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Variable-precision three-way concepts in L-contexts

Xuerong Zhao [a,b], Duoqian Miao [a,b,*], Hamido Fujita [c,d,e]

[a] Department of Computer Science and Technology, Tongji University, Shanghai 201804, China
[b] Key Laboratory of Embedded System and Service Computing, Ministry of Education, Shanghai 201804, China
[c] College of Mathematical Sciences, Harbin Engineering University, Harbin 150001, China
[d] Andalusian Research Institute in Data Science and Computational Intelligence (DaSCI), University of Granada, Granada, Spain
[e] Faculty of Software and Information Science, Iwate Prefectural University, Iwate, 020-0693, Japan

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ABSTRACT

The notion of fuzzy concept is proposed to deal with object-attribute data with **L**-values (where **L** is a truth-value structure). One disadvantage of fuzzy concept is that a fuzzy context contains a considerable number of fuzzy concepts. This makes it very time-consuming to generate a fuzzy concept lattice, and it is very difficult to find important concepts. In addition, the fuzzy concept shows great strictness when applying to crisp sets. To overcome these problems, we propose several new kinds of variable-precision concepts within **L**-contexts in this paper. First, we present two kinds of variable-precision two-way (VP2W) concepts: α-positive concept and β-negative concept. The family of each kind of VP2W concept forms a complete lattice. Next, considering both the positive and negative parts, we investigate two kinds of variable-precision three-way (VP3W) concepts: (α, β)-object-induced three-way concept and (α, β)-attribute-induced three-way concept. The family of each kind of VP3W concept forms a complete lattice. Then, we study the relationships between VP2W concepts and VP3W concepts. The results show that VP3W concept lattices can be directly generated by VP2W concept lattices. Finally, the experiments are preformed to verify the effectiveness of our model.

1. Introduction

Formal concept analysis (FCA) introduced by Wille [43] provides an effective way to unfold concepts from the context of bivalent data. Considering that human concepts have a graded structure (since whether a concept is applicable to a given object is a matter of degree instead of a yes-or-no question), **L**-concept analysis (**L**CA, or fuzzy concept analysis) generalizes FCA from the perspective of fuzzy set and the bivalent formal context is generalized to the **L**-context. An **L**-context consists of a universe of objects, a universe of attributes, and an **L**-relation between the two universes. The notation **L** represents a truth-value structure, like a complete lattice [9,10] or a residuated lattice [4,5]. An **L**-concept is a special pair of **L**-sets that mutually determine each other by derivation operators. The first paper relating to **L**CA is contributed by Burusco and Fuentes-González [9], followed by contributions by Pollandt [34] and Belohlavek [4]. The difference between their work is twofold: The approach proposed by Burusco and Fuentes-Gonzá adopts complete lattices as the truth-value structure while the latter uses residuated lattices; concepts of the former are defined based on t-conorms while the latter defines concepts by residual implications. Since then, **L**CA has been deeply researched and widely applied in various fields [1,2,5,10,13,15,21].

* Corresponding author at: Department of Computer Science and Technology, Tongji University, Shanghai 201804, China.
E-mail addresses: xrzhao@whu.edu.cn (X. Zhao), dqmiao@tongji.edu.cn (D. Miao), HFujita-799@acm.org (H. Fujita).

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One disadvantage of **L**CA is the considerable number of generated concepts for a given **L**-context. This makes it very time-consuming to generate a fuzzy concept lattice. Even though a variety of works were proposed to reduce the number of concepts [22,28,30–32,41,52], it is still very difficult to figure out important concepts from a set of **L**-concepts. On the other hand, an **L**-concept is a pair of **L**-sets. When it applies to crisp sets, it lays strong requirements on a pair of crisp sets $\langle O, A\rangle$ to be an **L**-concept: For all $o \in O$ and $a \in A$ they must be totally related; for $o \notin O$ there should exist an attribute a which is totally not related to o, and for $a \notin A$ there should exist an object which is totally not related to a (see Section 2). To overcome these problems, we propose several new kinds of variable-precision concepts within **L**-contexts in this paper.

In the framework of FCA, a concept is represented by a set of objects (called extent) and a set of attributes (called intent). The objects in the extent share all attributes in the intent, and the attributes in the intent are shared by all objects in the extent at the same time. This leads to a preference model of focusing only on commonly-shared information or positive information. In some cases, such as elections, we need not only positive information (such as supporters), but also negative information (such as opponents) for making decisions to promote the next step. To overcome this problem, Qi, Wei, and Yao [36] proposed three-way concept analysis (3WCA) by combining FCA with three-way decision theory [17,18,24,44, 46–48]. With 3WCA, different three-way concepts were investigated [14,19,25,36–38,42,51], for example, the OE concept and AE concept [36], the OEO concept and AEP concept [42], and the OEP concept and OED concept [51]. Note that the aforementioned three-way concepts were studied in complete formal contexts. Investigations of three-way concepts related to incomplete formal contexts (that is, according to current information, the information between some of the objects and attributes is unknown) can be found in [8,11,26,27,36,35,45]; these excellent works are omitted, since we only focus on complete contexts in the current paper.

Considering the respective advantages of **L**CA and 3WCA, it is natural to combine them together, which leads to the research of **L**-three-way concept analysis (**L**3WCA). Following this idea, He, Wei, and She [16] generalized OE concept and AE concept to **L**OE concept and **L**AE concept in **L**-contexts. Considering both positive and negative attributes, Bartl and Konecny [3] proposed two kinds of **L**-three-way concepts based on antitone and isotone concept-forming operators. Within the neutrosophic context, Singh [39] proposed the three-way fuzzy concept. The meaning of "three", however, is different from that in [36]. A three-way fuzzy concept of Singh's is a pair of neutrosophic sets, while a neutrosophic set N is characterized by a triple of functions (T_N, I_N, F_N) representing truth-membership function, indeterminacy-membership function, and falsity-membership function, respectively. Thus, the number "three" means that both the extent and intent of a three-way fuzzy concept are represented by "three" membership functions.

For **L**-three-way concept, it has similar disadvantages as **L**-concept, that is, a large number of generated concepts and the strict requirements when applying to crisp sets. In order to overcome these problems, we introduce a new method to deal with three-way concepts in **L**-contexts and propose the so-called variable-precision three-way (VP3W) concepts. The key idea of "variable-precision" is not first-born in this paper. Ma, Zhang, and Cai [29] introduced the notion of variable threshold concept in fuzzy contexts. Zhang, Ma, and Fan [49] introduced three kinds of variable threshold concepts in **L**-contexts, namely, crisp-crisp, crisp-fuzzy, and fuzzy-crisp variable threshold concepts. Based on the notion of α-satisfaction put forward by Pernelle [33], Ventos and Soldano [40] gave an overview of α-Galois lattice in a general sense without concrete formal contexts. Compared to these methods, the advantages of VP3W concepts proposed in current paper are listed as follows:

- The definition of VP3W concepts is closer to three-way concepts in form. In fact, the (1,0)-object-induced three-way concept is the OE concept and the (1,0)-attribute-induced three-way concept is the AE concept.
- The complexity of generating a VP3W concept lattice for an **L**-context is much lower than that of generating an **L**-three-way concept lattice [16] which has the same complexity as that of generating a three-way concept lattice [36].
- Due to the flexibility of the thresholds α and β, the VP3W concepts show more flexibility in applications. Besides, the important concepts can be found by setting the thresholds reasonably.

The rest of this paper is organized as follows. Section 2 is a brief review of **L**CA and indicates the strictness of **L**-concept when applying to crisp sets. Section 3 introduces the notion of variable-precision two-way (VP2W) concept, namely, α-positive concept and β-negative concept and investigates some related properties of them. Section 4 presents the main results: the study of VP3W concepts and the generalization of the main theorem of concept lattice which characterizes the hierarchical structure of VP3W concepts. Section 5 analyzes the relationships between VP2W concepts and VP3W concepts. In Section 6, we conduct several experiments to verify the effectiveness of our model. The last section concludes this paper.

2. L-concept analysis

This section recalls some basic notions related to **L**CA and analyzes a deficiency of **L**-concept when applying to crisp sets.

A complete residuated lattice **L** is a structure $(L, \vee, \wedge, \otimes, \rightarrow, 0_L, 1_L)$ such that (1) $(L, \vee, \wedge, 0_L, 1_L)$ is a complete lattice with the greatest element 1_L and the least element 0_L, (2) $(L, \otimes, 1_L)$ is a commutative monoid,[1] and (3) $(\otimes, \rightarrow)$ is an

[1] A commutative monoid is a triplet $(L, \otimes, 1_L)$ consisting of a set L, a binary operation $\otimes$ on L, and an identity element 1_L of L such that for $a, b, c \in L$, (1) $a \otimes b = b \otimes a$, (2) $a \otimes (b \otimes c) = (a \otimes b) \otimes c$, and (3) $a \otimes 1_L = 1_L \otimes a = a$.

adjoint pair[2] on L. In the rest of this paper, the notation **L** always denotes a complete residuated lattice. With **L**, one can establish the following notions: An **L**-set $\tilde{A}$ of a universe OB is a mapping $\tilde{A} : OB \longrightarrow L$ with $\tilde{A}(o)$ interpreting as "the truth degree of o belonging to $\tilde{A}$". The set of all **L**-sets in OB is denoted by L^{OB}. An **L**-context is a triplet $K = (OB, AT, \tilde{R})$ where OB is a set of objects, AT is a set of attributes, and $\tilde{R}$ is an **L**-relation from OB to AT, that is, a binary mapping $\tilde{R} : OB \times AT \longrightarrow L$ with $\tilde{R}(o, a)$ interpreting as "the truth degree of object o having attribute a". Some special cases of **L**-contexts are listed as follows:

(1) If $L = \{0, 1\}$, then an **L**-context is a formal context.
(2) If $L = [0, 1]$, then an **L**-context is a fuzzy context.
(3) If $L = \{[a^-, a^+] \mid 0 \le a^- \le a^+ \le 1\}$, then an **L**-context is an interval-valued fuzzy context.
(4) If $L = \{(u, v) \mid 0 \le u, v \le 1, 0 \le u + v \le 1\}$, then an **L**-context is an intuitionistic fuzzy context.

Bělohlávek [4,5,7] generalized formal concepts to **L**-concepts by introducing a pair of operators $(\tilde{*}, \tilde{*})$ defined by residual implication.

Definition 1. [4,5,7] Given an **L**-context $K = (OB, AT, \tilde{R})$, a pair of fuzzy subsets $\langle \tilde{O}, \tilde{A} \rangle$ with $\tilde{O} \in L^{OB}$ and $\tilde{A} \in L^{AT}$ is an **L**-concept if $\tilde{O}^{\tilde{*}} = \tilde{A}$ and $\tilde{A}^{\tilde{*}} = \tilde{O}$, where

$$\tilde{O}^{\tilde{*}}(a) = \bigwedge_{o \in OB} \big(\tilde{O}(o) \to \tilde{R}(o, a)\big), \quad a \in AT, \tag{1}$$

$$\tilde{A}^{\tilde{*}}(o) = \bigwedge_{a \in AT} \big(\tilde{A}(a) \to \tilde{R}(o, a)\big), \quad o \in OB. \tag{2}$$

According to basic rules of fuzzy logic, the value of $\tilde{O}^{\tilde{*}}(a)$ is interpreted as the truth degree of the proposition "a is shared by all objects from $\tilde{O}$" and $\tilde{A}^{\tilde{*}}(o)$ the truth degree of the proposition "o has all attributes from $\tilde{A}$". The following results show the strictness of **L**-concepts applied to crisp sets.

Theorem 1. *Given an* **L**-*context* $K = (OB, AT, \tilde{R})$ *with* $O \subseteq OB$ *and* $A \subseteq AT$, $\langle O, A \rangle$ *is an* **L**-*concept if and only if*

(1) *for each* $o \in O$ *and each* $a \in A$, $\tilde{R}(o, a) = 1_L$;
(2) *for* $a \notin A$, *there exists an* $o \in O$ *such that* $\tilde{R}(o, a) = 0_L$;
(3) *for* $o \notin O$, *there exists an* $a \in A$ *such that* $\tilde{R}(o, a) = 0_L$.

Proof. For $O \subseteq OB$ and $a \in AT$, it follows from Eq. (1) that

$$\begin{aligned} O^{\tilde{*}}(a) &= \bigwedge_{o \in OB} \big(O(o) \to \tilde{R}(o, a)\big) \\ &= \bigwedge_{o \in O} \big(O(o) \to \tilde{R}(o, a)\big) \wedge \bigwedge_{o \in O^c} \big(O(o) \to \tilde{R}(o, a)\big) \\ &= \bigwedge_{o \in O} \big(1_L \to \tilde{R}(o, a)\big) \wedge \bigwedge_{o \in O^c} \big(0_L \to \tilde{R}(o, a)\big) \\ &= \bigwedge_{o \in O} \tilde{R}(o, a). \quad \text{(The properties of} \to \text{can be found in [7].)} \end{aligned}$$

In a similar way, one can prove that $A^{\tilde{*}}(o) = \bigwedge_{a \in A} \tilde{R}(o, a)$ for $A \subseteq AT$ and $o \in OB$. Suppose $\langle O, A \rangle$ is an **L**-concept, then $O^{\tilde{*}} = A$. Accordingly, the following conditions must be satisfied:

(1) for $a \in A$, $\bigwedge_{o \in O} \tilde{R}(o, a) = 1_L$, namely, $\tilde{R}(o, a) = 1_L$, $\forall o \in O$;
(2) for $a \notin A$, $\bigwedge_{o \in O} \tilde{R}(o, a) = 0_L$, namely, there exists an $o \in O$ such that $\tilde{R}(o, a) = 0_L$.

Similarly, since $A^{\tilde{*}} = O$, the following hold:

(1) for $o \in O$ and $a \in A$, $\tilde{R}(o, a) = 1_L$;
(2) for $o \notin O$, there exists an $a \in A$ such that $\tilde{R}(o, a) = 0_L$.

[2] An adjoint pair $(\otimes, \to)$ is a pair of binary operations on L satisfying the property that $a \otimes b \leqslant c \Leftrightarrow a \leqslant b \to c$, for all $a, b, c \in L$. The operation $\otimes$ is called a multiplication on L and $\to$ is called a residuum or a residual implication on L.

Table 1
A fuzzy context.

	a	b	c	d	e	f
o_1	0.35	1	0.10	0.90	0.90	0
o_2	0.80	0.90	1	0.40	0.85	0
o_3	0.25	0.90	0.50	0.80	0.65	0.80
o_4	1	0.85	0.75	0.30	0.20	0.45

The converse is obvious. □

Theorem 1 shows the strictness of **L**-concept when applying to a pair of crisp sets $\langle O, A\rangle$: (1) For $o \in O$ and $a \in A$, they must be 1_L-related; (2) for $o \notin O$, there should exist a 0_L-related attribute a; (3) for $a \notin A$, there should exist a 0_L-related object o. On the other hand, one may face a large number of **L**-concepts generated from an **L**-context. This is not helpful for finding important concepts. To overcome these problems, we investigate several new kinds of variable-precision concepts in the sequel.

3. Variable-precision two-way concepts

This section introduces two kinds of VP2W concepts: α-positive concept and β-negative concept. Our aim in this section is to prepare necessary notions and facts to obtain VP3W concepts (which will be the subject of the next section).

3.1. α-positive concept

In some cases, people expect concepts to be determined by "some important attributes", namely, by a set of attributes which are key to the concepts. For $o \in OB$ and $a \in AT$, if $\tilde{R}(o,a) \geq \alpha$, we call a an α-positive attribute of o and o an α-positive object of a, where $\alpha \in L$. This leads to the following definition of α-positive operator.

Definition 2. Given an **L**-context $K = (OB, AT, \tilde{R})$ and $\alpha \in L$, for $O \subseteq OB$ and $A \subseteq AT$, we define

$$O^{*_\alpha} = \{a \in AT \mid \tilde{R}(o,a) \geq \alpha, \forall o \in O\} \tag{3}$$

the set of α-positive attributes shared by each object in O with a degree not less than α, and

$$A^{*_\alpha} = \{o \in OB \mid \tilde{R}(o,a) \geq \alpha, \forall a \in A\} \tag{4}$$

the set of α-positive objects sharing all attributes in A with a degree not less than α. The operator $*_\alpha$ is called the α-positive operator.

Obviously, it holds that

$$O^{*_\alpha} = \bigcap_{o \in O} \tilde{R}_\alpha(o), \quad A^{*_\alpha} = \bigcap_{a \in A} \tilde{R}_\alpha(a) \tag{5}$$

where $\tilde{R}_\alpha(o) = \{a \in AT \mid \tilde{R}(o,a) \geq \alpha\}$ and $\tilde{R}_\alpha(a) = \{o \in OB \mid \tilde{R}(o,a) \geq \alpha\}$.

Remark 1. If K is a fuzzy context, namely, $L = [0,1]$, then the operators defined in Eqs. (3) and (4) degenerate into those in [29].

Example 1. A fuzzy context is given in Table 1, where $OB = \{o_1, o_2, o_3, o_4\}$, $AT = \{a, b, c, d, e, f\}$, and $\tilde{R}$ is a fuzzy relation. Let $O = \{o_1, o_2\}$, $A = \{b, e\}$, and $\alpha = 0.8$. By computation, we have

$$O^{*_{0.8}} = \{b, e\}, \quad A^{*_{0.8}} = \{o_1, o_2\}. \tag{6}$$

Proposition 1. *The pair of operators $(*_\alpha, *_\alpha)$ forms a Galois connection between $(2^{OB}, \subseteq)$ and $(2^{AT}, \subseteq)$, namely,*

(1) $O_2^{*_\alpha} \subseteq O_1^{*_\alpha}$ *whenever* $O_1 \subseteq O_2 \subseteq OB$;
(2) $A_2^{*_\alpha} \subseteq A_1^{*_\alpha}$ *whenever* $A_1 \subseteq A_2 \subseteq AT$;
(3) $O \subseteq O^{*_\alpha *_\alpha}$ *for* $O \subseteq OB$;
(4) $A \subseteq A^{*_\alpha *_\alpha}$ *for* $A \subseteq AT$.

Proof. (1) Suppose $O_1 \subseteq O_2 \subseteq OB$ and $a \in O_2^{*_\alpha}$, then $\tilde{R}(o,a) \geq \alpha$, $\forall o \in O_2$. Considering that $O_1 \subseteq O_2$, it follows $\tilde{R}(o,a) \geq \alpha$, $\forall o \in O_1$, which means $a \in O_1^{*_\alpha}$. Therefore, $O_2^{*_\alpha} \subseteq O_1^{*_\alpha}$.

(2) Similarly, one proves that $A_2^{*\alpha} \subseteq A_1^{*\alpha}$ for $A_1 \subseteq A_2 \subseteq AT$.
(3) For $O \subseteq OB$, one obtains

$$O^{*\alpha *\alpha} = \{o \in OB \mid \tilde{R}(o,a) \geq \alpha, \forall a \in O^{*\alpha}\}$$

by Eq. (4). On the other hand, for a given $o \in O$, it follows that $\tilde{R}(o,a) \geq \alpha$ for $a \in O^{*\alpha}$ based on Eq. (3). This means $o \in O^{*\alpha *\alpha}$, or equivalently, $O \subseteq O^{*\alpha *\alpha}$.
(4) In a similar way, one proves that $A \subseteq A^{*\alpha *\alpha}$ for $A \subseteq AT$. $\square$

Properties in Items (1) and (2) exhibit the monotonic properties of $*_\alpha$. Properties in Items (3) and (4) illustrate the relationship between a set and the derived set by applying $*_\alpha$ twice. These properties ensure the pair of operators $(*_\alpha, *_\alpha)$ to be a Galois connection.

Proposition 2. *For O, $O_i \subseteq OB$, A, $A_i \subseteq AT$ ($i \in \Lambda$ where Λ is an index set), and $\alpha, \alpha_1, \alpha_2 \in L$, the following properties hold:*

(1) $O \subseteq A^{*\alpha} \Leftrightarrow A \subseteq O^{*\alpha}$;
(2) $O^{*\alpha} = O^{*\alpha *\alpha *\alpha}$, $\quad A^{*\alpha} = A^{*\alpha *\alpha *\alpha}$;
(3) $\left(\bigcup_{i\in\Lambda} O_i\right)^{*\alpha} = \bigcap_{i\in\Lambda} O_i^{*\alpha}$, $\quad \left(\bigcup_{i\in\Lambda} A_i\right)^{*\alpha} = \bigcap_{i\in\Lambda} A_i^{*\alpha}$;
(4) $\left(\bigcap_{i\in\Lambda} O_i\right)^{*\alpha} \supseteq \bigcup_{i\in\Lambda} O_i^{*\alpha}$, $\quad \left(\bigcap_{i\in\Lambda} A_i\right)^{*\alpha} \supseteq \bigcup_{i\in\Lambda} A_i^{*\alpha}$;
(5) $O^{*\alpha_2} \subseteq O^{*\alpha_1}$, $A^{*\alpha_2} \subseteq A^{*\alpha_1}$ *for* $\alpha_1 \leq \alpha_2$.

Proof. (1) Suppose $O \subseteq A^{*\alpha}$ with $O \subseteq OB$ and $A \subseteq AT$. For $a \in A$, it follows from Eq. (4) that $\tilde{R}(o,a) \geq \alpha$ for each $o \in O$, which means $a \in O^{*\alpha}$. Therefore, $A \subseteq O^{*\alpha}$. Similarly, one proves that $A \subseteq O^{*\alpha}$ implies $O \subseteq A^{*\alpha}$.
(2) For $O \subseteq OB$, since $O \subseteq O^{*\alpha *\alpha}$ by Proposition 1(3), it follows that $O^{*\alpha *\alpha *\alpha} \subseteq O^{*\alpha}$ by Proposition 1(1). On the other hand, let $A = O^{*\alpha}$. Then, $A \subseteq A^{*\alpha *\alpha}$ by Proposition 1(4), namely, $O^{*\alpha} \subseteq O^{*\alpha *\alpha *\alpha}$. Therefore, we have $O^{*\alpha} = O^{*\alpha *\alpha *\alpha}$. The other equation is similarly proved.
(3) For $O_i \subseteq OB$, the following equivalent statements hold:

$$\begin{aligned} a \in \left(\bigcup_{i\in\Lambda} O_i\right)^{*\alpha} &\Leftrightarrow \tilde{R}(o,a) \geq \alpha, \forall o \in \bigcup_{i\in\Lambda} O_i \\ &\Leftrightarrow \tilde{R}(o,a) \geq \alpha, \forall o \in O_i, \forall i \in \Lambda \\ &\Leftrightarrow a \in O_i^{*\alpha}, \forall i \in \Lambda \\ &\Leftrightarrow a \in \bigcap_{i\in\Lambda} O_i^{*\alpha} \end{aligned}$$

which means $(\bigcup_{i\in\Lambda} O_i)^{*\alpha} = \bigcap_{i\in\Lambda} O_i^{*\alpha}$. The other one is similarly proved.
(4) Since $\bigcap_{i\in\Lambda} O_i \subseteq O_j$ for all $j \in \Lambda$, it follows that $O_j^{*\alpha} \subseteq (\bigcap_{i\in\Lambda} O_i)^{*\alpha}$, $\forall j \in \Lambda$, moreover, $\bigcup_{i\in\Lambda} O_i^{*\alpha} \subseteq (\bigcap_{i\in\Lambda} O_i)^{*\alpha}$. The other one is similarly proved.
(5) It is obvious. $\square$

Item (1) is an equivalent statement of Galois connection. From Item (2), it can be found that the result of applying $*_\alpha$ three times in succession is the same as the result of applying it once. Properties in Items (3) and (4) indicate that the distributive property is applicable to set union but not to set intersection. Properties in Item (5) show the monotonicity about the threshold. With a pair of α-positive operators, one can define a new kind of variable-precision concept.

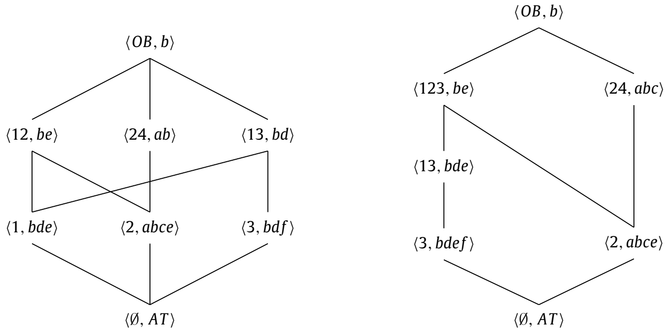
Definition 3. For $O \subseteq OB$, $A \subseteq AT$, and $\alpha \in L$, if $O^{*\alpha} = A$ and $A^{*\alpha} = O$, then $\langle O, A\rangle$ is called a variable-precision positive concept or an α-positive concept; O is called the extent and A the intent of $\langle O, A\rangle$.

Let $C^{*\alpha}(K)$ denote the set of all α-positive concepts of the **L**-context K. Taking Table 1 as an example, according to Eq. (6), we know that $\langle O, A\rangle = \langle\{o_1, o_2\}, \{b, e\}\rangle$ is a 0.8-positive concept.

Remark 2.

(1) If the **L**-relation $\tilde{R}$ degenerates into a binary relation R (namely, $L = \{0, 1\}$) and $\alpha = 1$, one obtains the sufficiency operator $*$ in [43], namely,

$$\begin{aligned} O^{*1} &= \{a \in AT \mid \tilde{R}(o,a) \geq 1, \forall o \in O\} = \{a \in AT \mid \forall o \in O\ (xRa)\} = O^*, \\ A^{*1} &= \{o \in OB \mid \tilde{R}(o,a) \geq 1, \forall a \in A\} = \{o \in OB \mid \forall a \in A\ (xRa)\} = A^*. \end{aligned}$$



(a) $\alpha = 0.8$. (b) $\alpha = 0.6$.

Fig. 1. α-positive concept lattices.

Thus, a 1-positive concept is a formal concept in [43].

(2) One needs to pay attention to the difference between similar concepts and α-positive concepts: The α concept in [28,30] is defined on the basis of inclusion degree within formal contexts; the variable threshold concept in [29] is proposed within fuzzy contexts; the variable threshold concepts in [49] are defined based on fuzzy implication operator.

For two α-positive concepts $\langle O_1, A_1\rangle$, $\langle O_2, A_2\rangle \in C^{*\alpha}(K)$, we say $\langle O_1, A_1\rangle$ is a sub-concept of $\langle O_2, A_2\rangle$ if and only if $\langle O_1, A_1\rangle \le_{*_\alpha} \langle O_2, A_2\rangle$ if and only if $O_1 \subseteq O_2$ (or equivalently, $A_2 \subseteq A_1$). Obviously, $\le_{*_\alpha}$ is a partial order on $C^{*\alpha}(K)$. According to Proposition 2(3), the intersection of any number of intents (respectively, extents) is always an intent (respectively, extent). However, the union of extents or intents does not generally result in an extent or an intent. Based on these properties and the order $\le_{*_\alpha}$, we can define the infimum and supremum of α-positive concepts.

Definition 4. For $\langle O_1, A_1\rangle$, $\langle O_2, A_2\rangle \in C^{*\alpha}(K)$, we define

$$\begin{aligned}
\langle O_1, A_1\rangle \wedge_{*_\alpha} \langle O_2, A_2\rangle &= \langle O_1 \cap O_2, (A_1 \cup A_2)^{*\alpha *\alpha}\rangle \\
&= \langle O_1 \cap O_2, (O_1 \cap O_2)^{*\alpha}\rangle, \\
\langle O_1, A_1\rangle \vee_{*_\alpha} \langle O_2, A_2\rangle &= \langle (O_1 \cup O_2)^{*\alpha *\alpha}, A_1 \cap A_2\rangle \\
&= \langle (A_1 \cap A_2)^{*\alpha}, A_1 \cap A_2\rangle. \qquad (7)
\end{aligned}$$

Obviously, we have $\langle O_1 \cap O_2, (A_1 \cup A_2)^{*\alpha *\alpha}\rangle$, $\langle (O_1 \cup O_2)^{*\alpha *\alpha}, A_1 \cap A_2\rangle \in C^{*\alpha}(K)$ for any $\langle O_1, A_1\rangle$, $\langle O_2, A_2\rangle \in C^{*\alpha}(K)$ according to Proposition 2 Items (2) and (3), which means $(C^{*\alpha}(K), \wedge_{*_\alpha}, \vee_{*_\alpha})$ is a lattice. The following is the main theorem of α-positive concept.

Theorem 2. *For $\alpha \in L$, $(C^{*\alpha}(K), \wedge_{*_\alpha}, \vee_{*_\alpha})$ is a complete lattice, called α-positive concept lattice.*

Proof. To prove a complete lattice, we assume $\langle O_i, A_i\rangle \in C^{*\alpha}(K)$, $i \in \Lambda$ with Λ being an index set. Obviously, we have $\langle \bigcap_{i\in\Lambda} O_i, (\bigcup_{i\in\Lambda} A_i)^{*\alpha *\alpha}\rangle \in C^{*\alpha}(K)$ and $\langle \bigcap_{i\in\Lambda} O_i, (\bigcup_{i\in\Lambda} A_i)^{*\alpha *\alpha}\rangle \le_{*_\alpha} \langle O_i, A_i\rangle$ for each $i \in \Lambda$. Next, we prove $\langle \bigcap_{i\in\Lambda} O_i, (\bigcup_{i\in\Lambda} A_i)^{*\alpha *\alpha}\rangle$ is the infimum. If not, suppose $\langle O, A\rangle \le_{*_\alpha} \langle O_i, A_i\rangle$ and $\langle \bigcap_{i\in\Lambda} O_i, (\bigcup_{i\in\Lambda} A_i)^{*\alpha *\alpha}\rangle \le_{*_\alpha} \langle O, A\rangle$. Then, it holds $O \subseteq O_i$ for $i \in \Lambda$ and $\bigcap_{i\in\Lambda} O_i \subseteq O$. This leads to $O = \bigcap_{i\in\Lambda} O_i$; besides, $A = O^{*\alpha} = (\bigcap_{i\in\Lambda} O_i)^{*\alpha} = (\bigcap_{i\in\Lambda} A_i^{*\alpha})^{*\alpha} = (\bigcup_{i\in\Lambda} A_i)^{*\alpha *\alpha}$ by Proposition 2(3). Together, we can say $\langle \bigcap_{i\in\Lambda} O_i, (\bigcup_{i\in\Lambda} A_i)^{*\alpha *\alpha}\rangle$ is the infimum.

In a similar way, one can prove that $\langle (\bigcup_{i\in\Lambda} O_i)^{*\alpha *\alpha}, \bigcap_{i\in\Lambda} A_i\rangle$ is the supremum of $\langle O_i, A_i\rangle$, $i \in \Lambda$. Therefore, $(C^{*\alpha}(K), \wedge_{*_\alpha}, \vee_{*_\alpha})$ is a complete lattice. □

Example 2 *(Continued from Example 1).* Fig. 1 exhibits the 0.8-positive concept lattice and 0.6-positive concept lattice by two Hasse diagrams, respectively. The number i in each node represents object o_i. A line connects two concepts, in which the lower concept is a sub-concept of the upper one.

An object set or an attribute set can generate an α-positive concept.

Proposition 3. *Given $O \subseteq OB$, $A \subseteq AT$, and $\alpha \in L$, $\langle O^{*\alpha *\alpha}, O^{*\alpha}\rangle$ and $\langle A^{*\alpha}, A^{*\alpha *\alpha}\rangle$ are α-positive concepts.*

Proof. It is obvious from Proposition 2(2). □

Example 3 *(Continued from Example 1).* Let $\alpha = 0.8$ and $O = \{o_2, o_3, o_4\}$. By computation, we have $O^{*0.8} = \{b\}$ and $O^{*0.8*0.8} = \{b\}^{*0.8} = OB$. Therefore, $\langle OB, \{b\}\rangle$ is a 0.8-positive concept. This can be easily verified from Fig. 1.

Proposition 4. *For a given* **L***-context* $K = (OB, AT, \tilde{R})$ *and* $\alpha \in L$*, let* $K_\alpha = (OB, AT, \tilde{R}_\alpha)$ *be the* α*-positive formal context of* K*, where* $\tilde{R}_\alpha = \{(o, a) \mid \tilde{R}(o, a) \geq \alpha\}$*. Then,* $\langle O, A\rangle$ *is an* α*-positive concept in* K *if and only if* $\langle O, A\rangle$ *is a formal concept in* K_α*.*

Proof. It is obvious. □

Proposition 4 provides us a convenient way to generate α-positive concept lattice: For an **L**-context K and $\alpha \in L$, one first computes α-positive formal context K_α, then applies the methods of generating formal concept lattice (e.g. [23]) to K_α. The obtained formal concept lattice is just the α-positive concept lattice $C^{*\alpha}(K)$. Therefore, the complexity of generating an α-positive concept lattice is the same as that of generating a formal concept lattice.

3.2. The relationship between fuzzy concepts and α-positive concepts

In this section, by analyzing the relationship between fuzzy concepts and α-positive concepts of fuzzy contexts, we further show the strictness of fuzzy concept when applying to crisp sets.

Theorem 3. *Suppose* $K = (OB, AT, \tilde{R})$ *is a fuzzy context and* $O \subseteq OB$*,* $A \subseteq AT$*, then*

$$(O^{\tilde{*}})_\alpha = O^{*\alpha}, \quad (A^{\tilde{*}})_\alpha = A^{*\alpha}, \quad \forall \alpha \in [0, 1]$$

where $O^{\tilde{*}}$ *and* $A^{\tilde{*}}$ *are defined by Eqs. (1) and (2).*

Proof. Given $O \subseteq OB$ and $a \in AT$, it follows from the proof of Theorem 1 that $O^{\tilde{*}}(a) = \bigwedge_{o\in O} \tilde{R}(o, a)$. Then, we have

$$a \in (O^{\tilde{*}})_\alpha \Leftrightarrow \bigwedge_{o\in O} \tilde{R}(o, a) \geq \alpha \Leftrightarrow \tilde{R}(o, a) \geq \alpha,\ \forall o \in O \Leftrightarrow a \in O^{*\alpha}$$

which leads to $(O^{\tilde{*}})_\alpha = O^{*\alpha}$, $\forall \alpha \in [0, 1]$. Similarly, one can prove that $(A^{\tilde{*}})_\alpha = A^{*\alpha}$. □

On the basis of Theorem 3, $O^{\tilde{*}}$ and $A^{\tilde{*}}$ can be expressed by $O^{*\alpha}$ and $A^{*\alpha}$, respectively.

Theorem 4. *Suppose* $K = (OB, AT, \tilde{R})$ *is a fuzzy context and* $O \subseteq OB$*,* $A \subseteq AT$*, then*

$$O^{\tilde{*}} = \bigcup_{\alpha\in[0,1]} \alpha O^{*\alpha}, \quad A^{\tilde{*}} = \bigcup_{\alpha\in[0,1]} \alpha A^{*\alpha}$$

where $(\alpha O^{*\alpha})(a) = \alpha \wedge O^{*\alpha}(a)$ *and* $(\alpha A^{*\alpha})(o) = \alpha \wedge A^{*\alpha}(o)$ *with* $o \in OB$ *and* $a \in AT$*.*

Proof. It is obvious according to the decomposition theorem of fuzzy set [20] and Theorem 3. □

The relationship between fuzzy concepts formed by crisp sets and α-positive concepts is demonstrated below.

Theorem 5. *Suppose* $K = (OB, AT, \tilde{R})$ *is a fuzzy context and* $O \subseteq OB$*,* $A \subseteq AT$*, then* $\langle O, A\rangle$ *is a fuzzy concept if and only if* $\langle O, A\rangle$ *is an* α*-positive concept for all* $\alpha \in (0, 1]$*.*

Proof. For $O \subseteq OB$ and $A \subseteq AT$, suppose $\langle O, A\rangle$ is a fuzzy concept, then $O^{\tilde{*}} = A$ and $A^{\tilde{*}} = O$ by Eq. (1) and, naturally, $(O^{\tilde{*}})_\alpha = A_\alpha$ and $(A^{\tilde{*}})_\alpha = O_\alpha$, $\forall \alpha \in [0, 1]$. Since O and A are crisp sets, it follows from Theorem 3 that $O^{*\alpha} = A$ and $A^{*\alpha} = O$, $\forall \alpha \in (0, 1]$, namely, $\langle O, A\rangle$ is an α-positive concept for all $\alpha \in (0, 1]$.

Conversely, suppose $O^{*\alpha} = A$ and $A^{*\alpha} = O$, $\forall \alpha \in (0, 1]$. By Theorem 4, it follows that

$$O^{\tilde{*}} = \bigcup_{\alpha\in[0,1]} \alpha O^{*\alpha} = \bigcup_{\alpha\in(0,1]} \alpha O^{*\alpha} = \bigcup_{\alpha\in(0,1]} \alpha A = A,$$

$$A^{\tilde{*}} = \bigcup_{\alpha\in[0,1]} \alpha A^{*\alpha} = \bigcup_{\alpha\in(0,1]} \alpha A^{*\alpha} = \bigcup_{\alpha\in(0,1]} \alpha A = A.$$

Therefore, $\langle O, A\rangle$ is a fuzzy concept (note that $0O^{*0} = \emptyset$ and $0A^{*0} = \emptyset$). □

Theorem 5 again shows the strictness of fuzzy concept applied to crisp sets.

3.3. β-negative concept

This section introduces two notions of β-negative operator and β-negative concept. For $o \in OB$ and $a \in AT$, if $\tilde{R}(o,a) \leq \beta$, we call a a β-negative attribute of o and o a β-negative object of a, where $\beta \in L$.

Definition 5. Given an **L**-context $K = (OB, AT, \tilde{R})$ and $\beta \in L$, for $O \subseteq OB$ and $A \subseteq AT$, we define

$$O^{\bar{*}_\beta} = \{a \in AT \mid \tilde{R}(o,a) \leq \beta, \forall o \in O\} \tag{8}$$

the set of β-negative attributes shared by each object in O with a degree not greater than β, and

$$A^{\bar{*}_\beta} = \{o \in OB \mid \tilde{R}(o,a) \leq \beta, \forall a \in A\} \tag{9}$$

the set of β-negative objects sharing all attributes in A with a degree not greater than β. The operator $\bar{*}_\beta$ is called the β-negative operator.

Obviously, the following results hold:

$$O^{\bar{*}_\beta} = \bigcap_{o \in O} R^{\mathrm{N}}_\beta(o), \quad A^{\bar{*}_\beta} = \bigcap_{a \in A} R^{\mathrm{N}}_\beta(a) \tag{10}$$

where $R^{\mathrm{N}}_\beta(o) = \{a \in AT \mid \tilde{R}(o,a) \leq \beta\}$ and $R^{\mathrm{N}}_\beta(a) = \{o \in OB \mid \tilde{R}(o,a) \leq \beta\}$.

The operators defined in Eqs. (8) and (9) are dually adjoint.

Proposition 5. *The pair of operators $(\bar{*}_\beta, \bar{*}_\beta)$ forms a Galois connection between $(2^{OB}, \subseteq)$ and $(2^{AT}, \subseteq)$, namely,*

(1) $O_2^{\bar{*}_\beta} \subseteq O_1^{\bar{*}_\beta}$ *whenever* $O_1 \subseteq O_2 \subseteq OB$;
(2) $A_2^{\bar{*}_\beta} \subseteq A_1^{\bar{*}_\beta}$ *whenever* $A_1 \subseteq A_2 \subseteq AT$;
(3) $O \subseteq O^{\bar{*}_\beta \bar{*}_\beta}$ *for* $O \subseteq OB$;
(4) $A \subseteq A^{\bar{*}_\beta \bar{*}_\beta}$ *for* $A \subseteq AT$.

Proof. The proof is similar to that of Proposition 1. $\square$

The basic properties of β-negative operators are presented in the following.

Proposition 6. *For $O, O_i \subseteq OB$, $A, A_i \subseteq AT$ ($i \in \Lambda$), and $\beta, \beta_1, \beta_2 \in L$, the following properties hold:*

(1) $O \subseteq A^{\bar{*}_\beta} \Leftrightarrow A \subseteq O^{\bar{*}_\beta}$;
(2) $O^{\bar{*}_\beta} = O^{\bar{*}_\beta \bar{*}_\beta \bar{*}_\beta}, \quad A^{\bar{*}_\beta} = A^{\bar{*}_\beta \bar{*}_\beta \bar{*}_\beta}$;
(3) $\left(\bigcup_{i \in \Lambda} O_i\right)^{\bar{*}_\beta} = \bigcap_{i \in \Lambda} O_i^{\bar{*}_\beta}, \quad \left(\bigcup_{i \in \Lambda} A_i\right)^{\bar{*}_\beta} = \bigcap_{i \in \Lambda} A_i^{\bar{*}_\beta}$;
(4) $\left(\bigcap_{i \in \Lambda} O_i\right)^{\bar{*}_\beta} \supseteq \bigcup_{i \in \Lambda} O_i^{\bar{*}_\beta}, \quad \left(\bigcap_{i \in \Lambda} A_i\right)^{\bar{*}_\beta} \supseteq \bigcup_{i \in \Lambda} A_i^{\bar{*}_\beta}$;
(5) $O^{\bar{*}_{\beta_1}} \subseteq O^{\bar{*}_{\beta_2}}$, $A^{\bar{*}_{\beta_1}} \subseteq A^{\bar{*}_{\beta_2}}$ *if* $\beta_1 \leq \beta_2$;

Proof. The proof is similar to that of Proposition 2. $\square$

With the pair of dually adjoint operators $(\bar{*}_\beta, \bar{*}_\beta)$, one can define another kind of variable-precision concept.

Definition 6. For $O \subseteq OB$, $A \subseteq AT$, and $\beta \in L$, if $O^{\bar{*}_\beta} = A$ and $A^{\bar{*}_\beta} = O$, then $\langle O, A \rangle$ is called a variable-precision negative concept or a β-negative concept; O is called the extent and A the intent of $\langle O, A \rangle$.

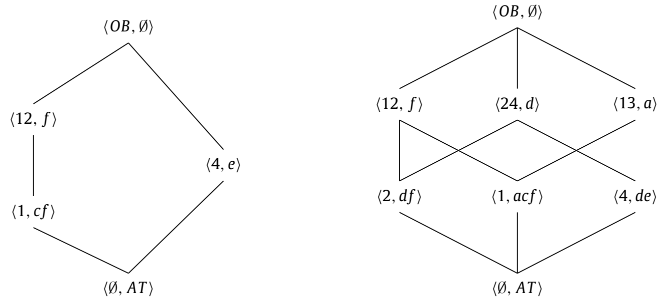
Denote by $C^{\bar{*}_\beta}(K)$ the set of all β-negative concepts of the **L**-context K. Taking Table 1 as an example, for $O = \{o_2, o_4\}$, $A = \{d\}$, and $\beta = 0.4$, we have $O^{\bar{*}_{0.4}} = \{o_2\}^{\bar{*}_{0.4}} \cap \{o_4\}^{\bar{*}_{0.4}} = \{d, f\} \cap \{d, e\} = \{d\}$ and $A^{\bar{*}_{0.4}} = \{d\}^{\bar{*}_{0.4}} = \{o_2, o_4\}$. Therefore, $\langle \{o_2, o_4\}, \{d\} \rangle$ is a 0.4-negative concept.

Remark 3. If the **L**-relation $\tilde{R}$ degenerates into a binary relation R (namely, $L = \{0, 1\}$) and $\beta = 0$, then one obtains the negative sufficiency operator $\bar{*}$ in [47], namely,

$$O^{\bar{*}_0} = \{a \in AT \mid \tilde{R}(o,a) \leq 0, \forall o \in O\} = \{a \in AT \mid \forall o \in O(\neg(xRa))\} = O^{\bar{*}},$$

$$A^{\bar{*}_0} = \{o \in OB \mid \tilde{R}(o,a) \leq 0, \forall a \in A\} = \{o \in OB \mid \forall a \in A(\neg(xRa))\} = A^{\bar{*}}.$$

Thus, a 0-negative concept is a negative formal concept in [47].



(a) $\beta = 0.2$. (b) $\beta = 0.4$.

Fig. 2. β-negative concept lattices.

For two β-negative concepts $\langle O_1, A_1\rangle, \langle O_2, A_2\rangle \in C^{\bar{*}_\beta}(K)$, we say $\langle O_1, A_1\rangle$ is a sub-concept of $\langle O_2, A_2\rangle$ if and only if $\langle O_1, A_1\rangle \le_{\bar{*}_\beta} \langle O_2, A_2\rangle$ if and only if $O_1 \subseteq O_2$ (or equivalently, $A_2 \subseteq A_1$). Based on the order $\le_{\bar{*}_\beta}$ and Proposition 6(3), we define the infimum and supremum of β-negative concepts as follows.

Definition 7. For $\langle O_1, A_1\rangle, \langle O_2, A_2\rangle \in C^{\bar{*}_\beta}(K)$, we define

$$\begin{aligned}\langle O_1, A_1\rangle \wedge_{\bar{*}_\beta} \langle O_2, A_2\rangle &= \langle O_1 \cap O_2, (A_1 \cup A_2)^{\bar{*}_\beta\bar{*}_\beta}\rangle\\ &= \langle O_1 \cap O_2, (O_1 \cap O_2)^{\bar{*}_\beta}\rangle,\\ \langle O_1, A_1\rangle \vee_{\bar{*}_\beta} \langle O_2, A_2\rangle &= \langle (O_1 \cup O_2)^{\bar{*}_\beta\bar{*}_\beta}, A_1 \cap A_2\rangle\\ &= \langle (A_1 \cap A_2)^{\bar{*}_\beta}, A_1 \cap A_2\rangle. \end{aligned} \tag{11}$$

The following is the main theorem of β-negative concepts.

Theorem 6. *For $\beta \in L$, $(C^{\bar{*}_\beta}(K), \wedge_{\bar{*}_\beta}, \vee_{\bar{*}_\beta})$ is a complete lattice, called β-negative concept lattice.*

Proof. The proof is similar to that of Theorem 2. □

Example 4 *(Continued from Example 1).* Fig. 2 exhibits two variable-precision negative concept lattices with $\beta = 0.4$ and $\beta = 0.2$, respectively. A line connects two concepts, in which the lower concept is a sub-concept of the upper one.

An object set or an attribute set can generate a β-negative concept.

Proposition 7. *Given $O \subseteq OB$, $A \subseteq AT$, and $\beta \in L$, $\langle O^{\bar{*}_\beta\bar{*}_\beta}, O^{\bar{*}_\beta}\rangle$ and $\langle A^{\bar{*}_\beta}, A^{\bar{*}_\beta\bar{*}_\beta}\rangle$ are β-negative concepts.*

Proof. It is obvious from Proposition 6(2). □

Proposition 8. *For a given* **L***-context $K = (OB, AT, \tilde{R})$ and $\beta \in L$, let $K^{\mathrm{N}}_\beta = (OB, AT, R^{\mathrm{N}}_\beta)$ be the β-negative formal context of K, where $R^{\mathrm{N}}_\beta = \{(o, a) \mid \tilde{R}(o, a) \le \beta\}$. Then, $\langle O, A\rangle$ is a β-negative concept in K if and only if $\langle O, A\rangle$ is a negative formal concept in K^{N}_β.*

Proof. It is obvious. □

Based on Proposition 8, one can construct β-negative concept lattices in the following way: For an **L**-context K and $\beta \in L$, first compute β-negative formal context K^{N}_β, then apply the methods of generating formal concept lattices to K^{N}_β. The obtained formal concept lattice is just the β-negative concept lattice $C^{\bar{*}_\beta}(K)$. Consequently, the complexity of generating a β-negative concept lattice is the same as that of generating a formal concept lattice.

Remark 4. Applying the α-positive operator or β-negative operator to an object set or an attribute set, one gets two disjoint parts of the corresponding universes. For example, for $O \subseteq OB$ and $\alpha \in L$, one obtains two disjoint regions of AT by applying the α-positive operator:

$$\begin{aligned}&\mathrm{POS}_\alpha(O) = O^{*_\alpha} = \{a \in AT \mid \tilde{R}(o, a) \ge \alpha, \forall o \in O\},\\ &\mathrm{NEG}_\alpha(O) = (O^{*_\alpha})^{\mathrm{c}}.\end{aligned}$$

Therefore, we call the α-positive operator and β-negative operator the VP2W operators, and the α-positive concept and β-negative concept the VP2W concepts.

4. Variable-precision three-way concepts

By generalizing the idea of three-way concepts [35,36], we investigate the notion of VP3W concept in this section.

4.1. Variable-precision three-way operators

Suppose (P, Q) and (Z, W) are two pairs of sets, we say $(P, Q) \subseteq (Z, W)$ if and only if $P \subseteq Z$ and $Q \subseteq W$. The intersection, union, and complement are defined as follows [36]:

$$\begin{aligned}&(P, Q) \cap (Z, W) = (P \cap Z, Q \cap W),\\&(P, Q) \cup (Z, W) = (P \cup Z, Q \cup W),\\&(P, Q)^{\mathrm{c}} = (P^{\mathrm{c}}, Q^{\mathrm{c}}).\end{aligned} \tag{12}$$

Based on VP2W operators, one can define VP3W operators and their inverses.

Definition 8. Given an **L**-context $K = (OB, AT, \tilde{R})$ and $\alpha, \beta \in L$ with $0_L \leq \beta < \alpha \leq 1_L$, for $O \subseteq OB$ and $A \subseteq AT$, we define

$$O^{\lessdot_\beta^\alpha} = (O^{*\alpha}, O^{\bar{*}\beta}) \tag{13}$$

the variable-precision object-induced three-way operator or (α, β)-object-induced three-way operator (short for VPO3W operator or (α, β)-O3W operator) and

$$A^{\lessdot_\beta^\alpha} = (A^{*\alpha}, A^{\bar{*}\beta}) \tag{14}$$

the variable-precision attribute-induced three-way operator or (α, β)-attribute-induced three-way operator (short for VPA3W operator or (α, β)-A3W operator).

Note that the condition $0_L \leq \beta < \alpha \leq 1_L$ is to make sure the disjointness of $O^{*\alpha}$ and $O^{\bar{*}\beta}$ as well as $A^{*\alpha}$ and $A^{\bar{*}\beta}$. The operator $\lessdot_\beta^\alpha$ combines the α-positive operator and β-negative operator which considers not only the positive attributes (or objects) but also the negative attributes (or objects). In addition, for any $O \subseteq OB$, $O^{\lessdot_\beta^\alpha}$ divides AT into three disjoint regions:

$$\begin{aligned}&\mathrm{POS}_\alpha(O) = O^{*\alpha} = \{a \in AT \mid \tilde{R}(o, a) \geq \alpha, \forall o \in O\},\\&\mathrm{NEG}_\beta(O) = O^{\bar{*}\beta} = \{a \in AT \mid \tilde{R}(o, a) \leq \beta, \forall o \in O\},\\&\mathrm{BND}_{(\alpha,\beta)}(O) = (\mathrm{POS}_\alpha(O) \cup \mathrm{NEG}_\beta(O))^{\mathrm{c}}.\end{aligned}$$

If the order $\leq$ on L is a total order, then $\mathrm{BND}_{(\alpha,\beta)}(O) = \{a \in AT \mid \beta < \tilde{R}(o, a) < \alpha\}$. Similarly, for any $A \subseteq AT$, $A^{\lessdot_\beta^\alpha}$ divides OB into three disjoint regions:

$$\begin{aligned}&\mathrm{POS}_\alpha(A) = A^{*\alpha} = \{o \in OB \mid \tilde{R}(o, a) \geq \alpha, \forall a \in A\},\\&\mathrm{NEG}_\beta(A) = A^{\bar{*}\beta} = \{o \in OB \mid \tilde{R}(o, a) \leq \beta, \forall a \in A\},\\&\mathrm{BND}_{(\alpha,\beta)}(A) = (\mathrm{POS}_\alpha(A) \cup \mathrm{NEG}_\beta(A))^{\mathrm{c}}.\end{aligned}$$

And $\mathrm{BND}_{(\alpha,\beta)}(A) = \{o \in OB \mid \beta < \tilde{R}(o, a) < \alpha\}$ for a total order $\leq$ on L.

Definition 9. Given an **L**-context $K = (OB, AT, \tilde{R})$ and $\alpha, \beta \in L$ with $0_L \leq \beta < \alpha \leq 1_L$, for $O_1, O_2 \subseteq OB$ and $A_1, A_2 \subseteq AT$, we define

$$(O_1, O_2)^{\gtrdot_\beta^\alpha} = O_1^{*\alpha} \cap O_2^{\bar{*}\beta}, \quad (A_1, A_2)^{\gtrdot_\beta^\alpha} = A_1^{*\alpha} \cap A_2^{\bar{*}\beta} \tag{15}$$

the object-induced inverse operator and attribute-induced inverse operator, respectively.

The set $(O_1, O_2)^{\gtrdot_\beta^\alpha}$ consists of attributes common to each object in O_1 with a degree not less than α and common to each object in O_2 with a degree not greater than β. The set $(A_1, A_2)^{\gtrdot_\beta^\alpha}$ consists of objects owning all attributes in A_1 with a degree not less than α and owning all attributes in A_2 with a degree not greater than β. The basic properties of operators $\lessdot_\beta^\alpha$ and $\gtrdot_\beta^\alpha$ are listed below.

Proposition 9. *For $O, O_j, O_{ij} \subseteq OB$ ($j = 1, 2, 3, 4$ and $i \in \Lambda$), the following properties hold:*

(1) $O_1 \subseteq O_2 \Rightarrow O_2^{\lessdot_\beta^\alpha} \subseteq O_1^{\lessdot_\beta^\alpha}$;
(2) $(O_1, O_2) \subseteq (O_3, O_4) \Rightarrow (O_3, O_4)^{\gtrdot_\beta^\alpha} \subseteq (O_1, O_2)^{\gtrdot_\beta^\alpha}$;
(3) $O \subseteq O^{\lessdot_\beta^\alpha \gtrdot_\beta^\alpha}$;
(4) $(O_1, O_2) \subseteq (O_1, O_2)^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha}$;
(5) $O^{\lessdot_\beta^\alpha} = O^{\lessdot_\beta^\alpha \gtrdot_\beta^\alpha \lessdot_\beta^\alpha}$;
(6) $(O_1, O_2)^{\gtrdot_\beta^\alpha} = (O_1, O_2)^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha \gtrdot_\beta^\alpha}$;
(7) $\left(\bigcup_{i\in\Lambda} O_i\right)^{\lessdot_\beta^\alpha} = \bigcap_{i\in\Lambda} O_i^{\lessdot_\beta^\alpha}$;
(8) $\left(\bigcup_{i\in\Lambda} (O_{i1}, O_{i2})\right)^{\gtrdot_\beta^\alpha} = \bigcap_{i\in\Lambda} (O_{i1}, O_{i2})^{\gtrdot_\beta^\alpha}$;
(9) $\left(\bigcap_{i\in\Lambda} O_i\right)^{\lessdot_\beta^\alpha} \supseteq \bigcup_{i\in\Lambda} O_i^{\lessdot_\beta^\alpha}$;
(10) $\left(\bigcap_{i\in\Lambda} (O_{i1}, O_{i2})\right)^{\gtrdot_\beta^\alpha} \supseteq \bigcup_{i\in\Lambda} (O_{i1}, O_{i2})^{\gtrdot_\beta^\alpha}$.

Proof. (1) Suppose $O_1 \subseteq O_2$, then we have $O_2^{\lessdot_\beta^\alpha} = (O_2^{*_\alpha}, O_2^{\bar{*}_\beta}) \subseteq (O_1^{*_\alpha}, O_1^{\bar{*}_\beta}) = O_1^{\lessdot_\beta^\alpha}$ by Propositions 1 and 5.
(2) Suppose $(O_1, O_2) \subseteq (O_3, O_4)$, then we have $(O_3, O_4)^{\gtrdot_\beta^\alpha} = O_3^{*_\alpha} \cap O_4^{\bar{*}_\beta} \subseteq O_1^{*_\alpha} \cap O_2^{\bar{*}_\beta} = (O_1, O_2)^{\gtrdot_\beta^\alpha}$ by Propositions 1 and 5.
(3) It follows from Propositions 1 and 5 that $O^{\lessdot_\beta^\alpha \gtrdot_\beta^\alpha} = (O^{*_\alpha}, O^{\bar{*}_\beta})^{\gtrdot_\beta^\alpha} = O^{*_\alpha *_\alpha} \cap O^{\bar{*}_\beta \bar{*}_\beta} \supseteq O \cap O = O$.
(4) It follows from Propositions 1, 2, 5, and 6 that $(O_1, O_2)^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha} = (O_1^{*_\alpha} \cap O_2^{\bar{*}_\beta})^{\lessdot_\beta^\alpha} = ((O_1^{*_\alpha} \cap O_2^{\bar{*}_\beta})^{*_\alpha}, (O_1^{*_\alpha} \cap O_2^{\bar{*}_\beta})^{\bar{*}_\beta}) \supseteq (O_1^{*_\alpha *_\alpha} \cup O_2^{\bar{*}_\beta *_\alpha}, O_1^{*_\alpha \bar{*}_\beta} \cup O_2^{\bar{*}_\beta \bar{*}_\beta}) \supseteq (O_1^{*_\alpha *_\alpha}, O_2^{\bar{*}_\beta \bar{*}_\beta}) \supseteq (O_1, O_2)$.
(5) According to Items (1) and (3), we have $O^{\lessdot_\beta^\alpha \gtrdot_\beta^\alpha \lessdot_\beta^\alpha} \subseteq O^{\lessdot_\beta^\alpha}$. On the other hand, it follows from Propositions 2 and 6 that $O^{\lessdot_\beta^\alpha \gtrdot_\beta^\alpha \lessdot_\beta^\alpha} = (O^{*_\alpha}, O^{\bar{*}_\beta})^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha} = (O^{*_\alpha *_\alpha} \cap O^{\bar{*}_\beta \bar{*}_\beta})^{\lessdot_\beta^\alpha} = ((O^{*_\alpha *_\alpha} \cap O^{\bar{*}_\beta \bar{*}_\beta})^{*_\alpha}, (O^{*_\alpha *_\alpha} \cap O^{\bar{*}_\beta \bar{*}_\beta})^{\bar{*}_\beta}) \supseteq (O^{*_\alpha *_\alpha *_\alpha}, O^{\bar{*}_\beta \bar{*}_\beta \bar{*}_\beta}) = (O^{*_\alpha}, O^{\bar{*}_\beta}) = O^{\lessdot_\beta^\alpha}$. Finally, it holds $O^{\lessdot_\beta^\alpha} = O^{\lessdot_\beta^\alpha \gtrdot_\beta^\alpha \lessdot_\beta^\alpha}$.
(6) According to Items (2) and (4), we have $(O_1, O_2)^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha \gtrdot_\beta^\alpha} \subseteq (O_1, O_2)^{\gtrdot_\beta^\alpha}$. On the other hand, it follows from Propositions 1 and 5 that $(O_1, O_2)^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha \gtrdot_\beta^\alpha} = ((O_1, O_2)^{\gtrdot_\beta^\alpha *_\alpha}, (O_1, O_2)^{\gtrdot_\beta^\alpha \bar{*}_\beta})^{\gtrdot_\beta^\alpha} = (O_1, O_2)^{\gtrdot_\beta^\alpha *_\alpha *_\alpha} \cap (O_1, O_2)^{\gtrdot_\beta^\alpha \bar{*}_\beta \bar{*}_\beta} \supseteq (O_1, O_2)^{\gtrdot_\beta^\alpha} \cap (O_1, O_2)^{\gtrdot_\beta^\alpha} = (O_1, O_2)^{\gtrdot_\beta^\alpha}$. Therefore, we have $(O_1, O_2)^{\gtrdot_\beta^\alpha} = (O_1, O_2)^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha \gtrdot_\beta^\alpha}$.
(7) It follows from Eq. (12) and Propositions 2 and 6 that $\left(\bigcup_{i\in\Lambda} O_i\right)^{\lessdot_\beta^\alpha} = \left(\left(\bigcup_{i\in\Lambda} O_i\right)^{*_\alpha}, \left(\bigcup_{i\in\Lambda} O_i\right)^{\bar{*}_\beta}\right) = \left(\bigcap_{i\in\Lambda} O_i^{*_\alpha}, \bigcap_{i\in\Lambda} O_i^{\bar{*}_\beta}\right) = \bigcap_{i\in\Lambda} (O_i^{*_\alpha}, O_i^{\bar{*}_\beta}) = \bigcap_{i\in\Lambda} O_i^{\lessdot_\beta^\alpha}$.
(8) It follows from Eq. (12) and Propositions 2 and 6 that $\left(\bigcup_{i\in\Lambda} (O_{i1}, O_{i2})\right)^{\gtrdot_\beta^\alpha} = \left(\bigcup_{i\in\Lambda} O_{i1}, \bigcup_{i\in\Lambda} O_{i2}\right)^{\gtrdot_\beta^\alpha} = \left(\bigcup_{i\in\Lambda} O_{i1}\right)^{*_\alpha} \cap \left(\bigcup_{i\in\Lambda} O_{i2}\right)^{\bar{*}_\beta} = \left(\bigcap_{i\in\Lambda} O_{i1}^{*_\alpha}\right) \cap \left(\bigcap_{i\in\Lambda} O_{i2}^{\bar{*}_\beta}\right) = \bigcap_{i\in\Lambda} (O_{i1}^{*_\alpha} \cap O_{i2}^{\bar{*}_\beta}) = \bigcap_{i\in\Lambda} (O_{i1}, O_{i2})^{\gtrdot_\beta^\alpha}$.
(9) The proof is similar to that of Item (7).
(10) The proof is similar to that of Item (8). □

For attribute sets, one gets similar properties.

Proposition 10. *For $A, A_j, A_{ij} \subseteq AT$ ($j = 1, 2, 3, 4$ and $i \in \Lambda$), the following properties hold:*

(1) $A_1 \subseteq A_2 \Rightarrow A_2^{\lessdot_\beta^\alpha} \subseteq A_1^{\lessdot_\beta^\alpha}$;
(2) $(A_1, A_2) \subseteq (A_3, A_4) \Rightarrow (A_3, A_4)^{\gtrdot_\beta^\alpha} \subseteq (A_1, A_2)^{\gtrdot_\beta^\alpha}$;
(3) $A \subseteq A^{\lessdot_\beta^\alpha \gtrdot_\beta^\alpha}$;
(4) $(A_1, A_2) \subseteq (A_1, A_2)^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha}$;
(5) $A^{\lessdot_\beta^\alpha} = A^{\lessdot_\beta^\alpha \gtrdot_\beta^\alpha \lessdot_\beta^\alpha}$;
(6) $(A_1, A_2)^{\gtrdot_\beta^\alpha} = (A_1, A_2)^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha \gtrdot_\beta^\alpha}$;
(7) $\left(\bigcup_{i\in\Lambda} A_i\right)^{\lessdot_\beta^\alpha} = \bigcap_{i\in\Lambda} A_i^{\lessdot_\beta^\alpha}$;
(8) $\left(\bigcup_{i\in\Lambda} (A_{i1}, A_{i2})\right)^{\gtrdot_\beta^\alpha} = \bigcap_{i\in\Lambda} (A_{i1}, A_{i2})^{\gtrdot_\beta^\alpha}$;
(9) $\left(\bigcap_{i\in\Lambda} A_i\right)^{\lessdot_\beta^\alpha} \supseteq \bigcup_{i\in\Lambda} A_i^{\lessdot_\beta^\alpha}$;
(10) $\left(\bigcap_{i\in\Lambda} (A_{i1}, A_{i2})\right)^{\gtrdot_\beta^\alpha} \supseteq \bigcup_{i\in\Lambda} (A_{i1}, A_{i2})^{\gtrdot_\beta^\alpha}$.

Proof. The proof is similar to that of Proposition 9. □

4.2. (α, β)-object-induced three-way concept

With (α, β)-O3W operator and attribute-induced inverse operator, one can define the (α, β)-object-induced three-way concept.

Definition 10. For $O \subseteq OB$ and $A_1, A_2 \subseteq AT$, if $O^{\lessdot_\beta^\alpha} = (A_1, A_2)$ and $(A_1, A_2)^{\gtrdot_\beta^\alpha} = O$, then $\langle O, (A_1, A_2)\rangle$ is called a variable-precision object-induced three-way concept or an (α, β)-object-induced three-way concept (short for VPO3W concept or (α, β)-O3W concept); O is called the extent and (A_1, A_2) the intent of $\langle O, (A_1, A_2)\rangle$.

Denote by $OC_3^{\lessdot_\beta^\alpha}(K)$ the set of all (α, β)-O3W concepts of the **L**-context K. Taking Table 1 as an example, let $\alpha = 0.8$, $\beta = 0.4$, and $O = \{o_1, o_2\}$. From Figs. 1 and 2, we have $O^{\lessdot_{0.4}^{0.8}} = (O^{*0.8}, O^{\bar{*}0.4}) = (\{b, e\}, \{f\})$; besides, $(\{b, e\}, \{f\})^{\gtrdot_{0.4}^{0.8}} = \{b, e\}^{*0.8} \cap \{f\}^{\bar{*}0.4} = \{o_1, o_2\} \cap \{o_1, o_2\} = \{o_1, o_2\}$. Thus, $\langle\{o_1, o_2\}, (\{b, e\}, \{f\})\rangle$ is a $(0.8, 0.4)$-O3W concept.

For two (α, β)-O3W concepts $\langle O_1, (A_{11}, A_{12})\rangle, \langle O_2, (A_{21}, A_{22})\rangle \in OC_3^{\lessdot_\beta^\alpha}(K)$, we say $\langle O_1, (A_{11}, A_{12})\rangle$ is a sub-concept of $\langle O_2, (A_{21}, A_{22})\rangle$ if and only if $\langle O_1, (A_{11}, A_{12})\rangle \leq_{\lessdot_\beta^\alpha} \langle O_2, (A_{21}, A_{22})\rangle$ if and only if $O_1 \subseteq O_2$ (or equivalently, $(A_{21}, A_{22}) \subseteq (A_{11}, A_{12})$). Obviously, $\leq_{\lessdot_\beta^\alpha}$ is a partial order on $OC_3^{\lessdot_\beta^\alpha}(K)$. With this order and Proposition 9 Items (7) and (8), one can define the infimum and supremum of (α, β)-O3W concepts.

Definition 11. For $\langle O_1, (A_{11}, A_{12})\rangle, \langle O_2, (A_{21}, A_{22})\rangle \in OC_3^{\lessdot_\beta^\alpha}(K)$, we define

$$\begin{aligned}
\langle O_1, (A_{11}, A_{12})\rangle \wedge_{\lessdot_\beta^\alpha} \langle O_2, (A_{21}, A_{22})\rangle &= \langle O_1 \cap O_2, ((A_{11}, A_{12}) \cup (A_{21}, A_{22}))^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha}\rangle \\
&= \langle O_1 \cap O_2, (O_1 \cap O_2)^{\lessdot_\beta^\alpha}\rangle, \\
\langle O_1, (A_{11}, A_{12})\rangle \vee_{\lessdot_\beta^\alpha} \langle O_2, (A_{21}, A_{22})\rangle &= \langle (O_1 \cup O_2)^{\lessdot_\beta^\alpha \gtrdot_\beta^\alpha}, (A_{11}, A_{12}) \cap (A_{21}, A_{22})\rangle \\
&= \langle ((A_{11}, A_{12}) \cap (A_{21}, A_{22}))^{\gtrdot_\beta^\alpha}, (A_{11}, A_{12}) \cap (A_{21}, A_{22})\rangle.
\end{aligned} \tag{16}$$

According to Proposition 10 Items (6) and (8), we have $\langle O_1 \cap O_2, ((A_{11}, A_{12}) \cup (A_{21}, A_{22}))^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha}\rangle, \langle (O_1 \cup O_2)^{\lessdot_\beta^\alpha \gtrdot_\beta^\alpha}, (A_{11}, A_{12}) \cap (A_{21}, A_{22})\rangle \in OC_3^{\lessdot_\beta^\alpha}(K)$, which means $(OC_3^{\lessdot_\beta^\alpha}(K), \wedge_{\lessdot_\beta^\alpha}, \vee_{\lessdot_\beta^\alpha})$ is a lattice. Actually, the set of all (α, β)-O3W concepts forms a complete lattice.

Theorem 7. *Given $\alpha, \beta \in L$ with $0_L \leq \beta < \alpha \leq 1_L$, $(OC_3^{\lessdot_\beta^\alpha}(K), \wedge_{\lessdot_\beta^\alpha}, \vee_{\lessdot_\beta^\alpha})$ is a complete lattice, called (α, β)-O3W concept lattice.*

Proof. To prove the result, we assume $\langle O_i, (A_{i1}, A_{i2})\rangle \in OC_3^{\lessdot_\beta^\alpha}(K)$, $i \in \Lambda$ with Λ being an index set. First, it is obvious from Proposition 10 Items (6) and (8) that $\langle \bigcap_{i\in\Lambda} O_i, (\bigcup_{i\in\Lambda}(A_{i1}, A_{i2}))^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha}\rangle$ is an (α, β)-O3W concept and $\langle \bigcap_{i\in\Lambda} O_i, (\bigcup_{i\in\Lambda}(A_{i1}, A_{i2}))^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha}\rangle \leq \langle O_i, (A_{i1}, A_{i2})\rangle$ for each $i \in \Lambda$. Next, we prove $\langle \bigcap_{i\in\Lambda} O_i, (\bigcup_{i\in\Lambda}(A_{i1}, A_{i2}))^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha}\rangle$ is the infimum. If not, suppose $\langle O, (A_1, A_2)\rangle \leq_{\lessdot_\beta^\alpha} \langle O_i, (A_{i1}, A_{i2})\rangle$ and $\langle \bigcap_{i\in\Lambda} O_i, (\bigcup_{i\in\Lambda}(A_{i1}, A_{i2}))^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha}\rangle \leq_{\lessdot_\beta^\alpha} \langle O, (A_1, A_2)\rangle$. Then, it follows $O \subseteq O_i$ for $i \in \Lambda$ and $\bigcap_{i\in\Lambda} O_i \subseteq O$. This leads to $O = \bigcap_{i\in\Lambda} O_i$; besides, $(A_1, A_2) = O^{\lessdot_\beta^\alpha} = (\bigcap_{i\in\Lambda} O_i)^{\lessdot_\beta^\alpha} = (\bigcap_{i\in\Lambda}(A_{i1}, A_{i2})^{\gtrdot_\beta^\alpha})^{\lessdot_\beta^\alpha} = (\bigcup_{i\in\Lambda}(A_{i1}, A_{i2}))^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha}$. Equivalently saying, $\langle \bigcap_{i\in\Lambda} O_i, (\bigcup_{i\in\Lambda}(A_{i1}, A_{i2}))^{\gtrdot_\beta^\alpha \lessdot_\beta^\alpha}\rangle$ is the infimum of $\langle O_i, (A_{i1}, A_{i2})\rangle$, $i \in \Lambda$.

In a similar way, one can prove that $\langle (\bigcup_{i\in\Lambda} O_i)^{\lessdot_\beta^\alpha \gtrdot_\beta^\alpha}, \bigcap_{i\in\Lambda}(A_{i1}, A_{i2})\rangle$ is an (α, β)-O3W concept and also the supremum of $\langle O_i, (A_{i1}, A_{i2})\rangle$, $i \in \Lambda$. Consequently, $(OC_3^{\lessdot_\beta^\alpha}(K), \wedge_{\lessdot_\beta^\alpha}, \vee_{\lessdot_\beta^\alpha})$ is a complete lattice. $\square$

4.3. (α, β)-attribute-induced three-way concept

With (α, β)-A3W operator and object-induced inverse operator, one can define the (α, β)-attribute-induced three-way concept.

Definition 12. For $O_1, O_2 \subseteq OB$ and $A \subseteq AT$, if $(O_1, O_2)^{\gtrdot_\beta^\alpha} = A$ and $A^{\lessdot_\beta^\alpha} = (O_1, O_2)$, then $\langle (O_1, O_2), A\rangle$ is called a variable-precision attribute-induced three-way concept or an (α, β)-attribute-induced three-way concept (short for VPA3W concept or (α, β)-A3W concept).

Denote by $AC_3^{\gtrdot_\beta^\alpha}(K)$ the set of all (α,β)-A3W concepts of the **L**-context $K=(OB,AT,\tilde{R})$. Taking Table 1 as an example, for $\alpha=0.8$, $\beta=0.4$, and $A=\{d,f\}$, we have $A^{\lessdot_{0.4}^{0.8}}=(A^{*0.8},A^{\bar{*}0.4})=(\{o_3\},\{o_2\})$; besides, $(\{o_3\},\{o_2\})^{\gtrdot_{0.4}^{0.8}}=\{o_3\}^{*0.8}\cap\{o_2\}^{\bar{*}0.4}=\{b,d,f\}\cap\{d,f\}=\{d,f\}$. Therefore, $\langle(\{o_3\},\{o_2\}),\{d,f\}\rangle$ is a (0.8, 0.4)-A3W concept.

Given two (α,β)-A3W concepts $\langle(O_{11},O_{12}),A_1\rangle,\langle(O_{21},O_{22}),A_2\rangle\in AC_3^{\gtrdot_\beta^\alpha}(K)$, we say $\langle(O_{11},O_{12}),A_1\rangle$ is a sub-concept of $\langle(O_{21},O_{22}),A_2\rangle$ if and only if $\langle(O_{11},O_{12}),A_1\rangle\leq_{\gtrdot_\beta^\alpha}\langle(O_{21},O_{22}),A_2\rangle$ if and only if $(O_{11},O_{12})\subseteq(O_{21},O_{22})$ (or equivalently, $A_2\subseteq A_1$). With the order $\leq_{\gtrdot_\beta^\alpha}$ and Proposition 10 Items (7) and (8), we now define the infimum and supremum of (α,β)-A3W concepts.

Definition 13. For $\langle(O_{11},O_{12}),A_1\rangle,\langle(O_{21},O_{22}),A_2\rangle\in AC_3^{\gtrdot_\beta^\alpha}(K)$, we define

$$\begin{aligned}\langle(O_{11},O_{12}),A_1\rangle\wedge_{\gtrdot_\beta^\alpha}\langle(O_{21},O_{22}),A_2\rangle&=\big\langle(O_{11},O_{12})\cap(O_{21},O_{22}),(A_1\cup A_2)^{\lessdot_\beta^\alpha\gtrdot_\beta^\alpha}\big\rangle\\&=\big\langle(O_{11},O_{12})\cap(O_{21},O_{22}),((O_{11},O_{12})\cap(O_{21},O_{22}))^{\gtrdot_\beta^\alpha}\big\rangle,\\\langle(O_{11},O_{12}),A_1\rangle\vee_{\gtrdot_\beta^\alpha}\langle(O_{21},O_{22}),A_2\rangle&=\big\langle((O_{11},O_{12})\cup(O_{21},O_{22}))^{\gtrdot_\beta^\alpha\lessdot_\beta^\alpha},A_1\cap A_2\big\rangle\\&=\big\langle(A_1\cap A_2)^{\lessdot_\beta^\alpha},A_1\cap A_2\big\rangle.\end{aligned}\tag{17}$$

The set of all (α,β)-A3W concepts forms a complete lattice.

Theorem 8. *Given $\alpha,\beta\in L$ with $0_L\leq\beta<\alpha\leq 1_L$, $(AC_3^{\gtrdot_\beta^\alpha}(K),\wedge_{\gtrdot_\beta^\alpha},\vee_{\gtrdot_\beta^\alpha})$ is a complete lattice, called (α,β)-A3W concept lattice.*

Proof. The proof is similar to that of Theorem 7. □

Note that we call (α,β)-O3W operator and (α,β)-A3W operator VP3W operators, and (α,β)-O3W concept and (α,β)-A3W concept VP3W concepts.

5. The relationships between VP2W concepts and VP3W concepts

This section mainly investigates the relationships between VP2W concepts and VP3W concepts.

5.1. The relationships between VP2W concepts and (α,β)-O3W concepts

An α-positive concept can produce an (α,β)-O3W concept; a β-negative concept can also produce an (α,β)-O3W concept.

Theorem 9. *Given $O\subseteq OB$, $A\subseteq AT$, and $\alpha,\beta\in L$ with $0_L\leq\beta<\alpha\leq 1_L$,*

(1) *if $\langle O,A\rangle$ is an α-positive concept, then $\langle O,(A,O^{\bar{*}\beta})\rangle$ is an (α,β)-O3W concept;*
(2) *if $\langle O,A\rangle$ is a β-negative concept, then $\langle O,(O^{*\alpha},A)\rangle$ is an (α,β)-O3W concept.*

Proof. (1) Suppose $\langle O,A\rangle$ is an α-positive concept, then $O^{*\alpha}=A$ and $A^{*\alpha}=O$. By Proposition 5, we have $O^{\lessdot_\beta^\alpha}=(O^{*\alpha},O^{\bar{*}\beta})=(A,O^{\bar{*}\beta})$ and $(A,O^{\bar{*}\beta})^{\gtrdot_\beta^\alpha}=A^{*\alpha}\cap O^{\bar{*}\beta\bar{*}\beta}=O\cap O^{\bar{*}\beta\bar{*}\beta}=O$. This proves that $\langle O,(A,O^{\bar{*}\beta})\rangle$ is an (α,β)-O3W concept.
(2) This is similarly proved as Item (1). □

Conversely, for a given (α,β)-O3W concept, one can naturally get an α-positive concept and a β-negative concept.

Theorem 10. *Given $O\subseteq OB$, $A_1,A_2\subseteq AT$, and $\alpha,\beta\in L$ with $0_L\leq\beta<\alpha\leq 1_L$, if $\langle O,(A_1,A_2)\rangle$ is an (α,β)-O3W concept, then $\langle A_1^{*\alpha},A_1\rangle$ is an α-positive concept and $\langle A_2^{\bar{*}\beta},A_2\rangle$ is a β-negative concept.*

Proof. Suppose $\langle O,(A_1,A_2)\rangle$ is an (α,β)-O3W concept, then $O^{*\alpha}=A_1$ and $O^{\bar{*}\beta}=A_2$. It, therefore, follows that $A_1^{*\alpha*\alpha}=O^{*\alpha*\alpha*\alpha}=O^{*\alpha}=A_1$, which means $\langle A_1^{*\alpha},A_1\rangle$ is an α-positive concept. Similarly, one proves that $\langle A_2^{\bar{*}\beta},A_2\rangle$ is a β-negative concept. □

Theorem 9 provides us a hint to form (α,β)-O3W concepts from α-positive concepts and β-negative concepts. Theorem 10 introduces a method to obtain α-positive concepts and β-negative concepts from (α,β)-O3W concepts. The following result establishes an equivalence between VP2W concepts and (α,β)-O3W concepts.

Algorithm 1: Generate (α, β)-O3W concept lattice.

input : α-positive concept lattice: $C^{*\alpha}(K) = \{\langle O_i^{\mathrm{P}}, A_i^{\mathrm{P}}\rangle\}$,
β-negative concept lattice: $C^{*\beta}(K) = \{\langle O_i^{\mathrm{N}}, A_i^{\mathrm{N}}\rangle\}$.

output: (α, β)-O3W concept lattice: $OC_3^{<^{\alpha}_{\beta}}(K) = \{\langle O_i, (A_i^{\alpha}, A_i^{\beta})\rangle\}$.

n = 0,
for $i = 1$ to $|C^{*\alpha}(K)|$ **do**
  for $j = 1$ to $|C^{*\beta}(K)|$ **do**
    n = n + 1,
    O_n = $O_i^{\mathrm{P}} \cap O_j^{\mathrm{N}}$,
  end
end
Delete repeated elements in $\{O_1, O_2, \cdots\}$,
for each O_i **do**
  compute A_i^{α}, A_i^{β}.
end

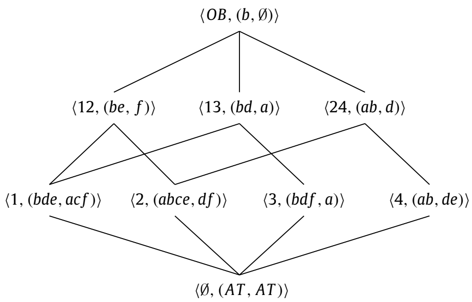


Fig. 3. (0.8, 0.4)-O3W concept lattice.

Theorem 11. *Given $\alpha, \beta \in L$ with $0_L \leq \beta < \alpha \leq 1_L$, $\langle O, (A_1, A_2)\rangle$ is an (α, β)-O3W concept if and only if there exist an α-positive concept $\langle O_1, A'\rangle$ and a β-negative concept $\langle O_2, A''\rangle$ such that $O = O_1 \cap O_2$, $A_1 = (O_1 \cap O_2)^{*\alpha}$, and $A_2 = (O_1 \cap O_2)^{\bar{*}\beta}$.*

Proof. Suppose $\langle O, (A_1, A_2)\rangle$ is an (α, β)-O3W concept. Let $O_1 = A_1^{*\alpha}$, $A' = A_1$ and $O_2 = A_2^{\bar{*}\beta}$, $A'' = A_2$. Then, according to Theorem 10, $\langle O_1, A'\rangle$ is an α-positive concept and $\langle O_2, A''\rangle$ is a β-negative concept. On the other hand, since $\langle O, (A_1, A_2)\rangle$ is an (α, β)-O3W concept, we have $O = (A_1, A_2)^{>^{\alpha}_{\beta}} = A_1^{*\alpha} \cap A_2^{\bar{*}\beta} = O_1 \cap O_2$, $(O_1 \cap O_2)^{*\alpha} = O^{*\alpha} = A_1$, and $(O_1 \cap O_2)^{\bar{*}\beta} = O^{\bar{*}\beta} = A_2$.

To prove the contrary, suppose $\langle O_1, A'\rangle$ is an α-positive concept and $\langle O_2, A''\rangle$ is a β-negative concept. Let $O = O_1 \cap O_2$, $A_1 = (O_1 \cap O_2)^{*\alpha}$, and $A_2 = (O_1 \cap O_2)^{\bar{*}\beta}$. Next, we prove $\langle O, (A_1, A_2)\rangle$ is an (α, β)-O3W concept. Obviously, $O^{<^{\alpha}_{\beta}} = (O^{*\alpha}, O^{\bar{*}\beta}) = (A_1, A_2)$ and $(A_1, A_2)^{>^{\alpha}_{\beta}} = A_1^{*\alpha} \cap A_2^{\bar{*}\beta} = (O_1 \cap O_2)^{*\alpha*\alpha} \cap (O_1 \cap O_2)^{\bar{*}\beta\bar{*}\beta}$. According to Propositions 1(3) and 5(3), it follows that $(O_1 \cap O_2)^{*\alpha*\alpha} \cap (O_1 \cap O_2)^{\bar{*}\beta\bar{*}\beta} \supseteq O_1 \cap O_2$. On the other hand, since $O_1 \cap O_2 \subseteq O_1$ and $O_1^{*\alpha*\alpha} = O_1$, we have $(O_1 \cap O_2)^{*\alpha*\alpha} \subseteq O_1$; in a similar way, we have $(O_1 \cap O_2)^{\bar{*}\beta\bar{*}\beta} \subseteq O_2$. Therefore, $(O_1 \cap O_2)^{*\alpha*\alpha} \cap (O_1 \cap O_2)^{\bar{*}\beta\bar{*}\beta} \subseteq O_1 \cap O_2$. Finally, it holds $(O_1 \cap O_2)^{*\alpha*\alpha} \cap (O_1 \cap O_2)^{\bar{*}\beta\bar{*}\beta} = O_1 \cap O_2 = O$, namely, $(A_1, A_2)^{>^{\alpha}_{\beta}} = O$. □

Theorem 11 provides us a way to produce (α, β)-O3W concept lattices from α-positive concept lattices and β-negative concept lattices. Each (α, β)-O3W concept can be obtained in the following way: Take an α-positive concept $\langle O_1, A_1\rangle$ from $C^{*\alpha}(K)$ and a β-negative concept $\langle O_2, A_2\rangle$ from $C^{\bar{*}\beta}(K)$, compute $O_1 \cap O_2$, $(O_1 \cap O_2)^{*\alpha}$, and $(O_1 \cap O_2)^{\bar{*}\beta}$, then $\langle O_1 \cap O_2, ((O_1 \cap O_2)^{*\alpha}, (O_1 \cap O_2)^{\bar{*}\beta})\rangle$ is an (α, β)-O3W concept. Algorithm 1 is applied to generate an (α, β)-O3W concept lattice from an α-positive concept lattice and a β-negative concept lattice. The time complexity of generating an (α, β)-O3W concept lattice is $O(|C^{*\alpha}(K)| \times |C^{\bar{*}\beta}(K)|)$.

Example 5 *(Continued from Examples 2 and 4).* Applying Algorithm 1, one gets the (0.8, 0.4)-O3W concept lattice from 0.8-positive concept lattice and 0.4-negative concept lattice. The result is shown in Fig. 3.

Remark 5. When $L = \{0, 1\}$, $\alpha = 1$, and $\beta = 0$, we get the relationships between OE-concept and two-way concepts (namely, formal concept and negative formal concept) [35].

Algorithm 2: Generate (α,β)-A3W concept lattice.

input : α-positive concept lattice: $C^{*\alpha}(K)=\{\langle O_i^{\mathrm{P}}, A_i^{\mathrm{P}}\rangle\}$,
β-negative concept lattice: $C^{*\beta}(K)=\{\langle O_i^{\mathrm{N}}, A_i^{\mathrm{N}}\rangle\}$.

output: (α,β)-A3W concept lattice: $AC_3^{\lessdot_\beta^\alpha}(K)=\{\langle (O_i^\alpha, O_i^\beta), A_i\rangle\}$.

n = 0,
for $i=1$ to $|C^{*\alpha}(K)|$ **do**
  for $j=1$ to $|C^{*\beta}(K)|$ **do**
    n = n + 1,
    $A_n = A_i^{\mathrm{P}} \cap A_j^{\mathrm{N}}$,
  end
end
Delete repeated elements in $\{A_1, A_2, \cdots\}$,
for each A_i **do**
  compute O_i^α, O_i^β.
end

5.2. The relationships between VP2W concepts and (α,β)-A3W concepts

An α-positive concept can produce an (α,β)-A3W concept; a β-negative concept can produce an (α,β)-A3W concept. Conversely, one can get an α-positive concept and a β-negative concept from a given (α,β)-A3W concept. The results are stated in Theorems 12 and 13, respectively.

Theorem 12. *Given $O \subseteq OB$, $A \subseteq AT$, and $\alpha,\beta \in L$ with $0_L \le \beta < \alpha \le 1_L$,*

(1) *if $\langle O, A\rangle$ is an α-positive concept, then $\langle (O, A^{\bar{*}_\beta}), A\rangle$ is an (α,β)-A3W concept;*
(2) *if $\langle O, A\rangle$ is a β-negative concept, then $\langle (A^{*\alpha}, O), A\rangle$ is an (α,β)-A3W concept.*

Proof. The proof is similar to that of Theorem 9. □

Theorem 13. *Given $O_1, O_2 \subseteq OB$, $A \subseteq AT$, and $\alpha,\beta \in L$ with $0_L \le \beta < \alpha \le 1_L$, if $\langle (O_1, O_2), A\rangle$ is an (α,β)-A3W concept, then $\langle O_1, O_1^{*\alpha}\rangle$ is an α-positive concept and $\langle O_2, O_2^{\bar{*}_\beta}\rangle$ is a β-negative concept.*

Proof. The proof is similar to that of Theorem 10. □

There also exists an equivalence between VP2W concepts and (α,β)-A3W concepts.

Theorem 14. *Given $\alpha,\beta \in L$ with $0_L \le \beta < \alpha \le 1_L$, $\langle (O_1, O_2), A\rangle$ is an (α,β)-A3W concept if and only if there exist an α-positive concept $\langle O', A_1\rangle$ and a β-negative concept $\langle O'', A_2\rangle$ such that $A = A_1 \cap A_2$, $O_1 = (A_1 \cap A_2)^{*\alpha}$, and $O_2 = (A_1 \cap A_2)^{\bar{*}_\beta}$.*

Proof. The proof is similar to that of Theorem 11. □

Theorem 14 provides us a convenient way to produce (α,β)-A3W concept lattices from α-positive concept lattices and β-negative concept lattices. Briefly speaking, for an α-positive concept $\langle O_1, A_1\rangle$ and a β-negative concept $\langle O_2, A_2\rangle$, $\langle A_1 \cap A_2, ((A_1 \cap A_2)^{*\alpha}, (A_1 \cap A_2)^{\bar{*}_\beta})\rangle$ is an (α,β)-A3W concept. We provide Algorithm 2 to generate an (α,β)-A3W concept lattice from an α-positive concept lattice and a β-negative concept lattice. The time complexity of generating an (α,β)-A3W concept lattice is $O(|C^{*\alpha}(K)| \times |C^{\bar{*}_\beta}(K)|)$.

Example 6 *(Continued from Examples 2 and 4).* Applying Algorithm 2, we obtain the (0.8, 0.4)-A3W concept lattice (exhibited in Fig. 4) from 0.8-positive concept lattice and 0.4-negative concept lattice.

6. Experiments

In this section, we conducted some experiments to verify the effectiveness of our model. **About datasets:** The datasets are shown in Table 2. The first dataset is from our example shown in Table 1. The second to the last are from UCI Machine Learning Repository [12]. **About algorithms:** The algorithm used to generate formal concept lattices is from [23]. Algorithms 1 and 2 were applied to generate (α,β)-O3W concept lattices and (α,β)-A3W concept lattices. To generate fuzzy concept lattices, we adopted the method in [6] which is based on a lexicographic order.

According to the method in [6], the time complexity to generate a fuzzy concept lattice for a fuzzy context is $O(|L|^{|AT|})$ where L is the truth-value set and AT is the attribute set of the fuzzy context. In the application, the truth-value set L

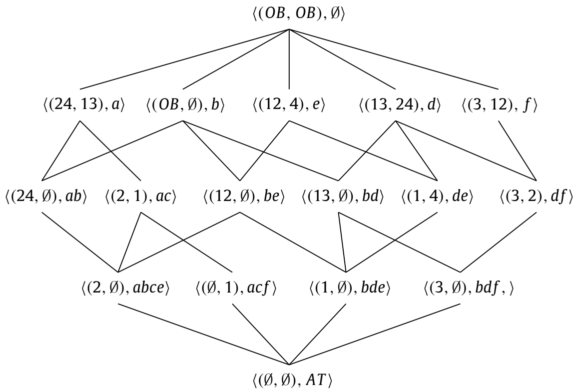


Fig. 4. $(0.8, 0.4)$-A3W concept lattice.

Table 2
Datasets.

Name	Object numbers	Attribute numbers	Missing values
Table 1	4	6	No
Breast **C**ancer **C**oimbra (BCC)	116	10	No
QCM	125	15	No
Speaker **A**ccent **R**ecognition (SAR)	329	12	No
Heart **F**ailure **C**linical **R**ecords (HFCR)	299	13	No

Table 3
The number of α-positive concepts.

Datasets \ α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Table 1	2	3	12	10	9	7	8	8	6	5
BBC	284	253	153	77	59	40	27	23	19	12
QCM	23	30	37	53	60	60	58	59	62	11
SAR	2816	2816	2944	983	340	195	111	55	24	12
HFCR	1179	1052	776	492	418	340	217	142	106	70

Table 4
Runtime with different α.

Datasets \ α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Table 1	0.0002	0.0002	0.0013	0.0013	0.0012	0.0009	0.0007	0.0006	0.0005	0.0004
BBC	0.5894	0.2558	0.1234	0.0727	0.0503	0.0391	0.0277	0.02515	0.0220	0.0172
QCM	0.2020	0.4104	0.3178	0.3370	0.2827	0.21662	0.2653	0.2383	0.2302	0.0253
SAR	29.1071	21.3583	11.0170	2.7607	0.9849	0.5330	0.3164	0.1770	0.0930	0.0614
HFCR	3.6456	2.4196	1.6124	1.0647	0.8920	0.6960	0.4406	0.3220	0.2736	0.2038

is generated by listing all different values that appear in an **L**-context. Therefore, when the dataset becomes larger, the base of L grows larger, and consequently, the time complexity is growing very high. With this in mind, we only conducted experiments on the first dataset in Table 2 to generate a fuzzy concept lattice. We found 224 fuzzy concepts, and the time to find all these concepts was 0.3611 seconds. In contrast, the number of concepts of each α-positive concept lattice for all datasets and the corresponding calculation time were listed in Tables 3 and 4, respectively. We set the initial value of α to 0.1 and the step to 0.1. The results in Table 3 show that the number of concepts in an α-concept lattice is much less than that of a fuzzy concept lattice. In addition, important concepts can be found by setting α to a high threshold.

In order to verify our proposed method of generating (α, β)-O3W and (α, β)-A3W concept lattices, we conducted another set of experiments. We set the initial values of α and β to 0.55 and 0.45, and the steps of α and β to 0.05 and -0.05, respectively. Table 5 shows the concept numbers of each kind of concept lattice. In order to show the trend of the concept numbers clearly, we first transformed each number in Table 5 with a logarithmic function with a base of 10, and then exhibited it in Fig. 5. The results illustrated that the number of concepts is decreasing with regard to α and increasing with regard to β, but not strictly monotonous.

Table 5
The number of concepts.

$(\alpha,\beta)=$		(0.55,0.45)	(0,60,0.40)	(0.65,0.35)	(0.70,0.30)	(0.75,0.25)	(0.80,0.20)	(0.85,0.15)	(0.90,0.10)	(0.95,0.05)
Table 1	α-positive	7	7	8	8	8	6	6	5	5
	β-negative	7	8	6	6	6	5	4	4	3
	(α,β)-O3W	11	10	9	9	9	6	7	6	6
	(α,β)-A3W	17	16	16	16	14	11	9	7	6
BBC	α-positive	49	40	29	27	24	23	21	19	16
	β-negative	264	302	270	246	240	175	100	66	36
	(α,β)-O3W	1085	987	747	594	499	338	197	124	64
	(α,β)-A3W	324	341	297	279	262	190	115	83	46
QCM	α-positive	43	60	47	58	49	59	63	62	51
	β-negative	287	186	127	75	75	53	41	43	44
	(α,β)-O3W	623	548	387	337	297	292	183	103	88
	(α,β)-A3W	672	410	328	209	227	151	116	120	137
SAR	α-positive	247	195	133	111	66	55	41	24	14
	β-negative	196	147	125	86	59	36	24	17	16
	(α,β)-O3W	3076	2515	1465	819	324	188	101	48	30
	(α,β)-A3W	435	295	220	164	104	71	49	29	19
HFCR	α-positive	393	331	301	217	172	138	122	106	78
	β-negative	1759	1831	1671	1579	1295	920	659	530	335
	(α,β)-O3W	24341	22872	19686	14619	9634	6322	4001	3335	1975
	(α,β)-A3W	2303	2256	2022	1768	1425	1011	724	589	357

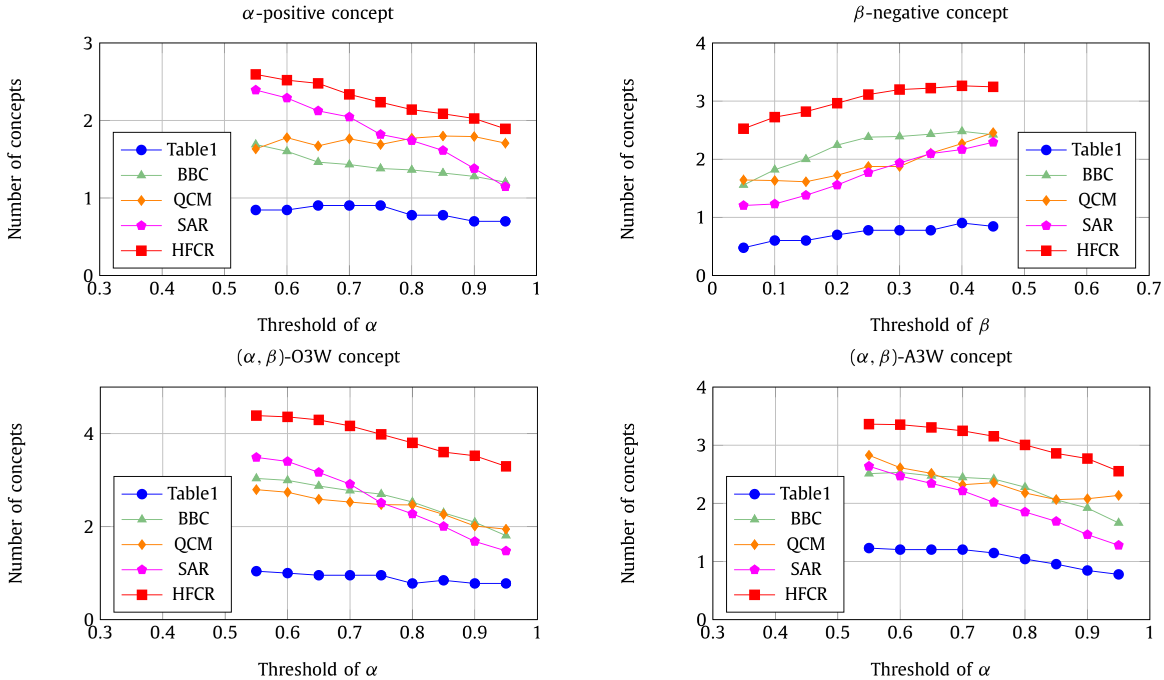


Fig. 5. The number of concepts.

7. Conclusion

The **L**-context refers to a formal context of which the relation is taking values on a truth-value structure **L**, usually a residuated lattice. Considering the disadvantages of **L**-concepts, we introduced two kinds of VP2W, namely, α-positive concept and β-negative concept, and two kinds of VP3W concepts, namely, (α,β)-O3W concept and (α,β)-A3W concept. The new model is more flexible in constructing different concepts with different thresholds. The family of α-positive concept (respectively, β-negative concept, (α,β)-O3W concept, and (α,β)-A3W concept) forms a complete lattice. We proved the equivalences between VP2W concepts and VP3W concepts and provided a way to generate (α,β)-O3W concept lattices and (α,β)-A3W concept lattices from α-positive concept lattices and β-negative concept lattices.

In order to have a clear understanding of variable-precision concepts, all examples are based on fuzzy contexts in this paper. From an application perspective, one may encounter different types of data; in this paper, the **L**-context is only an general notion. In addition, there are eight different kinds of two-way concepts and three-way concepts [47,50]; we only proposed two kinds of VP2W concepts and two kinds of VP3W concepts in this paper. For future study, we will investigate other kinds of VP2W and VP3W concepts and analyze with special **L**-contexts, for example, interval-valued fuzzy contexts, intuitionistic fuzzy contexts.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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