables V–VIII, respectively.

Theorem 4.5: Let S = (U, A, V, f) be a Pythagorean fuzzy information system, $R(a_P|x)$, $R(a_B|x)$, and $R(a_N|x)$ are the expected losses under the actions a_P, a_B , and a_N , respectively, for the agent $x \in U$, and $\bullet = I, S, \mathscr{P}, F$.

TABLE VII CLOSENESS TABLE PR(S)

Action	Р	В	Ν
x_1	$\mathcal{P}(R(a_P x_1))$	$\mathcal{P}(R(a_B x_1))$	$\mathcal{P}(R(a_N x_1))$
<i>x</i> ₂	$\mathcal{P}(R(a_P x_2))$	$\mathcal{P}(R(a_B x_2))$	$\mathcal{P}(R(a_N x_2))$
•		•	
x_n	$\mathcal{P}(R(a_P x_n))$	$\mathcal{P}(R(a_B x_n))$	$\mathcal{P}(R(a_N x_n))$

TABLE VIII PREFERRED TABLE FR(S)

Action	Р	В	Ν
x_1	$F(R(a_P x_1))$	$F(R(a_B x_1))$	$F(R(a_N x_1))$
<i>x</i> ₂	$F(R(a_P x_2))$	$F(R(a_B x_2))$	$F(R(a_N x_2))$
•	•	•	
•	•		
X _n	$F(R(a_P x_n))$	$F(R(a_B x_n))$	$F(R(a_N x_n))$

P: If $\bullet(R(a_P|x)) \leq \bullet(R(a_B|x))$ and $\bullet(R(a_P|x)) \leq \bullet(R(a_N|x))$, then we have $x \in POA(U)$.

B: If $\bullet(R(a_B|x)) \leq \bullet(R(a_P|x))$ and $\bullet(R(a_B|x)) \leq \bullet(R(a_N|x))$, then we have $x \in CTA(U)$.

N: If $\bullet(R(a_N|x)) \leq \bullet(R(a_P|x))$ and $\bullet(R(a_N|x)) \leq \bullet(R(a_B|x))$, then we have $x \in NEA(U)$.

Proof: It is straightforward by Bayesian minimum risk theory.

Example 4.6 (Continuation from Examples 3.2 and 4.2): We compute POA(U), CTA(U), and NEA(U) based on Definition 3.5 and Bayesian minimum risk theory as follows:

1) First, by Table III and Theorem 4.3, for $x_i \in U, 1 \le i \le 6$, we have

$$\begin{split} R(a_{P}|x_{i}) &= P\\ \left(\sqrt{1 - (1 - \mu_{\lambda_{PP}}^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - \mu_{\lambda_{PN}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}}, \\ (\nu_{\lambda_{PP}})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (\nu_{\lambda_{PN}})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}\right)\\ &= P\left(\sqrt{1 - (1 - 0.1^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - 0.9^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}}, \\ 0.8^{\mathscr{P}(\mathscr{R}(x_{i}))} * 0.2^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}\right)\\ R(a_{B}|x_{i}) &= P\\ \left(\sqrt{1 - (1 - \mu_{\lambda_{BP}}^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - \mu_{\lambda_{BN}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}}, \\ (\nu_{\lambda_{BP}})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (\nu_{\lambda_{BN}})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}\right)\\ &= P\left(\sqrt{1 - (1 - 0.6^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - 0.5^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}}, \\ 0.5^{\mathscr{P}(\mathscr{R}(x_{i}))} * 0.6^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}\right)\\ R(a_{N}|x_{i}) &= P\\ \left(\sqrt{1 - (1 - \mu_{\lambda_{NP}}^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - \mu_{\lambda_{NN}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}}, \\ \end{split}$$

TABLE IX EXPECTED LOSS TABLE IR(S)

Action	Р	В	N	Action	Р	В	Ν
x_1	P(0.4937,0.6380)	P(0.5859,0.5151)	P(0.8675,0.3521)	<i>x</i> ₁	-0.1633	0.0779	0.6286
<i>x</i> ₂	P(0.7920,0.3522)	P(0.5450,0.5570)	P(0.7103,0.5360)	<i>x</i> ₂	0.5031	-0.0132	0.2172
<i>x</i> ₃	P(0.8378,0.2919)	P(0.5308,0.5709)	P(0.6187,0.6122)	<i>x</i> ₃	0.6168	-0.0442	0.0080
<i>x</i> ₄	P(0.8101,0.3291)	P(0.5399,0.5620)	P(0.6808,0.5625)	<i>x</i> 4	0.5480	-0.0243	0.1471
<i>x</i> ₅	P(0.8065,0.3338)	P(0.5410,0.5609)	P(0.6873,0.5568)	<i>x</i> 5	0.5390	-0.0219	0.1623
x_6	P(0.7694,0.3800)	P(0.5505,0.5514)	P(0.7393,0.5080)	<i>x</i> ₆	0.4476	-0.0010	0.2885

$$\begin{aligned} (\nu_{\lambda_{NP}})^{\mathscr{P}(\mathscr{R}(x_i))} * (\nu_{\lambda_{NN}})^{1-\mathscr{P}(\mathscr{R}(x_i))}) \\ &= P\left(\sqrt{1 - (1 - 0.9^2)^{\mathscr{P}(\mathscr{R}(x_i))} * (1 - 0.2^2)^{1-\mathscr{P}(\mathscr{R}(x_i))}}, \\ 0.3^{\mathscr{P}(\mathscr{R}(x_i))} * 0.8^{1-\mathscr{P}(\mathscr{R}(x_i))} \right). \end{aligned}$$

Second, by Definition 4.4, we have the expected loss Table IR(S) shown by Table IX as follows.

Third, by Theorem 4.5, we have $POA(U) = \{x_1\}, CTA$ $(U) = \{x_2, x_5, x_6\}$ and $NEA(U) = \emptyset$. Obviously, it is difficult to compare the expected losses under the actions a_P, a_B , and a_N for the agents x_3 and x_4 . So, it fails to put the agents x_3 and x_4 into POA(U), CTA(U), and NEA(U). Therefore, $\{x_1\}$ and $\{x_2, x_5, x_6\}$ are different alliances, but we cannot identify x_3 and x_4 .

2) First, by Definition 2.3 and Table V, for $x_i \in U, 1 \le i \le 6$, we have

$$\begin{split} S(R(a_{P}|x_{i})) &= S \\ \left(P\left(\sqrt{1 - (1 - \mu_{\lambda_{PP}}^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - \mu_{\lambda_{PN}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}}, \\ (\nu_{\lambda_{PP}})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (\nu_{\lambda_{PN}})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))} \right) \right) &= S \\ \left(P\left(\sqrt{1 - (1 - 0.1^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - 0.9^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}}, \\ 0.8^{\mathscr{P}(\mathscr{R}(x_{i}))} * 0.2^{1 - \mathscr{P}(\mathscr{R}(x_{i}))} \right) \right) \\ S(R(a_{B}|x_{i})) &= S \\ \left(P\left(\sqrt{1 - (1 - \mu_{\lambda_{BP}}^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - \mu_{\lambda_{BN}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}}, \\ (\nu_{\lambda_{BP}})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (\nu_{\lambda_{BN}})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))} \right) \right) &= S \\ \left(P\left(\sqrt{1 - (1 - 0.6^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - 0.5^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}}, \\ 0.5^{\mathscr{P}(\mathscr{R}(x_{i}))} * 0.6^{1 - \mathscr{P}(\mathscr{R}(x_{i}))} \right) \right) \\ S(R(a_{N}|x_{i})) &= S \\ \left(P\left(\sqrt{1 - (1 - \mu_{\lambda_{NP}}^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - \mu_{\lambda_{NN}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}}, \\ (\nu_{\lambda_{NP}})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (\nu_{\lambda_{NN}})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))} \right) \right) &= S \end{split}$$

 $\left(P\left(\sqrt{1 - (1 - 0.9^2)^{\mathscr{P}(\mathscr{R}(x_i))} * (1 - 0.2^2)^{1 - \mathscr{P}(\mathscr{R}(x_i))}}, 0.3^{\mathscr{P}(\mathscr{R}(x_i))} * 0.8^{1 - \mathscr{P}(\mathscr{R}(x_i))}\right)\right).$

TABLE X

Score Table SR(S)

Second, by Definition 4.4, we have the score table SR(S) shown by Table X as follows.

Third, by Theorem 4.5, we have $POA(U) = \{x_1\}, CTA$ $(U) = \{x_2, x_3, x_4, x_5, x_6\}$ and $NEA(U) = \emptyset$. Moreover, we find that $\{x_1\}$ and $\{x_2, x_3, x_4, x_5, x_6\}$ are difference alliances. Especially, $\{x_1\}$ is a positive alliance, and $\{x_2, x_3, x_4, x_5, x_6\}$ is a neutral alliance.

3) First, by Definition 2.5, for $x_i \in U, 1 \le i \le 6$, we have

$$\begin{split} \mathscr{P}(R(a_{P}|x_{i}) &= \mathscr{P}\\ & \left(P\left(\sqrt{1 - (1 - \mu_{\lambda_{PP}}^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - \mu_{\lambda_{PN}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}}, \\ (\nu_{\lambda_{PP}})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (\nu_{\lambda_{PN}})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}\right)\right) &= \mathscr{P}\\ & \left(P\left(\sqrt{1 - (1 - 0.1^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - 0.9^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}}, \\ 0.8^{\mathscr{P}(\mathscr{R}(x_{i}))} * 0.2^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}\right)\right) \\ \mathscr{P}(R(a_{B}|x_{i})) &= \mathscr{P}\\ & \left(P\left(\sqrt{1 - (1 - \mu_{\lambda_{BP}}^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - \mu_{\lambda_{BN}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}}, \\ (\nu_{\lambda_{BP}})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (\nu_{\lambda_{BN}})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}\right)\right) &= \mathscr{P}\\ & \left(P\left(\sqrt{1 - (1 - 0.6^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - 0.5^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}, \\ 0.5^{\mathscr{P}(\mathscr{R}(x_{i}))} * 0.6^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}\right)\right) \\ & \mathscr{P}(R(a_{N}|x_{i})) &= \mathscr{P}\\ & \left(P\left(\sqrt{1 - (1 - \mu_{\lambda_{NP}}^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - \mu_{\lambda_{NN}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}, \\ (\nu_{\lambda_{NP}})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (\nu_{\lambda_{NN}})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}\right)\right) &= \mathscr{P}\\ & \left(P\left(\sqrt{1 - (1 - 0.9^{2})^{\mathscr{P}(\mathscr{R}(x_{i}))} * (1 - 0.2^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}, \\ (0.3^{\mathscr{P}(\mathscr{R}(x_{i}))} * 0.8^{1 - \mathscr{P}(\mathscr{R}(x_{i}))}\right)\right). \end{split}\right)$$

CTA(U)

 $\{x_2, x_5, x_6\}$

 $\{x_2, x_3, x_4, x_5, x_6\}$

 $\{x_2, x_3, x_4, x_5, x_6\}$

 $\{x_2, x_3, x_4, x_5, x_6\}$

NEA(U)

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TABLE XI CLOSENESS TABLE PR(S)

TABLE XIII THREE ALLIANCES BASED ON IR(S), SR(S), PR(S), and FR(S)

TABLE XIV Pythagorean Fuzzy Loss Functions $\{\lambda^{(i)}|i=1,2,\ldots,m\}$

POA(U)

 $\{x_1\}$

 ${x_1}$

 $\{x_1\}$

 $\{x_1\}$

Methods

IR(S)

SR(S)

PR(S)

FR(S)

Action	Р	В	Ν
<i>x</i> ₁	0.4395	0.5280	0.7797
<i>x</i> ₂	0.7015	0.4953	0.5899
<i>x</i> ₃	0.7543	0.4841	0.5032
<i>x</i> 4	0.7218	0.4913	0.5603
<i>x</i> 5	0.7176	0.4921	0.5666
<i>x</i> ₆	0.6771	0.4997	0.6207

TABLE XII PREFERRED TABLE FR(S)

Action	Р	В	Ν
x_1	0.4349	0.5319	0.7383
<i>x</i> ₂	0.7025	0.4946	0.5787
<i>x</i> ₃	0.7543	0.4819	0.5029
<i>x</i> ₄	0.7224	0.4901	0.5533
<i>x</i> 5	0.7184	0.4910	0.5588
<i>x</i> ₆	0.6784	0.4996	0.6047

Second, by Definition 4.4, we have the closeness Table PR(S) shown by Table XI as follows.

Third, by Theorem 4.5, we have $POA(U) = \{x_1\}, CTA(U) = \{x_2, x_3, x_4, x_5, x_6\}$ and $NEA(U) = \emptyset$. Furthermore, we find that $\{x_1\}$ and $\{x_2, x_3, x_4, x_5, x_6\}$ are difference alliances. Especially, $\{x_1\}$ is a positive alliance, and $\{x_2, x_3, x_4, x_5, x_6\}$ is a neutral alliance.

4) First, by Definition 4.4, we have the preferred table FR(S) shown by Table XII as follows.

Then, by Theorem 4.5, we have $POA(U) = \{x_1\}, CTA(U) = \{x_2, x_3, x_4, x_5, x_6\}$ and $NEA(U) = \emptyset$. Moreover, we see that $\{x_1\}$ and $\{x_2, x_3, x_4, x_5, x_6\}$ are difference alliances. Especially, $\{x_1\}$ is a positive alliance, and $\{x_2, x_3, x_4, x_5, x_6\}$ is a neutral alliance.

5) First, we list the alliances computed using IR(S), SR(S), PR(S), and FR(S) in Table XIII. Concretely, we have the same alliances with SR(S), PR(S), and FR(S), which are different from the results with IR(S); all agents are classified into the positive, neutral, and negative alliances with SR(S), PR(S), and FR(S), but we cannot put x_3 into any alliance with IR(S); almost all agents are classified into the neutral alliances with IR(S), SR(S), PR(S), and FR(S), and no agents are put into the negative alliances. Second, we have the positive alliance $\{x_1\}$ and the neutral alliance $\{x_2, x_5, x_6\}$ by IR(S); we get the positive alliance $\{x_1\}$ and the neutral alliance $\{x_2, x_3, x_4, x_5, x_6\}$ by SR(S), PR(S), and FR(S). It is obvious that most of countries belong to the neutral alliance, and x_1 holds different opinions on some issues, so it is a single alliance. If x_1 wants to get supports from other countries, then it must change opinions on some issues. Third, there are so many agents in the neutral alliance, so we should choose the appropriate thresholds and provide more effective approaches for studying Pythagorean fuzzy information systems for conflicts.

λ	Action	X	$\neg X$
a (1)	a_P	$\lambda_{PP}^{(1)} = P(\mu_{\lambda_{PP}^{(1)}}, \nu_{\lambda_{PP}^{(1)}})$	$\lambda_{PN}^{(1)} = P(\mu_{\lambda_{PN}^{(1)}}, \nu_{\lambda_{PN}^{(1)}})$
λ(*)	a_B	$\lambda_{BP}^{(1)} = P(\mu_{\lambda_{BP}^{(1)}}, \nu_{\lambda_{BP}^{(1)}})$	$\lambda_{BN}^{(1)} = P(\mu_{\lambda_{BN}^{(1)}}, \nu_{\lambda_{BN}^{(1)}})$
	a_N	$\lambda_{NP}^{(1)} = P(\mu_{\lambda_{NP}^{(1)}}, \nu_{\lambda_{NP}^{(1)}})$	$\lambda_{NN}^{(1)} = P(\mu_{\lambda_{NN}^{(1)}}, \nu_{\lambda_{NN}^{(1)}})$
a ⁽²⁾	a_P	$\lambda_{PP}^{(2)} = P(\mu_{\lambda_{PP}^{(2)}}, v_{\lambda_{PP}^{(2)}})$	$\lambda_{PN}^{(2)} = P(\mu_{\lambda_{PN}^{(2)}}, v_{\lambda_{PN}^{(2)}})$
$\mathcal{A}^{(-)}$	a_B	$\lambda_{BP}^{(2)} = P(\mu_{\lambda_{BP}^{(2)}}, \nu_{\lambda_{BP}^{(2)}})$	$\lambda_{BN}^{(2)} = P(\mu_{\lambda_{BN}^{(2)}}, \nu_{\lambda_{BN}^{(2)}})$
	a_N	$\lambda_{NP}^{(2)} = P(\mu_{\lambda_{NP}^{(2)}}, \nu_{\lambda_{NP}^{(2)}})$	$\lambda_{NN}^{(2)} = P(\mu_{\lambda_{NN}^{(2)}}, \nu_{\lambda_{NN}^{(2)}})$
		•	•
•	•		•
	•	•	•
ג (<i>m</i>)	a_P	$\lambda_{PP}^{(m)} = P(\mu_{\lambda_{PP}^{(m)}}, v_{\lambda_{PP}^{(m)}})$	$\lambda_{PN}^{(m)} = P(\mu_{\lambda_{PN}^{(m)}}, \nu_{\lambda_{PN}^{(m)}})$
Λ,	a_B	$\lambda_{BP}^{(m)} = P(\mu_{\lambda_{BP}^{(m)}}, v_{\lambda_{BP}^{(m)}})$	$\lambda_{BN}^{(m)} = P(\mu_{\lambda_{BN}^{(m)}}, \nu_{\lambda_{BN}^{(m)}})$
	a_N	$\lambda_{NP}^{(m)} = P(\mu_{\lambda_{NP}^{(m)}}, \nu_{\lambda_{NP}^{(m)}})$	$\lambda_{NN}^{(m)} = P(\mu_{\lambda_{NN}^{(m)}}, \nu_{\lambda_{NN}^{(m)}})$

V. THREE-WAY GROUP CONFLICT ANALYSIS OF PYTHAGOREAN FUZZY INFORMATION SYSTEMS

In this section, we investigate Pythagorean fuzzy information systems for conflicts with group decision theory.

Suppose there are m experts $\{E_1, E_2, \ldots, E_m\}$, who give the loss functions $\{\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}\}$ shown by Table XIV, where $\lambda^{(i)} = (\Omega, \mathscr{A}, \mathscr{L}^{(i)}), \Omega = \{X, \neg X\}$ is a set of two states, $\mathscr{A} = \{a_P, a_B, a_N\}, \qquad \mathscr{L}^{(i)} = \{\lambda_{PP}^{(i)}, \lambda_{BP}^{(i)}, \lambda_{NP}^{(i)}, \lambda_{PN}^{(i)}, \lambda_{BN}^{(i)}, \lambda_{NP}^{(i)}, \lambda_$ $\lambda_{NN}^{(i)}$, and $X(\subseteq U)$ and $\neg X(\subseteq U)$ indicate that an agent is in X and not in X, respectively. For simplicity, we take the same symbol to denote both the set C and the corresponding state. We also employ both the set $\neg C$ and the corresponding state as the same symbol. Furthermore, a_P , a_B , and a_N denote three actions in classifying an agent x into POA(U), CTA(U),and NEA(U), respectively; $\lambda_{PP}^{(i)}, \lambda_{BP}^{(i)}$, and $\lambda_{NP}^{(i)}$ stand for the losses of taking actions a_P, a_B , and a_N , respectively, when an agent belongs to X; $\lambda_{PN}^{(i)}, \lambda_{BN}^{(i)}$, and $\lambda_{NN}^{(i)}$ mean the losses of taking actions a_P, a_B , and a_N , respectively, when an agent belongs to $\neg X$, where $\lambda_{PP}^{(i)}, \lambda_{BP}^{(i)}, \lambda_{NP}^{(i)}, \lambda_{PN}^{(i)}, \lambda_{BN}^{(i)}$, and $\lambda_{NN}^{(i)}$ are PFNs, which satisfy $\lambda_{PP}^{(i)} \leq \lambda_{BP}^{(i)} \leq \lambda_{NP}^{(i)}$ and $\lambda_{NN}^{(i)} \leq \lambda_{BN}^{(i)} \leq \lambda_{PN}^{(i)}$. For the agent $x \in U$, the expected losses $R^{(i)}(a_P|x), R^{(i)}(a_B|x)$, and $R^{(i)}(a_N|x)$ under the actions a_P, a_B , and a_N with respect to the loss given by the expert E_i , respectively, as follows:

$$R^{(i)}(a_P|x) = \mathscr{P}(\mathscr{R}(x)) * \lambda_{PP}^{(i)} \oplus [1 - \mathscr{P}(\mathscr{R}(x))] * \lambda_{PN}^{(i)}$$

$$R^{(i)}(a_B|x) = \mathscr{P}(\mathscr{R}(x)) * \lambda_{BP}^{(i)} \oplus [1 - \mathscr{P}(\mathscr{R}(x))] * \lambda_{BN}^{(i)}$$

$$R^{(i)}(a_N|x) = \mathscr{P}(\mathscr{R}(x)) * \lambda_{NP}^{(i)} \oplus [1 - \mathscr{P}(\mathscr{R}(x))] * \lambda_{NN}^{(i)}.$$

λ	Action	X	$\neg X$
	a_P	$\lambda_{PP}^{(1)} = P(0.1, 0.8)$	$\lambda_{PN}^{(1)} = P(0.9, 0.2)$
$\lambda^{(1)}$	a_B	$\lambda_{BP}^{(1)} = P(0.6, 0.5)$	$\lambda_{BN}^{(1)} = P(0.5, 0.6)$
	a_N	$\lambda_{NP}^{(1)} = P(0.9, 0.3)$	$\lambda_{NN}^{(1)} = P(0.2, 0.8)$
	a_P	$\lambda_{PP}^{(2)} = P(0.2, 0.9)$	$\lambda_{PN}^{(2)} = P(0.8, 0.3)$
$\lambda^{(2)}$	a_B	$\lambda_{BP}^{(2)} = P(0.5, 0.7)$	$\lambda_{BN}^{(2)} = P(0.6, 0.5)$
	a_N	$\lambda_{NP}^{(2)} = P(0.8, 0.2)$	$\lambda_{NN}^{(2)} = P(0.1, 0.9)$
	a_P	$\lambda_{PP}^{(3)} = P(0.3, 0.9)$	$\lambda_{PN}^{(3)} = P(0.8, 0.1)$
$\lambda^{(3)}$	a_B	$\lambda_{BP}^{(3)} = P(0.5, 0.6)$	$\lambda_{BN}^{(3)} = P(0.6, 0.6)$
	a_N	$\lambda_{NP}^{(3)} = P(0.7, 0.1)$	$\lambda_{NN}^{(3)} = P(0.2, 0.9)$

TABLE XV Pythagorean Fuzzy Loss Functions $\{\lambda^{(i)} | i = 1, 2, 3\}$

Example 5.1 (Continuation from Example 4.6): Table XV depicts a collection of Pythagorean fuzzy loss functions $\{\lambda^{(i)} | i = 1, 2, 3\}$, which are given by three experts $\{E_1, E_2, E_3\}$.

By Theorem 4.3, we have the expected losses $R^{(i)}(a_P|x)$, $R^{(i)}(a_B|x)$, and $R^{(i)}(a_N|x)$ with respect to the Pythagorean fuzzy loss function $\lambda^{(i)}$ as follows:

$$\begin{split} &R^{(i)}(a_{P}|x) \\ &= P\left(\sqrt{1 - (1 - \mu_{\lambda_{PP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x))} * (1 - \mu_{\lambda_{PN}^{(i)}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x))}}, \\ & (\nu_{\lambda_{PP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x))} * (\nu_{\lambda_{PN}^{(i)}})^{1 - \mathscr{P}(\mathscr{R}(x))}\right) \\ &R^{(i)}(a_{B}|x) \\ &= P\left(\sqrt{1 - (1 - \mu_{\lambda_{BP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x))} * (1 - \mu_{\lambda_{BN}^{(i)}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x))}}, \\ & (\nu_{\lambda_{BP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x))} * (\nu_{\lambda_{BN}^{(i)}})^{1 - \mathscr{P}(\mathscr{R}(x))}\right) \\ &R^{(i)}(a_{N}|x) \\ &= P\left(\sqrt{1 - (1 - \mu_{\lambda_{NP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x))} * (1 - \mu_{\lambda_{NN}^{(i)}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x))}}, \end{split}$$

$$(\nu_{\lambda_{NP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x))} * (\nu_{\lambda_{NN}^{(i)}})^{1-\mathscr{P}(\mathscr{R}(x))}).$$
Theorem 5.2: Let $R^{(i)}(a_{D}|x) = R^{(i)}(a_{D}|x)$ and $R^{(i)}(a_{N}|x)$.

Theorem 5.2: Let $R^{(i)}(a_P|x)$, $R^{(i)}(a_B|x)$, and $R^{(i)}(a_N|x)$ be the expected losses under the actions a_P, a_B , and a_N using the Pythagorean fuzzy loss function $\lambda^{(i)}$, respectively, for the agent $x \in U$, and $\mathcal{W} = \{w_1, w_2, \dots, w_m\}$ be the weight vector of $R^{(i)}(a_{\bullet}|x)(i = 1, 2, \dots, m, \bullet = P, B, N)$. Then

$$\begin{split} \mathscr{R}(R^{(1)}(a_{P}|x),\ldots,R^{(m)}(a_{P}|x)) &= P\Big(\Sigma_{i=1}^{m}w_{i} \\ &* \sqrt{1 - (1 - \mu_{\lambda_{PP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x))} * (1 - \mu_{\lambda_{PN}^{(i)}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x))}}, \\ \Sigma_{i=1}^{m}w_{i} * (\nu_{\lambda_{PP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x))} * (\nu_{\lambda_{PN}^{(i)}})^{1 - \mathscr{P}(\mathscr{R}(x))}\Big); \\ \mathscr{R}(R^{(1)}(a_{B}|x),\ldots,R^{(m)}(a_{B}|x)) &= P\Big(\Sigma_{i=1}^{m}w_{i} \\ &* \sqrt{1 - (1 - \mu_{\lambda_{BP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x))} * (1 - \mu_{\lambda_{BN}^{(i)}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x))}}, \\ \Sigma_{i=1}^{m}w_{i} * (\nu_{\lambda_{BP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x))} * (\nu_{\lambda_{BN}^{(i)}})^{1 - \mathscr{P}(\mathscr{R}(x))}\Big); \end{split}$$

$$\begin{aligned} \mathscr{R}(R^{(1)}(a_N|x),\ldots,R^{(m)}(a_N|x)) &= P\Big(\sum_{i=1}^m w_i \\ *\sqrt{1-(1-\mu_{\lambda_{NP}^{(i)}}^2)^{\mathscr{P}(\mathscr{R}(x))} * (1-\mu_{\lambda_{NN}^{(i)}}^2)^{1-\mathscr{P}(\mathscr{R}(x))}}, \\ \sum_{i=1}^m w_i * (\nu_{\lambda_{NP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x))} * (\nu_{\lambda_{NN}^{(i)}})^{1-\mathscr{P}(\mathscr{R}(x))}\Big). \end{aligned}$$

Proof: It is straightforward by Theorem 4.3.

We see that Theorem 5.2 illustrates that the expected losses $\mathscr{R}(R^{(1)}(a_P|x),\ldots,R^{(m)}(a_P|x)), \quad \mathscr{R}(R^{(1)}(a_B|x),\ldots,R^{(m)}(a_B|x)),$ and $\mathscr{R}(R^{(1)}(a_N|x),\ldots,R^{(m)}(a_N|x))$ are PFNs. Especially, it implies that how to compute the expected losses $\mathscr{R}(R^{(1)}(a_P|x),\ldots,R^{(m)}(a_P|x)), \quad \mathscr{R}(R^{(1)}(a_B|x),\ldots,R^{(m)}(a_B|x)),$ and $\mathscr{R}(R^{(1)}(a_N|x),\ldots,R^{(m)}(a_N|x))$ with the closeness index function $\mathscr{P}(\mathscr{R}(x))$.

Definition 5.3: Let S = (U, A, V, f) be a Pythagorean fuzzy information system, $\mathscr{R}(R^{(1)}(a_P|x), \ldots, R^{(m)}(a_P|x)), \mathscr{R}(R^{(1)}(a_B|x), \ldots, R^{(m)}(a_B|x))$, and $\mathscr{R}(R^{(1)}(a_N|x), \ldots, R^{(m)}(a_N|x))$ are the expected losses under the actions a_P, a_B , and a_N , respectively, for the agent $x \in U$. Then, the group expected loss table $\mathscr{R}(R(S))$, group score table $\mathscr{S}(R(S))$, group closeness table $\mathscr{P}(R(S))$, and group preferred table $\mathscr{F}(R(S))$ are defined as Tables XVI–XIX, respectively.

Theorem 5.4: Let S = (U, A, V, f) be a Pythagorean fuzzy information system, $R(a_P|x)$, $R(a_B|x)$, and $R(a_N|x)$ are the expected losses under the actions a_P, a_B , and a_N , respectively, for the agent $x \in U$, and $\bullet = I, S, \mathscr{P}, F$. Then

$$\begin{split} P^* &: \text{If } \bullet (\mathscr{R}(R^{(1)}(a_P|x), \dots, R^{(m)}(a_P|x))) \leq \bullet (\mathscr{R}(R^{(1)}(a_B|x))) \\ & x), \dots, R^{(m)}(a_B|x))) \text{ and } \bullet (\mathscr{R}(R^{(1)}(a_P|x), \dots, R^{(m)}(a_P|x))) \\ & \leq \bullet (\mathscr{R}(R^{(1)}(a_N|x), \dots, R^{(m)}(a_N|x))), \text{ then we have } x \in POA \\ (U); \\ B^* &: \text{If } \bullet (\mathscr{R}(R^{(1)}(a_B|x), \dots, R^{(m)}(a_B|x))) \leq \bullet (\mathscr{R}(R^{(1)}(a_P|x))) \\ & x), \dots, R^{(m)}(a_P|x))) \text{ and } \bullet (\mathscr{R}(R^{(1)}(a_B|x), \dots, R^{(m)}(a_B|x))) \\ & \leq \bullet (\mathscr{R}(R^{(1)}(a_N|x), \dots, R^{(m)}(a_N|x))), \text{ then we have } x \in CTA(U); \end{split}$$

$$\begin{split} N^* &: \mathbf{If} \bullet (\mathscr{R}(R^{(1)}(a_N|x), \dots, R^{(m)}(a_N|x))) \leq \bullet (\mathscr{R}(R^{(1)}(a_P|x)), \dots, R^{(m)}(a_P|x))) \\ & x), \dots, R^{(m)}(a_P|x))) \text{ and } \bullet (\mathscr{R}(R^{(1)}(a_N|x), \dots, R^{(m)}(a_N|x))) \\ & \leq \bullet (\mathscr{R}(R^{(1)}(a_B|x), \dots, R^{(m)}(a_B|x))), \text{ then we have } x \in NEA(U). \end{split}$$

Proof: It is straightforward by Bayesian minimum risk theory.

Example 5.5 (Continuation from Example 5.1): Taking $w_1 = w_2 = w_3 = \frac{1}{3}$ for Pythagorean fuzzy loss functions $\{\lambda^{(i)} | i = 1, 2, 3\}$, we compute POA(U), CTA(U), and NEA(U) as follows.

1) First, for $x_j \in U$, by Theorem 5.2, we have

$$\begin{split} I(\mathscr{R}(R^{(1)}(a_{P}|x_{j}),\ldots,R^{(m)}(a_{P}|x_{j}))) &= P\left(\sum_{i=1}^{m}w_{i}\right) \\ &* \sqrt{1 - (1 - \mu_{\lambda_{PP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (1 - \mu_{\lambda_{PN}^{(i)}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{j}))}}, \\ \sum_{i=1}^{m}w_{i} * (\nu_{\lambda_{PP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (\nu_{\lambda_{PN}^{(i)}})^{1 - \mathscr{P}(\mathscr{R}(x_{j}))}\right); \\ I(\mathscr{R}(R^{(1)}(a_{B}|x_{j}),\ldots,R^{(m)}(a_{B}|x_{j}))) &= P\left(\sum_{i=1}^{m}w_{i}\right) \\ &* \sqrt{1 - (1 - \mu_{\lambda_{BP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (1 - \mu_{\lambda_{BN}^{(i)}}^{2})^{1 - \mathscr{P}(\mathscr{R}(x_{j}))}, \end{split}$$

TABLE XVI				
GROUP EXPECTED LOSS TABLE $\mathscr{R}(R(S))$				

Action	n		מ		λ.7	
Action	$\frac{l'}{\mathcal{O}(\mathcal{D}(1))(m+1)} = \frac{\mathcal{D}(m)(m+1)}{\mathcal{O}(m)(m+1)}$	(T) (T)	$\frac{B}{B}$	$\mathbf{p}(m)$ (m. 1.1.)		$\mathbf{p}(m)$ ($m \to \infty$
x_1	$\mathscr{H}(\mathbf{K}^{(m)}(a_{P} \mathbf{x}_{1}),,\mathbf{K}^{(m)}(a_{P} \mathbf{x}_{1}))$	$\mathscr{R}(R^{(1)}(a_B x_1),, R^{(m)}(a_B x_1))$		$\mathscr{R}(\mathbf{K}^{(n)}(a_N x_1), \dots, \mathbf{K}^{(m)}(a_N x_1))$ $\mathscr{R}(\mathbf{P}(1)(a_N x_1), \dots, \mathbf{P}^{(m)}(a_N x_1))$		
<i>x</i> ₂	$\mathscr{K}(K^{(n)}(a_P x_2),,K^{(m)}(a_P x_2))$	$\mathscr{K}(K)$	$(a_B x_2),,$	$K^{(m)}(a_B x_2))$	$\mathscr{K}(R^{(1)}(a_N x_2),\ldots$	$, \kappa^{(n)}(a_N x_2))$
•					•	
•						
•	· · · ·				· ·	()
Xn	$\mathscr{R}(R^{(1)}(a_P x_n),, R^{(m)}(a_P x_n))$	$\mathscr{R}(R)$	$\mathbf{R}^{(1)}(a_B x_n),,$	$R^{(m)}(a_B x_n))$	$\mathscr{R}(\mathbb{R}^{(1)}(a_N x_n),\ldots)$	$, R^{(m)}(a_N x_n))$
		TABLE	F XVII			
	Gr	OUP SCORE T	ABLE $\mathscr{S}(R)$	S))		
Action	р		В		N	
rı	$S(\mathcal{R}(R^{(1)}(a_{p} x_{1}) - R^{(m)}(a_{p} x_{1}))))$	$S(\mathcal{R}(R))$	$(1)(a_p \mathbf{r}_1)$	$R^{(m)}(a_{p} x_{1})))$	$S(\mathcal{R}(\mathcal{R}^{(1)}(a_{N} x_{1})))$	$R^{(m)}(a_N(x_1)))$
x_1 x_2	$S\left(\mathscr{R}(R^{(1)}(a_P x_2),, R^{(m)}(a_P x_2))\right)$	$S(\mathcal{R}(R^{0}$	$^{(1)}(a_B x_2),,$	$R^{(m)}(a_B x_2)))$	$S(\mathscr{R}(R^{(1)}(a_N x_2),$	$(a_N(x_1)))$
•	•		•			
			•			
• r	$S(\mathscr{R}(R^{(1)}(a_{p} \mathbf{x}_{r}) - R^{(m)}(a_{p} \mathbf{x}_{r}))))$	$S(\mathcal{R}(R))$	$^{(1)}(a_{\mathbf{p}} \mathbf{r}_{*})$	$R^{(m)}(a_{p} \mathbf{x}_{n})))$	$S(\mathcal{R}(\mathcal{R}^{(1)}(a_{N} \mathbf{x}_{n})))$	$R^{(m)}(a_{M} \mathbf{r}_{n})))$
~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~~	$S(\mathcal{B}(\mathbf{A} (up \mathbf{A}_n),, \mathbf{A} (up \mathbf{A}_n)))$	5 (20 (R	( <i>uBnn</i> ),,	$(u_B(x_n)))$	$\mathcal{D}(\mathcal{M}(\mathbf{A} \mid (u_N \mid x_n)), \dots)$	$(u_N(x_n)))$
	~	TABLE	XVIII			
	Grou	JP CLOSENESS	TABLE $\mathscr{P}(F)$	$\mathfrak{L}(\mathcal{S})$		
Action	Р		В		N	
$x_1$	$\mathscr{P}(\mathscr{R}(R^{(1)}(a_P x_1),, R^{(m)}(a_P x_1)))$	$\mathcal{P}(\mathcal{R}(R$	$^{(1)}(a_B x_1),,$	$R^{(m)}(a_B x_1)))$	$\mathscr{P}(\mathscr{R}(\mathbb{R}^{(1)}(a_N x_1),$	$., R^{(m)}(a_N x_1)))$
$x_2$	$\mathcal{P}(\mathcal{R}(R^{(1)}(a_P x_2),, R^{(m)}(a_P x_2)))$	$\mathcal{P}(\mathcal{R}(R$	$^{(1)}(a_B x_2),,$	$R^{(m)}(a_B x_2)))$	$\mathscr{P}(\mathscr{R}(\mathbb{R}^{(1)}(a_N x_2),$	$(R^{(m)}(a_N x_2)))$
	•					
$x_n$	$\mathscr{P}(\mathscr{R}(\mathbb{R}^{(1)}(a_P x_n),,\mathbb{R}^{(m)}(a_P x_n)))$	$\mathcal{P}(\mathcal{R}(R$	$^{(1)}(a_B x_n),,$	$R^{(m)}(a_B x_n)))$	$\mathscr{P}(\mathscr{R}(\mathbb{R}^{(1)}(a_N x_n),$	$., R^{(m)}(a_N x_n)))$
	Grou	TABLI JP Preferred	E XIX • Table <i>F</i> (F	R(S))		
			× ×	< <i>//</i>		
Action	Р		В		N	
$x_1$	$F(\mathscr{R}(R^{(1)}(a_P x_1),, R^{(m)}(a_P x_1)))$	$F(\mathscr{R}(R^{(1)}))$	$^{(1)}(a_B x_1),, 1$	$R^{(m)}(a_B x_1)))$	$F(\mathscr{R}(R^{(1)}(a_N x_1),$	$(R^{(m)}(a_N x_1)))$
<i>x</i> ₂	$F(\mathscr{R}(R^{(1)}(a_P x_2),, R^{(m)}(a_P x_2))))$	$F(\mathscr{R}(R^{(1)}))$	$^{(1)}(a_B x_2),, .$	$R^{(m)}(a_B x_2)))$	$F(\mathscr{R}(R^{(1)}(a_N x_2),$	$(R^{(m)}(a_N x_2)))$
$x_n$	$F(\mathscr{R}(R^{(1)}(a_P x_n),, R^{(m)}(a_P x_n)))$	$F(\mathscr{R}(R^{(1)}))$	$^{(1)}(a_B x_n),, .$	$R^{(m)}(a_B x_n)))$	$F(\mathscr{R}(R^{(1)}(a_N x_n),\ldots))$	$, R^{(m)}(a_N x_n)))$
		`		-	ΓΑΒΙΕΧΧ	
$\sum_{i=1}^{m} w_i * ($	$\sum_{i=1}^{m} w_i * (\nu_{\lambda_{BP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x_j))} * (\nu_{\lambda_{BN}^{(i)}})^{1-\mathscr{P}(\mathscr{R}(x_j))} \Big);$			GROUP EXPECT	ED LOSS TABLE $\mathscr{R}(R($	(S))
$I(\mathcal{R}(R^{(1)}))$	$(a_N r_1) = R^{(m)}(a_N r_1)) - P(\Sigma$	m $m$	Action	Р	B	N
1 100 110	$(\omega_{IV}   \omega_{J}), \dots, IC \qquad (\omega_{IV}   \omega_{J}))) = I $	-i=1 ^w i	X1	P(0.4623.0.6731)	P(0.5412.0.5926)	P(0.7617.0.2503)
/			ro	P(0.7203.0.3558)	P(0.5571.0.5769)	P(0.6020.0.4633)
$* \sqrt{1 - (1 - 1)}$	$(1-\mu_{j(i)}^2)^{\mathscr{P}(\mathscr{R}(x_j))} * (1-\mu_{j(i)}^2)^{1-\varepsilon}$	$\mathscr{P}(\mathscr{R}(x_j)),$	¥2	P(0.7660.0.2929)	P(0.5609.0.5730)	P(0.5186.0.5679)
V `	$\lambda_{NP}^{(2)}$		л <u>з</u>	P(0.7380.0.3314)	P(0.5586.0.5754)	P(0 5746 0 4986)
		\	л4 х-	P(0.7344.0.3364)	P(0.5583.0.5757)	P(0.5806.0.4000)
$\sum_{i=1}^{m} w_i * i$	$(\nu_{\lambda^{(i)}})^{\mathscr{F}(\mathscr{K}(x_j))} * (\nu_{\lambda^{(i)}})^{1-\mathscr{F}(\mathscr{K}(x_j))}$	).	ng re	P(0.6987.0.3852)	P(0 5554 0 5786)	P(0.6295.0.4273)
	NP NN	/	~6	x (0.0707,0.3032)	. (0.3334,0.3700)	x (0.0275,0.7275)
					•.• 11•	1.(
Second by	Definition 5.3 we have the group	expected	The	erefore, $\{x_1\}$ is a p	ositive alliance, an	a $\{x_2, x_4, x_5, x_6\}$
	$\mathcal{D}(\mathcal{D}(\mathcal{O})) \rightarrow \mathcal{T}(\mathcal{D}(\mathcal{O}))$	expected	is a	neutral alliance. W	/e find that $\{x_1\}$ and	d $\{x_2, x_4, x_5, x_6\}$
loss table g	loss table $\mathscr{R}(R(S))$ as Table XX.			are difference alliances.		

2) First, for  $x_j \in U$ , by Theorem 5.2, we have

$$S(\mathscr{R}(R^{(1)}(a_P|x_j), \dots, R^{(m)}(a_P|x_j))) = S\left(P\left(\sum_{i=1}^m w_i\right) * \sqrt{1 - (1 - \mu_{\lambda_{PP}^{(i)}}^2)^{\mathscr{P}(\mathscr{R}(x_j))} * (1 - \mu_{\lambda_{PN}^{(i)}}^2)^{1 - \mathscr{P}(\mathscr{R}(x_j))}},$$

Third, by Theorem 5.4, we have  $POA(U) = \{x_1\}, CTA$ 

 $(U) = \{x_2, x_4, x_5, x_6\}$  and  $NEA(U) = \emptyset$ . It is difficult to compare the expected losses under the actions  $a_P, a_B$ , and  $a_N$  for the agent  $x_3$  by Definition 2.2. So, we cannot classify the agent  $x_3$  into POA(U), CTA(U), and

NEA(U).

TABLE XXI GROUP SCORE TABLE  $\mathscr{S}(R(S))$ 

Action	Р	В	Ν
$x_1$	-0.2393	-0.0583	0.5176
<i>x</i> ₂	0.3922	-0.0224	0.1478
<i>x</i> ₃	0.5009	-0.0137	-0.0536
<i>x</i> ₄	0.4349	-0.0191	0.0816
<i>x</i> 5	0.4263	-0.0198	0.0961
<i>x</i> ₆	0.3398	-0.0262	0.2137

$$\begin{split} & \Sigma_{i=1}^{m} w_{i} * (\nu_{\lambda_{PP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (\nu_{\lambda_{PN}^{(i)}})^{1-\mathscr{P}(\mathscr{R}(x_{j}))} \Big) \Big); \\ & S(\mathscr{R}(R^{(1)}(a_{B}|x_{j}), \dots, R^{(m)}(a_{B}|x_{j}))) = S\Big(P\Big(\Sigma_{i=1}^{m} w_{i} \\ & * \sqrt{1 - (1 - \mu_{\lambda_{BP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (1 - \mu_{\lambda_{BN}^{(i)}}^{2})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}}, \\ & \Sigma_{i=1}^{m} w_{i} * (\nu_{\lambda_{BP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (\nu_{\lambda_{BN}^{(i)}})^{1-\mathscr{P}(\mathscr{R}(x_{j}))} \Big) \Big); \\ & S(\mathscr{R}(R^{(1)}(a_{N}|x_{j}), \dots, R^{(m)}(a_{N}|x_{j}))) = S\Big(P\Big(\Sigma_{i=1}^{m} w_{i} \\ & * \sqrt{1 - (1 - \mu_{\lambda_{NP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (1 - \mu_{\lambda_{NN}^{(i)}}^{2})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}, \\ & \Sigma_{i=1}^{m} w_{i} * (\nu_{\lambda_{NP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (\nu_{\lambda_{NN}^{(i)}})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}\Big)\Big). \end{split}$$

Second, by Definition 5.3, we have the group score table  $\mathscr{S}(R(S))$  shown in Table XXI as follows. Third, by Theorem 5.4, we have  $POA(U) = \{x_1\}, CTA(U) = \{x_2, x_4, x_5, x_6\}$  and  $NEA(U) = \{x_3\}$ . Therefore,  $\{x_1\}$  is a positive alliance,  $\{x_2, x_4, x_5, x_6\}$  is a neutral alliance, and  $\{x_3\}$  is a negative alliance.

3) First, for  $x_j \in U$ , by Theorem 5.2, we have

$$\begin{split} \mathscr{P}(\mathscr{R}(R^{(1)}(a_{P}|x_{j}),\ldots,R^{(m)}(a_{P}|x_{j}))) &= \mathscr{P}\Big(P\Big(\Sigma_{i=1}^{m}w_{i} \\ &* \sqrt{1-(1-\mu_{\lambda_{PP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x_{j}))}*(1-\mu_{\lambda_{PN}^{(i)}}^{2})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}}, \\ \Sigma_{i=1}^{m}w_{i}*(\nu_{\lambda_{PP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x_{j}))}*(\nu_{\lambda_{PN}^{(i)}})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}\Big)\Big); \\ \mathscr{P}(\mathscr{R}(R^{(1)}(a_{B}|x_{j}),\ldots,R^{(m)}(a_{B}|x_{j}))) &= \mathscr{P}\Big(P\Big(\Sigma_{i=1}^{m}w_{i} \\ &* \sqrt{1-(1-\mu_{\lambda_{BP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x_{j}))}*(1-\mu_{\lambda_{BN}^{(i)}}^{2})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}, \\ \Sigma_{i=1}^{m}w_{i}*(\nu_{\lambda_{BP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x_{j}))}*(\nu_{\lambda_{BN}^{(i)}})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}\Big)\Big); \\ \mathscr{P}(\mathscr{R}(R^{(1)}(a_{N}|x_{j}),\ldots,R^{(m)}(a_{N}|x_{j}))) &= \mathscr{P}\Big(P\Big(\Sigma_{i=1}^{m}w_{i} \\ &* \sqrt{1-(1-\mu_{\lambda_{NP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x_{j}))}}*(1-\mu_{\lambda_{NN}^{(i)}}^{2})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}, \\ \Sigma_{i=1}^{m}w_{i}*(\nu_{\lambda_{NP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x_{j}))}*(\nu_{\lambda_{NN}^{(i)}})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}\Big)\Big). \end{split}$$

Second, by Definition 5.3, we have the group closeness table  $\mathscr{P}(R(S))$  shown in Table XXII as follows.

TABLE XXII GROUP CLOSENESS TABLE  $\mathscr{P}(R(S))$ 

Action	Р	В	Ν
<i>x</i> ₁	0.4103	0.4785	0.6907
$x_2$	0.6448	0.4918	0.5519
<i>x</i> ₃	0.6887	0.4950	0.4810
$x_4$	0.6616	0.4930	0.5287
<i>x</i> 5	0.6582	0.4927	0.5338
<i>x</i> ₆	0.6246	0.4903	0.5752

TABLE XXIII GROUP PREFERRED TABLE  $\mathscr{F}(R(S))$ 

Action	Р	В	N
$x_1$	0.4046	0.4768	0.7389
<i>x</i> ₂	0.6670	0.4911	0.5626
<i>x</i> ₃	0.7193	0.4946	0.4778
$x_4$	0.6871	0.4924	0.5343
$x_5$	0.6830	0.4922	0.5404
$x_6$	0.6430	0.4896	0.5916

Third, by Theorem 5.4, we have  $POA(U) = \{x_1\}, CTA$  $(U) = \{x_2, x_4, x_5, x_6\}$  and  $NEA(U) = \{x_3\}$ . Therefore,  $\{x_1\}$  is a positive alliance,  $\{x_2, x_4, x_5, x_6\}$  is a neutral alliance, and  $\{x_3\}$  is a negative alliance.

4) First, for 
$$x_j \in U$$
, by Theorem 5.2, we have

$$\begin{split} F(\mathscr{R}(R^{(1)}(a_{P}|x_{j}),\ldots,R^{(m)}(a_{P}|x_{j}))) &= F\left(P\left(\sum_{i=1}^{m}w_{i}\right)^{*} \sqrt{1-(1-\mu_{\lambda_{PP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (1-\mu_{\lambda_{PN}^{(i)}}^{2})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}}, \\ \Sigma_{i=1}^{m}w_{i}*(\nu_{\lambda_{PP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (\nu_{\lambda_{PN}^{(i)}})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}\right)); \\ F(\mathscr{R}(R^{(1)}(a_{B}|x_{j}),\ldots,R^{(m)}(a_{B}|x_{j}))) &= F\left(P\left(\sum_{i=1}^{m}w_{i}\right)^{*} \sqrt{1-(1-\mu_{\lambda_{BP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (1-\mu_{\lambda_{BN}^{(i)}}^{2})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}, \\ \Sigma_{i=1}^{m}w_{i}*(\nu_{\lambda_{BP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (\nu_{\lambda_{BN}^{(i)}})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}\right)); \\ F(\mathscr{R}(R^{(1)}(a_{N}|x_{j}),\ldots,R^{(m)}(a_{N}|x_{j}))) &= F\left(P\left(\sum_{i=1}^{m}w_{i},\sqrt{1-(1-\mu_{\lambda_{NP}^{(i)}}^{2})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (1-\mu_{\lambda_{NN}^{(i)}}^{2})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}, \\ \Sigma_{i=1}^{m}w_{i}*(\nu_{\lambda_{NP}^{(i)}})^{\mathscr{P}(\mathscr{R}(x_{j}))} * (\nu_{\lambda_{NN}^{(i)}})^{1-\mathscr{P}(\mathscr{R}(x_{j}))}\right)). \end{split}$$

Second, by Definition 5.3, we have the group preferred table  $\mathscr{F}(R(S))$  shown in Table XXIII as follows.

Third, by Theorem 5.4, we have  $POA(U) = \{x_1\}, CTA$  $(U) = \{x_2, x_4, x_5, x_6\}$  and  $NEA(U) = \{x_3\}$ . Therefore,  $\{x_1\}$  is a positive alliance,  $\{x_2, x_4, x_5, x_6\}$  is a neutral alliance, and  $\{x_3\}$  is a negative alliance.

5) First, we list the alliances computed using ℛ(R(S)), 𝒴 (R(S)), 𝒴(R(S)), and 𝒴(R(S)) in Table XXIV. Concretely, we have the same alliances with 𝒴(R(S)), 𝒴(R (S)), and 𝒴(R(S)), which are different from the results with 𝔅(R(S)); all agents are classified into the positive,

TABLE XXIV THREE ALLIANCES BASED ON  $\mathscr{R}(R(S)), \mathscr{S}(R(S)), \mathscr{P}(R(S)),$  and  $\mathscr{F}(R(S))$ 

Method	POA(U)	CTA(U)	NEA(U)
$\mathscr{R}(R(S))$	$\{x_1\}$	$\{x_2, x_4, x_5, x_6\}$	Ø
$\mathcal{S}(R(S))$	$\{x_1\}$	$\{x_2, x_4, x_5, x_6\}$	$\{x_3\}$
$\mathcal{P}(R(S))$	$\{x_1\}$	$\{x_2, x_4, x_5, x_6\}$	$\{x_3\}$
$\mathscr{F}(R(S))$	$\{x_1\}$	$\{x_2, x_4, x_5, x_6\}$	${x_3}$

neutral, and negative alliances with  $\mathscr{S}(R(S)), \mathscr{P}(R(S)),$ and  $\mathscr{F}(R(S))$ , but we cannot put  $x_3$  into any alliance with  $\mathscr{R}(R(S))$ ; almost all agents are classified into the neutral alliances with  $\mathscr{R}(R(S)), \mathscr{S}(R(S)), \mathscr{P}(R(S)),$ and  $\mathscr{F}(R(S))$ , and less agents are put into the positive and negative alliances. Second, we have the positive alliance  $\{x_1\}$  and the neutral alliance  $\{x_2, x_4, x_5, x_6\}$  by  $\mathscr{R}(R(S))$ ; we get the positive alliance  $\{x_1\}$ , the neutral alliance  $\{x_2, x_4, x_5, x_6\}$ , and the negative alliance  $\{x_3\}$ by  $\mathscr{S}(R(S)), \mathscr{P}(R(S))$ , and  $\mathscr{F}(R(S))$ . So, we find that  $\{x_1\}$  and  $\{x_3\}$  belong to the positive alliance and the neutral alliance, respectively, so they hold different opinions on most of issues. In other words, they are opponents. We also see that  $\{x_1\}$  and  $\{x_3\}$  are single alliances, if they want to get supports from other countries, then they must change opinions on some issues. Third, we put the agent  $x_3$  into the neutral alliance in Example 4.6 with a loss function, and we assign the agent  $x_3$  to the negative alliance in Example 5.5 with three loss functions. So, we find that three-way group method is more effective than threeway method for conflict analysis of Pythagorean fuzzy information systems. Therefore, we should study how to compute the expected losses of actions with more loss functions or other types of Pythagorean fuzzy loss functions and provide more effective approaches for studying Pythagorean fuzzy information systems for conflicts in the future.

#### VI. CONCLUSIONS AND FUTURE WORK

In the era of big data, the study of conflicts is of greatest importance both practically and theoretically for human society. In this paper, we have presented the concepts of positive, neutral, and negative alliances with two thresholds, and employed examples to illustrate how to construct the positive, neutral, and negative alliances in Pythagorean fuzzy information systems for conflicts. Moreover, we have studied three-way conflict analysis of Pythagorean fuzzy information systems based on Bayesian minimum risk theory and employed examples to illustrate how to compute different alliances with a Pythagorean fuzzy loss function given by an expert. Finally, we have investigated threeway group conflict analysis of Pythagorean fuzzy information systems and explored examples to illustrate how to calculate different alliances with a group of Pythagorean fuzzy loss functions given by more experts. In the future, we will study dynamic Pythagorean fuzzy information systems for conflicts. Furthermore, we will provide effective algorithms for conflict analysis of dynamic Pythagorean fuzzy information systems.

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