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Related families-based attribute reduction of dynamic covering decision information systems

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ABSTRACT

Many efforts have focused on studying techniques for selecting most informative features from data sets. Especially, the related family-based approaches have been provided for attribute reduction of covering information systems. However, the existing related family-based methods have to recompute reducts for dynamic covering decision information systems. In this paper, firstly, we investigate the mechanisms of updating the related families and attribute reducts by the utilization of previously learned results in dynamic covering decision information systems with variations of attributes. Then, we design incremental algorithms for attribute reduction of dynamic covering decision information systems in terms of attribute arriving and leaving using the related families and employ examples to demonstrate that how to update attribute reducts with the proposed algorithms. Finally, experimental comparisons with the non-incremental algorithms on UCI data sets illustrate that the proposed incremental algorithms are feasible and efficient to conduct attribute reduction of dynamic covering decision information systems with immigration and emigration of attributes.

1. Introduction

Covering rough set theory, pioneered by Zakowski[62] in 1983, has become a useful mathematical tool for dealing with uncertain and imprecise information in practical situations. As a substantial constituent of granular computing, covering-based rough set theory has been applied to many fields such as feature selection and data mining without any prior knowledge. Especially, covering rough set theory is being attracting more and more attention in the era of artificial intelligence, which provides powerful supports for the development of data processing technique.

Many researchers[1,5,8–11,15,17–22,25,27,28,31,34,41,42,48–50, 52–56,58–61,64–69] have studied covering-based rough set theory. For example, Hu et al.[8] proposed a matrix representation of multi-granulation approximations in optimistic and pessimistic multi-granulation rough sets and matrix-based dynamic approaches for updating approximations in multigranulation rough sets when a single granular structure evolves over time. Lang et al.[15] presented incremental approaches to computing the second and sixth lower and upper approximations of sets in dynamic covering approximation spaces. Luo et al.[29] investigated the updating properties for dynamic maintenance of approximations when the criteria values in the set-valued decision system evolve with time and proposed two incremental algorithms for computing rough approximations with the addition and removal of criteria values. Wang et al.[49] transformed the set approximation computation into products of the type-1 and type-2 characteristic matrices and the characteristic function of the set in covering approximation spaces. Yang et al.[54] provided a new type of fuzzy covering-based rough set model by introducing the notion of fuzzy β-minimal description and generalized the model to L-fuzzy covering-based rough set which is defined over fuzzy lattices. Yang et al.[56] provided related family-based methods for computing attribute reducts and relative attribute reducts for covering rough sets, which remove superfluous attributes while keeping the approximation space of covering information system unchanged. Yao et al.[61] classified all approximation operators into element-based approximation operators, granule-based approximation operators, and subsystem-based approximation operators.

Knowledge reduction of dynamic information systems[2–4,6,7, 12–14,16,19,23,24,26,29,30,32,33,35,36,38–40,43–47,51,57,63,66] has attracted more attention. For example, Chen et al.[3] employed an incremental manner to update minimal elements in the discernibility

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matrices at the arrival of an incremental sample. Huang et al.[12] presented the extended variable precision rough set model based on the λ-tolerance relation in terms of Bhattacharyya distance and incremental mechanisms by the utilization of previously learned approximation results and region relation matrices for updating rough approximations in set-valued information systems. Lang et al.[14] focused on knowledge reduction of dynamic covering information systems with variations of objects using the type-1 and type-2 characteristic matrices. Li et al.[19] discussed the principles of updating P-dominating sets and P-dominated sets when some attributes are added into or deleted from the attribute set P. Luo et. al[30] analyzed the dynamic characteristics of conditional partition and decision classification on the universe when the insertion or deletion of objects occurs and presented incremental algorithms for updating probabilistic approximations, which are proficient to efficiently classify the incremental objects into decision regions by avoiding re-computation efforts. Qian et al.[35] defined a new attribute reduct for sequential three-way decisions and designed attribute reduction algorithms satisfying the monotonicity of the probabilistic positive region, which provide a new insight into the attribute reduction problem of sequential three-way decisions. Xu et al.[51] introduced the stream computing learning method on the basis of existing incremental learning studies and solved the challenges resulted from simultaneous addition and deletion of objects. Yang et al.[57] provided an insight into the incremental process of attribute reduction with fuzzy rough sets which reveals how to add new attributes into the current reduct and delete existing attributes from the current reduct and two incremental algorithms of attribute reduction with fuzzy rough sets for one incoming sample and multiple incoming samples, respectively. Zhang et al.[63] proposed incremental approaches for computing the lower and upper approximations with dynamic attribute variation in set-valued information systems.

In real-world decision making, there are many covering decision information systems such as incomplete information systems and set-valued information systems, and researchers have proposed many methods for attribute reduction of covering decision information systems on the basis of discernibility matrices. Especially, we observe that the third lower and upper approximation operators are regarded as the most reasonable in covering-based rough sets, and discernibility matrices-based attribute reduction methods can not work for constructing attribute reducts of covering decision information systems with respect to the third type approximation operators. Meanwhile, we see that the related families-based methods proposed by Yang[56] are very effective for knowledge reduction of covering decision information systems, which bridge the gap where the discernibility matrix is not applicable. In practical situations, covering decision information systems are varying with time. Especially, there are many dynamic covering decision information systems with variations of object sets, attribute sets and attribute values, and knowledge reduction of dynamic covering decision information systems is a significant challenge of covering-based rough sets. Furthermore, we find that there are few researches on knowledge reduction of dynamic covering decision information systems using the related families, and non-incremental approaches are time-consuming for knowledge reduction of dynamic covering decision information systems with respect to the third lower and upper approximation operators. Therefore, the incremental learning technique is desired to improve computational efficiency of attribute reduction of dynamic covering decision information systems by employing the previous reduct results.

The purpose of this paper is to investigate attribute reduction of dynamic covering decision information systems. First, we study attribute reduction of dynamic covering decision information systems with immigrations of attributes. Concretely, we analyze the mechanisms of updating the related sets of objects in dynamic covering decision information systems with attribute arriving, and construct the related families of dynamic covering decision information systems based on those of original covering decision information systems. Meanwhile, we investigate the relationship between attribute reducts of dynamic covering decision information systems and those of original covering decision information systems, and provide incremental algorithms for updating attribute reducts of dynamic covering decision information systems with attribute arriving. Second, we investigate attribute reduction of dynamic covering decision information systems with emigrations of attributes. Concretely, we illustrate the mechanisms of constructing the related families of dynamic covering decision information systems based on those of original covering decision information systems. After that, we investigate the relationship between attribute reducts of dynamic covering decision information systems and those of original covering decision information systems and propose incremental algorithms for updating attribute reducts of dynamic covering decision information systems with attribute leaving. Finally, we provide heuristic incremental algorithms for updating attribute reducts of dynamic covering decision information systems with immigration and emigration of attributes and employ the experimental results on UCI data sets to indicate that the proposed algorithms outperform the static algorithms while inserting into or removing from attribute sets in dynamic covering decision information systems.

The rest of this paper is organized as follows: In Section 2, we briefly review the basic concepts of covering-based rough set theory. In Section 3, we study updated mechanisms for constructing attribute reducts of dynamic covering decision information systems with variations of attribute sets. In Section 4, we provide heuristic algorithms for computing reducts of dynamic covering decision information systems. In Section 5, we employ the experimental results to illustrate that the related families-based incremental approaches are effective to perform attribute reduction of dynamic covering decision information systems. Concluding remarks and further research are given in Section 6.

2. Preliminaries

In this section, we briefly review some concepts of covering-based rough sets.

Suppose $S=(U,A,V,f)$ is an information system, where $U=\{x_1,x_2,\cdots,x_n\}$ is a finite set of objects, A is a finite set of attributes, $V=\{V_a|a\in A\}$, where V_a is the set of values of attribute a, and $card(V_a)>1$, f is a function from $U\times A$ into V, and the indiscernibility relation $IND(B)\subseteq U\times U$ is defined as follows: $IND(B)=\{(x,y)\in U\times U|\ \forall\ b\in B, b(x)=b(y)\}$, where $b(x)$ and $b(y)$ denote the values of objects x and y on $b\in B\subseteq A$, respectively. Especially, we have the equivalence class $[x]_B=\{y\in U|(x,y)\in IND(B)\}$ for $x\in U$.

Definition 2.1. [37] Let $S=(U,A,V,f)$ be an information system, and $B\subseteq C$. Then the Pawlak upper and lower approximations of $X\subseteq U$ with respect to $IND(B)$ are defined as follows:

$$\overline{R}(X)=\{x\in U|[x]_B\cap X\neq\emptyset\},\ \underline{R}(X)=\{x\in U|[x]_B\subseteq X\}.$$

According to Definition 2.1, we see that Pawlak rough set model is constructed on an indiscernibility relation or a family of equivalence classes, and each object is classified into a certain concept in the Pawlak rough set model. But different concepts of the universe usually overlap, and the condition of the equivalence relation is so strict that limit its application in practical situations.

Subsequently, Zakowski[62] employed the covering of the universe to establish a covering based generalized rough sets as follows.

Definition 2.2. [62] Let U be a finite universe of discourse, and $\mathscr{C}$ a family of subsets of U. Then $\mathscr{C}$ is called a covering of U if none of elements of $\mathscr{C}$ is empty and $\bigcup\{C|C \in \mathscr{C}\} = U$. Furthermore, $(U, \mathscr{C})$ is referred to as a covering approximation space.

If U is a finite universe of discourse, and $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, where $\mathscr{C}_i$ $(1 \leq i \leq m)$ is a covering of U, then (U, Δ) is called a covering information system; $(U, \Delta, \mathscr{D})$ is called a covering decision information system, where Δ and $\mathscr{D}$ denote coverings and partition based on conditional attributes and decision attributes, respectively.

Example 2.3. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ be eight houses, $C = \{quality\}$ the attribute set, the domain of *quality* is {*high, middle, low*}. We employ the specialists A and B to evaluate these houses and show their evaluation reports as follows:

$$high_A = \{x_1, x_4, x_5, x_7\}, middle_A = \{x_2, x_8\}, low_A = \{x_3, x_6\};$$
$$high_B = \{x_1, x_2, x_4, x_7\}, middle_B = \{x_5\}, low_B = \{x_3, x_6, x_8\},$$

where $high_A$ denotes the houses belonging to high quality by the specialist A, and the meanings of other symbols are similar. Since their evaluations are of equal importance, we consider all their advice and derive the covering approximation space $(U, \mathscr{C}_{price})$, where $\mathscr{C}_{price} = \{high_{A\vee B}, middle_{A\vee B}, low_{A\vee B}\}$, and

$$high_{A\vee B} = high_A \cup high_B = \{x_1, x_2, x_4, x_5, x_7\};$$
$$middle_{A\vee B} = middle_A \cup middle_B = \{x_2, x_5, x_8\};$$
$$low_{A\vee B} = low_A \cup low_B = \{x_3, x_6, x_8\}.$$

Definition 2.4. [69] Let $(U, \mathscr{C})$ be a covering approximation space, and $Md_{\mathscr{C}}(x) = \{K \in \mathscr{C}|x \in K \wedge (\forall S \in \mathscr{C} \wedge x \in S \wedge S \subseteq K \Rightarrow K = S)\}$ for $x \in U$. Then $Md_{\mathscr{C}}(x)$ is called the minimal description of x.

By Definition 2.4, we observe that the minimal description of x is a set of the minimal elements containing x in $\mathscr{C}$. For a covering $\mathscr{C}$ of U, K is a union reducible element of $\mathscr{C}$, $\mathscr{C} - \{K\}$ and $\mathscr{C}$ have the same $Md(x)$ for $x \in U$. If K is a union reducible element of $\mathscr{C}$ if and only if $K \notin Md(x)$ for any $x \in U$, and denote $\mathscr{M}_{\cup\Delta} = \{Md_{\cup\Delta}(x)|x \in U\}$ with respect to a family of coverings Δ.

Definition 2.5. [69] Let $(U, \mathscr{C})$ be a covering approximation space, and $Md_{\mathscr{C}}(x)$ the minimal description of $x \in U$. Then the third lower and upper approximations of $X \subseteq U$ with respect to $\mathscr{C}$ are defined as follows:

$$CL_{\mathscr{C}}(X) = \cup\{K \in \mathscr{C}|K \subseteq X\};$$
$$CH_{\mathscr{C}}(X) = \cup\{K \in Md_{\mathscr{C}}(x)|x \in X\}.$$

According to Definition 2.5, we see that the third lower and upper approximation operators are typical representatives of non-dual approximation operators for covering approximation spaces. Furthermore, we have $CL_{\mathscr{C}}(X) = \bigcup\{K \in \mathscr{C}| \exists\, x \in U,$ s.t. $(K \in Md_{\mathscr{C}}(x)) \wedge (K \subseteq X)\}$ with the minimal descriptions. Especially, we have $CL_{\cup\Delta}(X) = \cup\{K \in Md_{\cup\Delta}(x)|K \subseteq X\}$ and $CH_{\cup\Delta}(X) = \cup\{K \in Md_{\cup\Delta}(x)|x \in X\}$. For simplicity, we denote $POS_{\cup\Delta}(X) = CL_{\cup\Delta}(X)$, $NEG_{\cup\Delta}(X) = U \setminus CH_{\cup\Delta}(X)$ and $BND_{\cup\Delta}(X) = CH_{\cup\Delta}(X) \setminus CL_{\cup\Delta}(X)$.

Definition 2.6. [56] Let $(U, \Delta, \mathscr{D})$ be a covering decision information system, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, and $\mathscr{D} = \{D_1, D_2, ..., D_k\}$. Then

(1) if there exists $K \in Md_{\cup\Delta}(y)$ and $D_j \in \mathscr{D}$ such that $x \in K \subseteq D_j$ for any $x \in U$, where $y \in U$, then $(U, \Delta, \mathscr{D})$ is called a consistent covering decision information system.

(2) if there exists $x \in U$ but $\nexists\, K \in \cup\Delta$ and $D_j \in \mathscr{D}$ such that $x \in K \subseteq D_j$, then $(U, \Delta, \mathscr{D})$ is called an inconsistent covering decision information system.

By Definition 2.6, we see that all covering information systems are classified into consistent covering decision information systems and inconsistent covering decision information systems. For simplicity, the symbols $\mathscr{M}_{\cup\Delta} \preceq \mathscr{D}$ and $\mathscr{M}_{\cup\Delta} \npreceq \mathscr{D}$ denote $(U, \Delta, \mathscr{D})$ is a consistent covering decision information system and an inconsistent covering decision information system, respectively.

Definition 2.7. [56] Let $(U, \Delta, \mathscr{D})$ be a covering decision information system, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, $\mathscr{D} = \{D_1, D_2, ..., D_k\}$, and $POS_{\cup\Delta}(\mathscr{D}) = \bigcup\{POS_{\cup\Delta}(D_i)\,|D_i \in \mathscr{D}\}$. Then

(1) if $POS_{\cup\Delta}(\mathscr{D}) = POS_{\cup\Delta-\{\mathscr{C}_i\}}(\mathscr{D})$ for $\mathscr{C}_i \in \Delta$, then $\mathscr{C}_i$ is called superfluous relative to $\mathscr{D}$; Otherwise, $\mathscr{C}_i$ is called indispensable relative to $\mathscr{D}$;

(2) if every element of $P \subseteq \Delta$ satisfying $POS_{\cup P}(\mathscr{D}) = POS_{\cup\Delta}(\mathscr{D})$ is indispensable relative to $\mathscr{D}$, then P is called a reduct of Δ relative to $\mathscr{D}$.

By Definition 2.7, we observe that a reduct P satisfies two conditions as follows: (1) $POS_{\cup P}(\mathscr{D}) = POS_{\cup\Delta}(\mathscr{D})$; (2) $POS_{\cup\Delta}(\mathscr{D}) \neq POS_{\cup P-\{\mathscr{C}_i\}}(\mathscr{D})$ for any $\mathscr{C}_i \in \Delta$. Furthermore, the first condition indicates the joint sufficiency of the covering set P, and the second condition means that each covering in P is individually necessary. Therefore, P is the minimum covering set keeping the positive regions of decision classes.

Definition 2.8. [56] Let $(U, \Delta, \mathscr{D})$ be a covering decision information system, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, $\mathscr{A}_\Delta = \{C_k \in \cup\Delta|\, \exists\, D_j \in \mathscr{D},$ s.t. $C_k \subseteq D_j\}$, and $r(x) = \{\mathscr{C} \in \Delta|\, \exists\, C_k \in \mathscr{A}_\Delta,$ s.t. $x \in C_k \in \mathscr{C}\}$, and $R(U, \Delta, \mathscr{D}) = \{r(x)|x \in POS_{\cup\Delta}(\mathscr{D})\}$ the related family of $(U, \Delta, \mathscr{D})$. Then

(1) $f(U, \Delta, \mathscr{D}) = \bigwedge\{\bigvee r(x)|r(x) \in R(U, \Delta, \mathscr{D})\}$ is the related function, where $\bigvee r(x)$ is the disjunction of all elements in $r(x)$;

(2) $g(U, \Delta, \mathscr{D}) = \bigvee_{i=1}^{l}\{\bigwedge\Delta_i|\Delta_i \subseteq \Delta\}$ is the reduced disjunctive form of $f(U, \Delta, \mathscr{D})$ with the multiplication and absorption laws.

By Definition 2.8, we get the related function $f(U, \Delta, \mathscr{D}) = \bigwedge\{\bigvee r(x)|r(x) \in R(U, \Delta, \mathscr{D})\}$ and its reduced disjunctive form $g(U, \Delta, \mathscr{D}) = \bigvee_{i=1}^{l}\{\bigwedge\Delta_i|\Delta_i \subseteq \Delta\}$. Especially, we obtain attribute reducts $\mathscr{R}(U, \Delta, \mathscr{D}) = \{\Delta_1, \Delta_1, ..., \Delta_l\}$ for the covering decision information system $(U, \Delta, \mathscr{D})$ with the reduced disjunctive form $g(U, \Delta, \mathscr{D})$.

Algorithm 2.9. (Non-Incremental Algorithm of Computing $\mathscr{R}(U, \Delta, \mathscr{D})$ of $(U, \Delta, \mathscr{D})$)

Step 1: Input $(U, \Delta, \mathscr{D})$;

Step 2: Construct $POS_{\cup\Delta}(\mathscr{D}) = \bigcup\{POS_{\cup\Delta}(D_i)\,|D_i \in \mathscr{D}\}$;

Step 3: Compute $R(U, \Delta, \mathscr{D}) = \{r(x)|x \in POS_{\cup\Delta}(\mathscr{D})\}$;

Step 4: Construct $f(U, \Delta, \mathscr{D}) = \bigwedge\{\bigvee r(x)|r(x) \in R(U, \Delta, \mathscr{D})\} = \bigvee_{i=1}^{l}\{\bigwedge\Delta_i|\Delta_i \subseteq \Delta\}$;

Step 5: Output $\mathscr{R}(U, \Delta, \mathscr{D})$.

We employ two examples to illustrate how to construct attribute reducts of consistent and inconsistent covering decision information systems as follows.

Example 2.10. (1) Let $(U, \Delta, \mathscr{D})$ be a consistent covering decision information system, where $U = \{x_1, x_2, ..., x_8\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5\}$, and $\mathscr{D} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8\}\}$, where

$$\mathscr{C}_1 = \{\{x_1, x_2\}, \{x_2, x_3, x_4\}, \{x_3\}, \{x_4\}, \{x_5, x_6\}, \{x_6, x_7, x_8\}\};$$
$$\mathscr{C}_2 = \{\{x_1, x_3, x_4\}, \{x_2, x_3\}, \{x_4, x_5\}, \{x_5, x_6\}, \{x_6\}, \{x_7, x_8\}\};$$
$$\mathscr{C}_3 = \{\{x_1\}, \{x_1, x_2, x_3\}, \{x_2, x_3\}, \{x_3, x_4, x_5, x_6\}, \{x_5, x_7, x_8\}\};$$
$$\mathscr{C}_4 = \{\{x_1, x_2, x_4\}, \{x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_6\}, \{x_7, x_8\}\};$$
$$\mathscr{C}_5 = \{\{x_1, x_2, x_3\}, \{x_4\}, \{x_5, x_6\}, \{x_5, x_6, x_8\}, \{x_4, x_7, x_8\}\}.$$

By Definition 2.8, firstly, we have $r(x_1) = \{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_5\}$, $r(x_2) = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5\}$, $r(x_3) = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5\}$, $r(x_4) = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_5\}$, $r(x_5) = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_5\}$, $r(x_6) = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_5\}$, $r(x_7) = \{\mathscr{C}_2, \mathscr{C}_4\}$ and $r(x_8) = \{\mathscr{C}_2, \mathscr{C}_4\}$. Secondly, we get $R(U, \Delta, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_5\}, \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5\}, \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_5\}, \{\mathscr{C}_2, \mathscr{C}_4\}\}$ and

$$\begin{aligned} f(U, \Delta, \mathscr{D}) &= \bigwedge\{\bigvee r(x)|r(x) \in R(U, \Delta, \mathscr{D})\} \\ &= (\mathscr{C}_1 \vee \mathscr{C}_3 \vee \mathscr{C}_5) \wedge (\mathscr{C}_1 \vee \mathscr{C}_2 \vee \mathscr{C}_3 \vee \mathscr{C}_4 \vee \mathscr{C}_5) \\ &\quad \wedge (\mathscr{C}_1 \vee \mathscr{C}_2 \vee \mathscr{C}_4 \vee \mathscr{C}_5) \wedge (\mathscr{C}_2 \vee \mathscr{C}_4) \\ &= (\mathscr{C}_1 \vee \mathscr{C}_3 \vee \mathscr{C}_5) \wedge (\mathscr{C}_2 \vee \mathscr{C}_4) \\ &= (\mathscr{C}_1 \wedge \mathscr{C}_2) \vee (\mathscr{C}_1 \wedge \mathscr{C}_4) \vee (\mathscr{C}_2 \wedge \mathscr{C}_3) \vee (\mathscr{C}_3 \wedge \mathscr{C}_4) \\ &\quad \vee (\mathscr{C}_2 \wedge \mathscr{C}_5) \vee (\mathscr{C}_4 \wedge \mathscr{C}_5). \end{aligned}$$

Therefore, we have $\mathscr{R}(U, \Delta, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_2\}, \{\mathscr{C}_1, \mathscr{C}_4\}, \{\mathscr{C}_2, \mathscr{C}_3\}, \{\mathscr{C}_3, \mathscr{C}_4\}, \{\mathscr{C}_2, \mathscr{C}_5\}, \{\mathscr{C}_4, \mathscr{C}_5\}\}$.

(2) Let $(U, \Delta, \mathscr{D})$ be an inconsistent covering decision information system, where $U = \{x_1, x_2, ..., x_8\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\}$, and $\mathscr{D} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8\}\}$, where

$$\begin{aligned}
\mathscr{C}_1 &= \{\{x_1, x_2, x_3, x_4\}, \{x_3, x_6, x_7\}, \{x_4, x_5\}, \{x_6\}, \{x_7, x_8\}\};\\
\mathscr{C}_2 &= \{\{x_1\}, \{x_2, x_3, x_4\}, \{x_4, x_5\}, \{x_4, x_5, x_6\}, \{x_6, x_7, x_8\}\};\\
\mathscr{C}_3 &= \{\{x_1\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4, x_8\}, \{x_3, x_4, x_5, x_6, x_7\}\};\\
\mathscr{C}_4 &= \{\{x_1, x_4, x_5\}, \{x_2, x_3, x_4, x_5\}, \{x_4, x_5, x_6, x_7, x_8\}\}.
\end{aligned}$$

By Definition 2.8, firstly, we get $r(x_1) = \{\mathscr{C}_2, \mathscr{C}_3\}$, $r(x_2) = \emptyset$, $r(x_3) = \emptyset$, $r(x_4) = \{\mathscr{C}_1, \mathscr{C}_2\}$, $r(x_5) = \{\mathscr{C}_1, \mathscr{C}_2\}$, $r(x_6) = \{\mathscr{C}_1, \mathscr{C}_2\}$, $r(x_7) = \{\mathscr{C}_1\}$ and $r(x_8) = \{\mathscr{C}_1\}$. After that, we obtain $R(U, \Delta, \mathscr{D}) = \{\{\mathscr{C}_2, \mathscr{C}_3\}, \{\mathscr{C}_1, \mathscr{C}_2\}, \{\mathscr{C}_1\}\}$ and

$$\begin{aligned}
f(U, \Delta, \mathscr{D}) &= \bigwedge\{\bigvee r(x) | r(x) \in R(U, \Delta, \mathscr{D})\}\\
&= (\mathscr{C}_2 \vee \mathscr{C}_3) \wedge (\mathscr{C}_1 \vee \mathscr{C}_2) \wedge \mathscr{C}_1\\
&= (\mathscr{C}_2 \vee \mathscr{C}_3) \wedge \mathscr{C}_1\\
&= (\mathscr{C}_1 \wedge \mathscr{C}_2) \vee (\mathscr{C}_1 \wedge \mathscr{C}_3).
\end{aligned}$$

Therefore, we get $\mathscr{R}(U, \Delta, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_2\}, \{\mathscr{C}_1, \mathscr{C}_3\}\}$.

3. Related family-based attribute reduction of dynamic covering decision information systems

In this section, we study the related family-based attribute reduction of dynamic covering decision information systems with variations of attribute sets.

Definition 3.1. Let $(U, \Delta, \mathscr{D})$ and $(U, \Delta^+, \mathscr{D})$ be covering decision information systems, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, and $\Delta^+ = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m, \mathscr{C}_{m+1}\}$. Then $(U, \Delta^+, \mathscr{D})$ is called a dynamic covering decision information system of $(U, \Delta, \mathscr{D})$.

According to Definition 3.1, if $(U, \Delta, \mathscr{D})$ is a consistent covering decision information system, then we see that $(U, \Delta^+, \mathscr{D})$ is consistent when adding $\mathscr{C}_{m+1}$ into $(U, \Delta, \mathscr{D})$. Furthermore, if $(U, \Delta, \mathscr{D})$ is an inconsistent covering decision information system, we notice that $(U, \Delta^+, \mathscr{D})$ is consistent or inconsistent when adding $\mathscr{C}_{m+1}$ into $(U, \Delta, \mathscr{D})$. In practical situations, there are many dynamic covering information systems, and we only discuss dynamic covering decision information systems with variations of attribute sets in this section.

Example 3.2. (Continuation from Example 2.10) (1) Let $(U, \Delta, \mathscr{D})$ and $(U, \Delta^+, \mathscr{D})$ be covering decision information systems, where $U = \{x_1, x_2, ..., x_8\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5\}$, $\Delta^+ = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5, \mathscr{C}_6\}$, and $\mathscr{D} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8\}\}$, where $\mathscr{C}_6 = \{\{x_1, x_4, x_5\}, \{x_2\}, \{x_3, x_4, x_6\}, \{x_3, x_5, x_7\}, \{x_7, x_8\}\}$. According to Definitions 2.6 and 3.1, we see that $(U, \Delta^+, \mathscr{D})$ is a dynamic covering decision information system of $(U, \Delta, \mathscr{D})$. Especially, we observe that $(U, \Delta, \mathscr{D})$ and $(U, \Delta^+, \mathscr{D})$ are consistent covering decision information systems.

(2) Let $(U, \Delta, \mathscr{D})$ and $(U, \Delta^+, \mathscr{D})$ be covering decision information systems, where $U = \{x_1, x_2, ..., x_8\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\}$, $\Delta^+ = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5\}$, and $\mathscr{D} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8\}\}$, where $\mathscr{C}_5 = \{\{x_1, x_5, x_6\}, \{x_4, x_5\}, \{x_2, x_3, x_4\}, \{x_5, x_6, x_7, x_8\}\}$. By Definitions 2.6 and 3.1, we observe that $(U, \Delta^+, \mathscr{D})$ is a dynamic covering decision information system of $(U, \Delta, \mathscr{D})$. Specially, we see that $(U, \Delta, \mathscr{D})$ and $(U, \Delta^+, \mathscr{D})$ are inconsistent covering decision information systems.

Suppose $(U, \Delta^+, \mathscr{D})$ and $(U, \Delta, \mathscr{D})$ are covering decision information systems, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, and $\Delta^+ = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m, \mathscr{C}_{m+1}\}$, $\mathscr{A}_\Delta = \{C \in \cup\Delta | \exists D_j \in \mathscr{D}, \text{ s.t. } C \subseteq D_j\}$, $\mathscr{A}_{\Delta^+} = \{C \in \cup\Delta^+ | \exists D_j \in \mathscr{D}, \text{ s.t. } C \subseteq D_j\}$, $\mathscr{A}_{\mathscr{C}_{m+1}} = \{C \in \mathscr{C}_{m+1} | \exists D_j \in \mathscr{D}, \text{ s.t. } C \subseteq D_j\}$, $r(x) = \{\mathscr{C} \in \Delta | \exists C \in \mathscr{A}_\Delta, \text{ s.t. } x \in C \in \mathscr{C}\}$, and $r^+(x) = \{\mathscr{C} \in \Delta^+ | \exists C \in \mathscr{A}_{\Delta^+}, \text{ s.t. } x \in C \in \mathscr{C}\}$.

Theorem 3.3. *Let $(U, \Delta, \mathscr{D})$ and $(U, \Delta^+, \mathscr{D})$ be covering decision information systems, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, and $\Delta^+ = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m, \mathscr{C}_{m+1}\}$. Then we have*

$$r^+(x) = \begin{cases} r(x) \cup \{\mathscr{C}_{m+1}\}, & \text{if } x \in \cup\mathscr{A}_{\mathscr{C}_{m+1}};\\ r(x), & \text{otherwise}. \end{cases}$$

Proof. By Definition 2.8, we have $r(x) = \{\mathscr{C} \in \Delta | \exists C \in \mathscr{A}_\Delta, \text{ s.t. } x \in C \in \mathscr{C}\}$, and $r^+(x) = \{\mathscr{C} \in \Delta^+ | \exists C \in \mathscr{A}_{\Delta^+}, \text{ s.t. } x \in C \in \mathscr{C}\}$. Since $\Delta^+ = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m, \mathscr{C}_{m+1}\}$, it follows that $r^+(x) = \{\mathscr{C} \in \Delta | \exists C \in \mathscr{A}_\Delta, \text{ s.t. } x \in C \in \mathscr{C}\} \cup \{\mathscr{C}_{m+1} | \exists C \in \mathscr{C}_{m+1}, \text{ s.t. } x \in C \in \mathscr{C}_{m+1}\}$ for $x \in U$. For simplicity, we denote $\cup\mathscr{A}_{\mathscr{C}_{m+1}} = \cup\{C | C \in \mathscr{C}_{m+1}, \exists D_i \in \mathscr{D}, \text{ s.t. } C \subseteq D_i\}$. So we get $r^+(x) = r(x) \cup \{\mathscr{C}_{m+1}\}$ and $r^+(y) = r(y)$ for $x \in \cup\mathscr{A}_{\mathscr{C}_{m+1}}$ and $y \notin \cup\mathscr{A}_{\mathscr{C}_{m+1}}$, respectively. Therefore, we obtain

$$r^+(x) = \begin{cases} r(x) \cup \{\mathscr{C}_{m+1}\}, & \text{if } x \in \cup\mathscr{A}_{\mathscr{C}_{m+1}};\\ r(x), & \text{otherwise}. \end{cases}$$

□

Theorem 3.3 illustrates the relationship between $r(x)$ of $(U, \Delta, \mathscr{D})$ and $r^+(x)$ of $(U, \Delta^+, \mathscr{D})$. Concretely, we construct the related set $r^+(x)$ on the basis of the related set $r(x)$, which reduces time complexity of computing the related family $R(U, \Delta^+, \mathscr{D})$. Especially, we only need to compute $\mathscr{A}_{\mathscr{C}_{m+1}}$ for attribute reduction of dynamic covering decision information system $(U, \Delta^+, \mathscr{D})$, and we get $r^+(x) = r(x)$ and $r^+(x) = r(x) \cup \{\mathscr{C}_{m+1}\}$ when $\cup\mathscr{A}_{\mathscr{C}_{m+1}} = \emptyset$ and $\cup\mathscr{A}_{\mathscr{C}_{m+1}} = U$, respectively, for $x \in U$.

Theorem 3.4. *Let $(U, \Delta, \mathscr{D})$ and $(U, \Delta^+, \mathscr{D})$ be covering decision information systems, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, $\Delta^+ = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m, \mathscr{C}_{m+1}\}$, $\blacktriangle f(U, \Delta, \mathscr{D}) = (\bigwedge_{x \in POS_{\cup\Delta}(\mathscr{D}) \wedge x \notin \cup\mathscr{A}_{\mathscr{C}_{m+1}}} \bigvee r(x)) \bigwedge \mathscr{C}_{m+1} = \bigvee_{i=1}^{l}\{\bigwedge \Delta'_i | \Delta'_i \subseteq \Delta\}$, and $\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D}) = \{\Delta'_j | \not\exists \Delta_i \in \mathscr{R}(U, \Delta, \mathscr{D}), \text{ s.t. } \Delta_i \subset \Delta'_j, 1 \leq j \leq l\}$. If $POS_{\cup\Delta^+}(\mathscr{D}) = POS_{\cup\Delta}(\mathscr{D})$, then $\mathscr{R}(U, \Delta^+, \mathscr{D}) = \mathscr{R}(U, \Delta, \mathscr{D}) \cup (\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D}))$.*

Proof. Taking $\Delta_i \in \mathscr{R}(U, \Delta, \mathscr{D})$, by Definition 2.5, we have $POS_{\cup\Delta}(\mathscr{D}) = POS_{\cup\Delta_i}(\mathscr{D})$ and $POS_{\cup\Delta_i}(\mathscr{D}) \neq POS_{\cup\Delta_i - \{\mathscr{C}_i\}}(\mathscr{D})$ for $\mathscr{C}_i \in \Delta_i$. We also get $POS_{\cup\Delta^+}(\mathscr{D}) = POS_{\cup\Delta_i}(\mathscr{D})$ and $POS_{\cup\Delta_i}(\mathscr{D}) \neq POS_{\cup\Delta_i - \{\mathscr{C}_i\}}(\mathscr{D})$ for $\mathscr{C}_i \in \Delta_i$. So $\Delta_i \in \mathscr{R}(U, \Delta^+, \mathscr{D})$. Thus, we obtain $\mathscr{R}(U, \Delta, \mathscr{D}) \subseteq \mathscr{R}(U, \Delta^+, \mathscr{D})$. Furthermore, taking $\Delta'_j \in \blacktriangle\mathscr{R}(U, \Delta, \mathscr{D})$, it implies that $POS_{\cup\Delta}(\mathscr{D}) = POS_{\cup\Delta'_j}(\mathscr{D})$ and $POS_{\cup\Delta'_j}(\mathscr{D}) \neq POS_{\cup\Delta'_j - \{\mathscr{C}_i\}}(\mathscr{D})$ for $\mathscr{C}_i \in \Delta'_j$. It follows that $\Delta'_j \in \mathscr{R}(U, \Delta^+, \mathscr{D})$. So we get $\mathscr{R}(U, \Delta, \mathscr{D}) \cup (\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D})) \subseteq \mathscr{R}(U, \Delta^+, \mathscr{D})$. Furthermore, we have $\mathscr{R}(U, \Delta^+, \mathscr{D}) = \mathscr{R}_1(U, \Delta^+, \mathscr{D}) \cup \mathscr{R}_2(U, \Delta^+, \mathscr{D})$, where $\mathscr{R}_1(U, \Delta^+, \mathscr{D}) = \{\Delta_i | \mathscr{C}_{m+1} \notin \Delta_i, \Delta_i \in \mathscr{R}(U, \Delta^+, \mathscr{D})\}$ and $\mathscr{R}_2(U, \Delta^+, \mathscr{D}) = \{\Delta_i | \mathscr{C}_{m+1} \in \Delta_i, \Delta_i \in \mathscr{R}(U, \Delta^+, \mathscr{D})\}$. Obviously, we obtain $\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D}) \subseteq \mathscr{R}_2(U, \Delta^+, \mathscr{D})$. To prove $\mathscr{R}_2(U, \Delta^+, \mathscr{D}) \subseteq \blacktriangle\mathscr{R}(U, \Delta, \mathscr{D})$, we only need to prove $\mathscr{R}_2(U, \Delta^+, \mathscr{D}) \setminus (\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D})) = \emptyset$. Suppose we have $\Delta' = \{\mathscr{C}_{1'}, \mathscr{C}_{2'}, ..., \mathscr{C}_{l'}, \mathscr{C}_{m+1}\} \in \mathscr{R}_2(U, \Delta^+, \mathscr{D}) \setminus \blacktriangle\mathscr{R}(U, \Delta, \mathscr{D})$, there exists $x \in \mathscr{R}(U, \Delta^+, \mathscr{D})$ such that $\mathscr{C}_{i'} \in r^+(x) (1' \leq i' \leq l')$. If $\mathscr{C}_{m+1} \in r^+(x)$, then $\mathscr{C}_{i'}$ is superfluous relative to $\mathscr{D}$. It implies that $\mathscr{C}_{m+1} \notin r^+(x)$. It follows that $\Delta' \in \blacktriangle\mathscr{R}(U, \Delta, \mathscr{D})$, which is contradicted. So $\mathscr{R}_2(U, \Delta^+, \mathscr{D}) \setminus (\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D})) = \emptyset$. Thus $\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D}) = \mathscr{R}_2(U, \Delta^+, \mathscr{D})$. Since $\mathscr{R}_1(U, \Delta^+, \mathscr{D}) \subseteq \mathscr{R}(U, \Delta, \mathscr{D})$, so we have $\mathscr{R}(U, \Delta^+, \mathscr{D}) \subseteq \mathscr{R}(U, \Delta, \mathscr{D}) \cup (\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D}))$. Therefore, $\mathscr{R}(U, \Delta^+, \mathscr{D}) = \mathscr{R}(U, \Delta, \mathscr{D}) \cup (\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D}))$. □

Theorem 3.4 illustrates the relationship between $\mathscr{R}(U, \Delta^+, \mathscr{D})$ of $(U, \Delta^+, \mathscr{D})$ and $\mathscr{R}(U, \Delta, \mathscr{D})$ of $(U, \Delta, \mathscr{D})$. Concretely, we construct the attribute reducts $\mathscr{R}(U, \Delta^+, \mathscr{D})$ on the basis of $\mathscr{R}(U, \Delta, \mathscr{D})$, which reduces time complexities of computing attribute reducts of $(U, \Delta^+, \mathscr{D})$. Especially, we only need to construct $\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D})$ for attribute reduction of $(U, \Delta^+, \mathscr{D})$. Furthermore, each reduct of $\mathscr{R}(U, \Delta, \mathscr{D})$ belongs to $\mathscr{R}(U, \Delta^+, \mathscr{D})$, so there is no need of computation if we want to get only a reduct of $(U, \Delta^+, \mathscr{D})$, and we get reducts containing $\mathscr{C}_{m+1}$ for $(U, \Delta^+, \mathscr{D})$ by Theorem 3.4.

Theorem 3.5. *Let* $(U, \Delta, \mathscr{D})$ *and* $(U, \Delta^+, \mathscr{D})$ *be covering decision information systems, where* $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, *and* $\Delta^+ = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m, \mathscr{C}_{m+1}\}$. *If* $POS_{\cup\Delta^+}(\mathscr{D}) \neq POS_{\cup\Delta}(\mathscr{D})$, *then we have* $r^+(x) = \{\mathscr{C}_{m+1}\}$ *for* $x \in POS_{\cup\Delta^+}(\mathscr{D}) \setminus POS_{\cup\Delta}(\mathscr{D})$.

Theorem 3.5 illustrates some properties of the related family $R(U, \Delta^+, \mathscr{D})$ when $POS_{\cup\Delta^+}(\mathscr{D}) \neq POS_{\cup\Delta}(\mathscr{D})$. Concretely, we get the related set $r^+(x) = \{\mathscr{C}_{m+1}\}$ for $x \in POS_{\cup\Delta^+}(\mathscr{D}) \setminus POS_{\cup\Delta}(\mathscr{D})$. In other words, $\mathscr{C}_{m+1}$ belongs to each reduct of $\mathscr{R}(U, \Delta^+, \mathscr{D})$ when $POS_{\cup\Delta^+}(\mathscr{D}) \neq POS_{\cup\Delta}(\mathscr{D})$.

Theorem 3.6. *Let* $(U, \Delta^+, \mathscr{D})$ *and* $(U, \Delta, \mathscr{D})$ *be covering decision information systems, where* $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, $\Delta^+ = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m, \mathscr{C}_{m+1}\}$, $\blacktriangle f(U, \Delta, \mathscr{D}) = \mathscr{C}_{m+1} \bigwedge (\bigwedge_{x \in POS_{\cup\Delta}(\mathscr{D}) \wedge x \notin \cup \mathscr{A}_{\mathscr{C}_{m+1}}} \bigvee r(x)) = \bigvee_{i=1}^{l} \{\bigwedge \Delta_i' | \Delta_i' \subseteq \Delta\}$, *and* $\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D}) = \{\Delta_j' | 1 \leq j \leq l\}$. *If* $POS_{\cup\Delta^+}(\mathscr{D}) \neq POS_{\cup\Delta}(\mathscr{D})$, *then* $\mathscr{R}(U, \Delta^+, \mathscr{D}) = \blacktriangle\mathscr{R}(U, \Delta, \mathscr{D})$.

Proof. It is similar to the proof of Theorem 3.4. □

Theorem 3.6 illustrates the relationship between $\mathscr{R}(U, \Delta^+, \mathscr{D})$ and $\mathscr{R}(U, \Delta, \mathscr{D})$ when $POS_{\cup\Delta^+}(\mathscr{D}) \neq POS_{\cup\Delta}(\mathscr{D})$. Especially, there is no direct relationship between $\mathscr{R}(U, \Delta, \mathscr{D})$ and $\mathscr{R}(U, \Delta^+, \mathscr{D})$, and we should compute $\mathscr{R}(U, \Delta^+, \mathscr{D})$ with the related family $R(U, \Delta^+, \mathscr{D})$.

We provide an incremental algorithm for computing attribute reducts of dynamic covering decision information systems as follows.

Algorithm 3.7. (Incremental Algorithm of Computing $\mathscr{R}(U, \Delta^+, \mathscr{D})$ of $(U, \Delta^+, \mathscr{D})$)

Step 1: Input $(U, \Delta^+, \mathscr{D})$;

Step 2: Construct $POS_{\cup\Delta^+}(\mathscr{D})$;

Step 3: Compute $R(U, \Delta^+, \mathscr{D}) = \{r^+(x) | x \in POS_{\cup\Delta^+}(\mathscr{D})\}$, where

$$r^+(x) = \begin{cases} r(x) \cup \{\mathscr{C}_{m+1}\}, & x \in \cup\mathscr{A}_{\mathscr{C}_{m+1}}; \\ r(x), & otherwise. \end{cases}$$

Step 4: Construct $\blacktriangle f(U, \Delta, \mathscr{D}) = \mathscr{C}_{m+1} \bigwedge \left(\bigwedge_{x \in POS_{\cup\Delta}(\mathscr{D}) \wedge x \notin \cup\mathscr{A}_{\mathscr{C}_{m+1}}} \bigvee r(x)\right) = \bigvee_{i=1}^{l} \{\bigwedge \Delta_i' | \Delta_i' \subseteq \Delta\}$;

Step 5: Compute $\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D}) = \{\Delta_j' | \exists \Delta_i \in \mathscr{R}(U, \Delta, \mathscr{D})$, s.t. $\Delta_i \subset \Delta_j', 1 \leq j \leq k'\}$;

Step 6: Output $\mathscr{R}(U, \Delta^+, \mathscr{D})$.

We employ an example to illustrate how to construct attribute reducts of dynamic covering decision information systems by Algorithm 3.7 as follows.

Example 3.8. (Continuation from Examples 2.10) (1) By Definition 2.8, firstly, we have $r^+(x_1) = r(x_1)$, $r^+(x_2) = r(x_2) \cup \{\mathscr{C}_6\}$, $r^+(x_3) = r(x_3)$, $r^+(x_4) = r(x_4)$, $r^+(x_5) = r(x_5)$, $r^+(x_6) = r(x_6)$, $r^+(x_7) = r(x_7) \cup \{\mathscr{C}_6\}$ and $r^+(x_8) = r(x_8) \cup \{\mathscr{C}_6\}$. Consequently, we get $R(U, \Delta^+, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_5\}, \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5\}, \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5, \mathscr{C}_6\}, \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_5\}, \{\mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_6\}\}$, and

$$\begin{aligned} f(U, \Delta^+, \mathscr{D}) &= \bigwedge\{\bigvee r^+(x) | r^+(x) \in R(U, \Delta^+, \mathscr{D})\} \\ &= (\mathscr{C}_1 \vee \mathscr{C}_3 \vee \mathscr{C}_5) \wedge (\mathscr{C}_1 \vee \mathscr{C}_2 \vee \mathscr{C}_3 \vee \mathscr{C}_4 \vee \mathscr{C}_5 \vee \mathscr{C}_6) \\ &\quad \wedge (\mathscr{C}_1 \vee \mathscr{C}_2 \vee \mathscr{C}_3 \vee \mathscr{C}_4 \vee \mathscr{C}_5) \wedge (\mathscr{C}_1 \\ &\quad \vee \mathscr{C}_2 \vee \mathscr{C}_4 \vee \mathscr{C}_5) \wedge (\mathscr{C}_2 \vee \mathscr{C}_4 \vee \mathscr{C}_6) \\ &= (\mathscr{C}_1 \vee \mathscr{C}_3 \vee \mathscr{C}_5) \wedge (\mathscr{C}_1 \vee \mathscr{C}_2 \vee \mathscr{C}_4 \vee \mathscr{C}_5) \wedge (\mathscr{C}_2 \vee \mathscr{C}_4 \vee \mathscr{C}_6) \\ &= (\mathscr{C}_1 \wedge \mathscr{C}_2) \vee (\mathscr{C}_1 \wedge \mathscr{C}_4) \vee (\mathscr{C}_1 \wedge \mathscr{C}_6) \vee (\mathscr{C}_2 \wedge \mathscr{C}_3) \\ &\quad \vee (\mathscr{C}_2 \wedge \mathscr{C}_5) \vee (\mathscr{C}_3 \wedge \mathscr{C}_4) \vee (\mathscr{C}_4 \wedge \mathscr{C}_5) \vee (\mathscr{C}_5 \wedge \mathscr{C}_6). \end{aligned}$$

So we have $\mathscr{R}(U, \Delta^+, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_2\}, \{\mathscr{C}_1, \mathscr{C}_4\}, \{\mathscr{C}_1, \mathscr{C}_6\}, \{\mathscr{C}_2, \mathscr{C}_3\}, \{\mathscr{C}_2, \mathscr{C}_5\}, \{\mathscr{C}_3, \mathscr{C}_4\}, \{\mathscr{C}_4, \mathscr{C}_5\}, \{\mathscr{C}_5, \mathscr{C}_6\}\}$.

Secondly, by Theorem 3.4, we get

$$\begin{aligned} \blacktriangle f(U, \Delta^+, \mathscr{D}) &= \mathscr{C}_6 \wedge (\mathscr{C}_1 \vee \mathscr{C}_3 \vee \mathscr{C}_5) \wedge (\mathscr{C}_1 \vee \mathscr{C}_2 \vee \mathscr{C}_3 \vee \mathscr{C}_4 \vee \mathscr{C}_5) \\ &\quad \wedge (\mathscr{C}_1 \vee \mathscr{C}_2 \vee \mathscr{C}_4 \vee \mathscr{C}_5) \\ &= \mathscr{C}_6 \wedge (\mathscr{C}_1 \vee \mathscr{C}_3 \vee \mathscr{C}_5) \wedge (\mathscr{C}_1 \vee \mathscr{C}_2 \vee \mathscr{C}_4 \vee \mathscr{C}_5) \\ &= (\mathscr{C}_1 \wedge \mathscr{C}_6) \vee (\mathscr{C}_5 \wedge \mathscr{C}_6) \vee (\mathscr{C}_2 \wedge \mathscr{C}_3 \wedge \mathscr{C}_6) \\ &\quad \vee (\mathscr{C}_3 \wedge \mathscr{C}_4 \wedge \mathscr{C}_6). \end{aligned}$$

It follows that $\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_6\}, \{\mathscr{C}_5, \mathscr{C}_6\}\}$. Therefore, we have $\mathscr{R}(U, \Delta^+, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_2\}, \{\mathscr{C}_1, \mathscr{C}_4\}, \{\mathscr{C}_2, \mathscr{C}_3\}, \{\mathscr{C}_2, \mathscr{C}_5\}, \{\mathscr{C}_3, \mathscr{C}_4\}, \{\mathscr{C}_4, \mathscr{C}_5\}, \{\mathscr{C}_1, \mathscr{C}_6\}, \{\mathscr{C}_5, \mathscr{C}_6\}\}$.

(2) By Definition 2.8, we have that $r^+(x_1) = \{\mathscr{C}_2, \mathscr{C}_3\}$, $r^+(x_2) = \emptyset$, $r^+(x_3) = \emptyset$, $r^+(x_4) = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_5\}$, $r^+(x_5) = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_5\}$, $r^+(x_6) = \{\mathscr{C}_1, \mathscr{C}_2\}$, $r^+(x_7) = \{\mathscr{C}_1\}$ and $r^+(x_8) = \{\mathscr{C}_1\}$. Subsequently, we have

$$\begin{aligned} f(U, \Delta^+, \mathscr{D}) &= \bigwedge\{\bigvee r^+(x) | r^+(x) \in R(U, \Delta^+, \mathscr{D})\} \\ &= (\mathscr{C}_1 \vee \mathscr{C}_2) \wedge (\mathscr{C}_2 \vee \mathscr{C}_3) \wedge (\mathscr{C}_1 \vee \mathscr{C}_2 \vee \mathscr{C}_5) \wedge \mathscr{C}_1 \\ &= (\mathscr{C}_2 \vee \mathscr{C}_3) \wedge \mathscr{C}_1 \\ &= (\mathscr{C}_1 \wedge \mathscr{C}_2) \vee (\mathscr{C}_1 \wedge \mathscr{C}_3). \end{aligned}$$

So we obtain $\mathscr{R}(U, \Delta^+, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_2\}, \{\mathscr{C}_1, \mathscr{C}_3\}\}$.

Secondly, by Theorem 3.6, we have

$$\begin{aligned} \blacktriangle f(U, \Delta, \mathscr{D}) &= \mathscr{C}_5 \wedge (\bigvee r^+(x_1)) \wedge (\bigvee r^+(x_6)) \wedge (\bigvee r^+(x_7)) \\ &\quad \wedge (\bigvee r^+(x_8)) \\ &= \mathscr{C}_5 \wedge (\mathscr{C}_2 \vee \mathscr{C}_3) \wedge (\mathscr{C}_1 \vee \mathscr{C}_2) \wedge \mathscr{C}_1 \\ &= (\mathscr{C}_1 \wedge \mathscr{C}_2 \wedge \mathscr{C}_5) \vee (\mathscr{C}_1 \wedge \mathscr{C}_3 \wedge \mathscr{C}_5). \end{aligned}$$

It follows that $\blacktriangle\mathscr{R}(U, \Delta, \mathscr{D}) = \emptyset$. Therefore, we get $\mathscr{R}(U, \Delta^+, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_2\}, \{\mathscr{C}_1, \mathscr{C}_3\}\}$.

Example 3.8 illustrates how to compute attribute reducts of dynamic covering decision information systems by Algorithms 2.9 and 3.7. We see that the incremental algorithm is more effective than the non-incremental algorithm for attribute reduction of consistent and inconsistent dynamic covering decision information systems.

In practical situations, there are a lot of dynamic covering decision information systems caused by deleting attributes, and we study attribute reduction of covering decision information systems when deleting attributes as follows.

Definition 3.9. Let $(U, \Delta, \mathscr{D})$ and $(U, \Delta^-, \mathscr{D})$ be covering decision information systems, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, and $\Delta^- = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_{m-1}\}$. Then $(U, \Delta^-, \mathscr{D})$ is called a dynamic covering decision information system of $(U, \Delta, \mathscr{D})$.

By Definition 3.9, we see that $(U, \Delta^-, \mathscr{D})$ is consistent or inconsistent when deleting $\mathscr{C}_m$ from $(U, \Delta, \mathscr{D})$ since $(U, \Delta, \mathscr{D})$ is a consistent covering decision information system. Furthermore, we notice that $(U, \Delta^-, \mathscr{D})$ is inconsistent when deleting $\mathscr{C}_m$ from $(U, \Delta, \mathscr{D})$ since $(U, \Delta, \mathscr{D})$ is an inconsistent covering decision information system. In the following, we study two types of dynamic covering decision information systems when deleting attributes as follows: (1)$(U, \Delta^-, \mathscr{D})$ is a consistent covering decision information system; (2)$(U, \Delta^-, \mathscr{D})$ is an inconsistent covering decision information system.

Example 3.10. (Continuation from Examples 2.10) (1) Let $(U, \Delta, \mathscr{D})$ and $(U, \Delta^-, \mathscr{D})$ be covering decision information systems, where $U = \{x_1, x_2, ..., x_8\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5\}$, $\Delta^- = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\}$, and $\mathscr{D} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8\}\}$, where

$$\begin{aligned} \mathscr{C}_1 &= \{\{x_1, x_2\}, \{x_2, x_3, x_4\}, \{x_3\}, \{x_4\}, \{x_5, x_6\}, \{x_6, x_7, x_8\}\}; \\ \mathscr{C}_2 &= \{\{x_1, x_3, x_4\}, \{x_2, x_3\}, \{x_4, x_5\}, \{x_5, x_6\}, \{x_6\}, \{x_7, x_8\}\}; \\ \mathscr{C}_3 &= \{\{x_1\}, \{x_1, x_2, x_3\}, \{x_2, x_3\}, \{x_3, x_4, x_5, x_6\}, \{x_5, x_7, x_8\}\}; \\ \mathscr{C}_4 &= \{\{x_1, x_2, x_4\}, \{x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_6\}, \{x_7, x_8\}\}; \\ \mathscr{C}_5 &= \{\{x_1, x_2, x_3\}, \{x_4\}, \{x_5, x_6\}, \{x_5, x_6, x_8\}, \{x_4, x_7, x_8\}\}. \end{aligned}$$

By Definition 3.9, we see that $(U, \Delta^-, \mathscr{D})$ is a dynamic covering decision information system of $(U, \Delta, \mathscr{D})$. Especially, we observe that $(U, \Delta, \mathscr{D})$ and $(U, \Delta^-, \mathscr{D})$ are consistent covering decision information systems.

(2) Let $(U, \Delta, \mathscr{D})$ and $(U, \Delta^{-}, \mathscr{D})$ be covering decision information systems, where $U = \{x_1, x_2, ..., x_8\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\}$, $\Delta^{-} = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3\}$, and $\mathscr{D} = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8\}\}$. By Definition 3.9, we see that $(U, \Delta^{-}, \mathscr{D})$ is a dynamic covering decision information system of $(U, \Delta, \mathscr{D})$. Specially, we observe that $(U, \Delta, \mathscr{D})$ and $(U, \Delta^{-}, \mathscr{D})$ are inconsistent covering decision information systems.

Suppose $(U, \Delta, \mathscr{D})$ and $(U, \Delta^{-}, \mathscr{D})$ are covering decision information systems, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, and $\Delta^{-} = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_{m-1}\}$, $\mathscr{A}_{\Delta} = \{C \in \cup\Delta \mid \exists D_j \in \mathscr{D}, \text{ s.t. } C \subseteq D_j\}$, $r(x) = \{\mathscr{C} \in \Delta \mid \exists C \in \mathscr{A}_{\Delta}, \text{ s.t. } x \in C \in \mathscr{C}\}$, and $r^{-}(x) = \{\mathscr{C} \in \Delta^{-} \mid \exists C \in \mathscr{A}_{\Delta^{-}}, \text{ s.t. } x \in C \in \mathscr{C}\}$.

Theorem 3.11. *Let $(U, \Delta, \mathscr{D})$ and $(U, \Delta^{-}, \mathscr{D})$ be covering decision information systems, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, and $\Delta^{-} = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_{m-1}\}$. Then we have*

$$r^{-}(x) = \begin{cases} r(x) \setminus \{\mathscr{C}_m\}, & \text{if } x \in \cup\mathscr{A}_{\mathscr{C}_m}; \\ r(x), & \text{otherwise.} \end{cases}$$

Proof. By Definitions 2.8, for any $x \in U$, we have $r(x) = \{\mathscr{C} \in \Delta \mid \exists C \in \mathscr{A}_{\Delta}, \text{ s.t. } x \in C \in \mathscr{C}\}$, and $r^{-}(x) = \{\mathscr{C} \in \Delta^{-} \mid \exists C \in \mathscr{A}_{\Delta^{-}}, \text{ s.t. } x \in C \in \mathscr{C}\}$. If $\mathscr{C}_m \in r(x)$, then we have $r^{-}(x) = r(x) \setminus \{\mathscr{C}_m\}$. After that, if $\mathscr{C}_m \notin r(x)$, then we have $r^{-}(x) = r(x)$. Therefore, we have $r^{-}(x) = r(x) \setminus \{\mathscr{C}_m\}$. □

Theorem 3.12 illustrates the relationship between $r(x)$ of $(U, \Delta, \mathscr{D})$ and $r^{-}(x)$ of $(U, \Delta^{-}, \mathscr{D})$. Concretely, we construct the related set $r^{-}(x)$ on the basis of the related set $r(x)$, which reduces time complexity of computing the related families for attribute reduction of dynamic covering decision information systems.

Theorem 3.12. *Let $(U, \Delta, \mathscr{D})$ and $(U, \Delta^{-}, \mathscr{D})$ be covering decision information systems, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, and $\Delta^{-} = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_{m-1}\}$. If $POS_{\cup\Delta^{-}}(\mathscr{D}) = POS_{\cup\Delta}(\mathscr{D})$, then we have $\mathscr{R}(U, \Delta^{-}, \mathscr{D}) = \{\Delta_i \mid \mathscr{C}_m \notin \Delta_i \in \mathscr{R}(U, \Delta, \mathscr{D})\}$.*

Proof. The proof is straightforward by Definition 2.6. □

Theorem 3.12 illustrates the relationship between $\mathscr{R}(U, \Delta^{-}, \mathscr{D})$ of $(U, \Delta, \mathscr{D})$ and $\mathscr{R}(U, \Delta, \mathscr{D})$ of $(U, \Delta^{-}, \mathscr{D})$, and we have $\mathscr{R}(U, \Delta^{-}, \mathscr{D}) = \{\Delta_i \mid \mathscr{C}_m \notin \Delta_i \in \mathscr{R}(U, \Delta, \mathscr{D})\}$ when $POS_{\cup\Delta^{-}}(\mathscr{D}) = POS_{\cup\Delta}(\mathscr{D})$. Furthermore, each element of $\mathscr{R}(U, \Delta, \mathscr{D})$ which does not contain $\mathscr{C}_m$ belongs to $\mathscr{R}(U, \Delta^{-}, \mathscr{D})$, so there is no need of computation if we want to get only a reduct of $(U, \Delta^{-}, \mathscr{D})$, which reduces time complexities of computing attribute reducts of dynamic covering decision information systems.

Theorem 3.13. *Let $(U, \Delta, \mathscr{D})$ and $(U, \Delta^{-}, \mathscr{D})$ be covering decision information systems, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, and $\Delta^{-} = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_{m-1}\}$. If $POS_{\cup\Delta^{-}}(\mathscr{D}) \neq POS_{\cup\Delta}(\mathscr{D})$, then we have $r(x) = \{\mathscr{C}_m\}$ for $x \in POS_{\cup\Delta}(\mathscr{D}) \setminus POS_{\cup\Delta^{-}}(\mathscr{D})$. Especially, we get $\mathscr{C}_m \in \Delta_i$ for any $\Delta_i \in \mathscr{R}(U, \Delta, \mathscr{D})$.*

Proof. The proof is straightforward by Definition 2.8. □

Theorem 3.13 illustrates some properties of the related family $R(U, \Delta, \mathscr{D})$ when $POS_{\cup\Delta^{-}}(\mathscr{D}) \neq POS_{\cup\Delta}(\mathscr{D})$. Especially, there is no direct relationship between $\mathscr{R}(U, \Delta, \mathscr{D})$ and $\mathscr{R}(U, \Delta^{-}, \mathscr{D})$, and we should compute $\mathscr{R}(U, \Delta^{-}, \mathscr{D})$ by Definition 2.8 after constructing the related family $R(U, \Delta^{-}, \mathscr{D})$, which reduces time complexities of computing attribute reducts of dynamic covering decision information systems.

Theorem 3.14. *Let $(U, \Delta, \mathscr{D})$ and $(U, \Delta^{-}, \mathscr{D})$ be covering decision information systems, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, and $\Delta^{-} = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_{m-1}\}$, and $f(U, \Delta^{-}, \mathscr{D}) = \bigwedge\{\bigvee r^{-}(x) \mid r^{-}(x) \in R(U, \Delta^{-}, \mathscr{D})\} = \bigvee_{i=1}^{l} \{\bigwedge \Delta_i \mid \Delta_i \subseteq \Delta^{-}\}$. If $POS_{\cup\Delta^{-}}(\mathscr{D}) \neq POS_{\cup\Delta}(\mathscr{D})$, then we have $\mathscr{R}(U, \Delta^{-}, \mathscr{D}) = \{\Delta_1, \Delta_1, ..., \Delta_l\}$ for $(U, \Delta^{-}, \mathscr{D})$.*

Proof. The proof is straightforward by Definition 2.8. □

We provide an incremental algorithm for computing attribute reducts of dynamic covering decision information systems as follows.

Algorithm 3.15. Incremental Algorithm of Computing $\mathscr{R}(U, \Delta^{-}, \mathscr{D})$ of $(U, \Delta^{-}, \mathscr{D})$)

Step 1: Input $(U, \Delta^{-}, \mathscr{D})$;

Step 2: Construct $POS_{\cup\Delta^{-}}(\mathscr{D})$;

Step 3: Compute $R(U, \Delta^{-}, \mathscr{D}) = \{r^{-}(x) \mid x \in POS_{\cup\Delta^{-}}(\mathscr{D})\}$, where

$$r^{-}(x) = \begin{cases} r(x) \setminus \{\mathscr{C}_m\}, & \text{if } x \in \cup\mathscr{A}_{\mathscr{C}_m}; \\ r(x), & \text{otherwise.} \end{cases}$$

Step 4: Compute $\mathscr{R}(U, \Delta^{-}, \mathscr{D})$ by Theorems 3.12 and 3.14;

Step 5: Output $\mathscr{R}(U, \Delta^{-}, \mathscr{D})$.

We employ an example to illustrate how to construct attribute reducts of dynamic covering decision information systems by Algorithm 3.15 as follows.

Example 3.16. (Continuation from Example 3.8) (1) By Definition 2.8, firstly, we have $r^{-}(x_1) = \{\mathscr{C}_1, \mathscr{C}_3\}$, $r^{-}(x_2) = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\}$, $r^{-}(x_3) = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\}$, $r^{-}(x_4) = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_4\}$, $r^{-}(x_5)$ $\mathscr{C}_2, \mathscr{C}_4\}$, r^{-} $= \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_4\}$, $r^{-}(x_6)$=$\{\mathscr{C}_1,$ $(x_7) = \{\mathscr{C}_2, \mathscr{C}_4\}$ and $r^{-}(x_8) = \{\mathscr{C}_2, \mathscr{C}_4\}$. Secondly, we derive

$$\begin{aligned} f(U, \Delta^{-}, \mathscr{D}) &= \bigwedge\{\bigvee r^{-}(x) \mid r^{-}(x) \in R(U, \Delta, \mathscr{D})\} \\ &= (\mathscr{C}_1 \vee \mathscr{C}_3) \wedge (\mathscr{C}_2 \vee \mathscr{C}_4) \\ &= (\mathscr{C}_1 \wedge \mathscr{C}_2) \vee (\mathscr{C}_1 \wedge \mathscr{C}_4) \vee (\mathscr{C}_2 \wedge \mathscr{C}_3) \vee (\mathscr{C}_3 \wedge \mathscr{C}_4), \end{aligned}$$

and $\mathscr{R}(U, \Delta^{-}, \mathscr{D})$=$\{\{\mathscr{C}_1, \mathscr{C}_2\}, \{\mathscr{C}_1, \mathscr{C}_4\}, \{\mathscr{C}_2, \mathscr{C}_3\}, \{\mathscr{C}_3, \mathscr{C}_4\}\}$. Thirdly, by Theorem 3.12, we get $\mathscr{R}(U, \Delta^{-}, \mathscr{D})$=$\{\{\mathscr{C}_1, \mathscr{C}_2\}, \{\mathscr{C}_1, \mathscr{C}_4\}, \{\mathscr{C}_2, \mathscr{C}_3\}, \{\mathscr{C}_3, \mathscr{C}_4\}\}$.

(2) By Definition 2.8, firstly, we have $r^{-}(x_1) = \{\mathscr{C}_2, \mathscr{C}_3\}$, $r^{-}(x_2) = \emptyset$, $r^{-}(x_3) = \emptyset$, $r^{-}(x_4) = \{\mathscr{C}_1, \mathscr{C}_2\}$, $r^{-}(x_5) = \{\mathscr{C}_1, \mathscr{C}_2\}$, $r^{-}(x_6)$ and r^{-} $= \{\mathscr{C}_1, \mathscr{C}_2\}$, $r^{-}(x_7) = \{\mathscr{C}_1\}$ $(x_8) = \{\mathscr{C}_1\}$. Secondly, we obtain

$$\begin{aligned} f(U, \Delta^{-}, \mathscr{D}) &= \bigwedge\{\bigvee r^{-}(x) \mid r^{-}(x) \in R(U, \Delta, \mathscr{D})\} \\ &= (\mathscr{C}_2 \vee \mathscr{C}_3) \wedge (\mathscr{C}_1 \vee \mathscr{C}_2) \wedge \mathscr{C}_1 \\ &= (\mathscr{C}_2 \vee \mathscr{C}_3) \wedge \mathscr{C}_1 \\ &= (\mathscr{C}_1 \wedge \mathscr{C}_2) \vee (\mathscr{C}_1 \wedge \mathscr{C}_3), \end{aligned}$$

and $\mathscr{R}(U, \Delta^{-}, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_2\}, \{\mathscr{C}_1, \mathscr{C}_3\}\}$. Thirdly, by Theorem 3.14, we have $\mathscr{R}(U, \Delta^{-}, \mathscr{D})$=$\{\{\mathscr{C}_1, \mathscr{C}_2\}, \{\mathscr{C}_1, \mathscr{C}_3\}\}$.

Example 3.16 illustrates how to compute attribute reducts of dynamic covering decision information systems when deleting attributes by Algorithms 2.9 and 3.14. We see that the incremental algorithm is more effective than the non-incremental algorithm for attribute reduction of consistent and inconsistent dynamic covering decision information systems.

4. Heuristic algorithms for attribute reduction of dynamic covering decision information systems

In this section, we present heuristic algorithms for computing attribute reducts of dynamic covering decision information systems with variations of attribute sets.

Suppose $(U, \Delta, \mathscr{D})$ is a covering decision information system, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, $\mathscr{D} = \{D_1, D_2, ..., D_k\}$, $R(U, \Delta, \mathscr{D}) = \{r(x) \mid x \in POS_{\cup\Delta}(\mathscr{D})\}$,

$$SR(U, \Delta, \mathscr{D}) = \{r(x) \in R(U, \Delta, \mathscr{D}) \mid x \in POS_{\cup\Delta}(\mathscr{D}) \wedge (\forall y \in POS_{\cup\Delta}(\mathscr{D}), r(y) \nsubseteq r(x) \wedge r(y) \in R(U, \Delta, \mathscr{D}))\},$$

and $\|\mathscr{C}\|$ denotes the number of times for a covering $\mathscr{C}$ appeared in $SR(U, \Delta, \mathscr{D})$.

Algorithm 4.1. (Heuristic Algorithm of Computing a Reduct of $(U, \Delta, \mathscr{D})$)(NIHA).

Step 1: Input $(U, \Delta, \mathscr{D})$;

Step 2: Construct $POS_{\cup\Delta}(\mathscr{D}) = \bigcup\{POS_{\cup\Delta}(D_i) \mid D_i \in \mathscr{D}\}$;

Step 3: Compute $R(U, \Delta, \mathscr{D}) = \{r(x)|x \in POS_{\cup\Delta}(\mathscr{D})\}$;

Step 4: Construct a reduct $\triangle^* = \{\mathscr{C}_{i_1}, \mathscr{C}_{i_2}, ..., \mathscr{C}_{i_j}\}$, where $SR_1(U, \Delta, \mathscr{D}) = SR(U, \Delta, \mathscr{D})$, $\|\mathscr{C}_{i_1}\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r(x) \in SR_1(U, \Delta, \mathscr{D})\}$; $SR_2(U, \Delta, \mathscr{D}) = \{r(x) \in SR(U, \Delta, \mathscr{D})|\mathscr{C}_{i_1} \notin r(x)\}$, $\|\mathscr{C}_{i_2}\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r(x) \in SR_2(U, \Delta, \mathscr{D})\}$; $SR_3(U, \Delta, \mathscr{D}) = \{r(x) \in SR(U, \Delta, \mathscr{D})|\mathscr{C}_{i_1} \notin r(x) \vee \mathscr{C}_{i_2} \notin r(x)\}$, $\|\mathscr{C}_{i_3}\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r(x) \in SR_3(U, \Delta, \mathscr{D})\}$; ... ; $SR_j(U, \Delta, \mathscr{D}) = \{r(x) \in SR(U, \Delta, \mathscr{D})|\mathscr{C}_{i_1} \notin r(x) \vee \mathscr{C}_{i_2} \notin r(x) \vee ... \vee \mathscr{C}_{i_{j-1}} \notin r(x)\}$, $\|\mathscr{C}_{i_j}\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r(x) \in SR_j(U, \Delta, \mathscr{D})\}$, and $SR(U, \Delta, \mathscr{D}) = \{r(x)|\mathscr{C}_{i_1} \in r(x) \vee \mathscr{C}_{i_2} \in r(x) \vee ... \vee \mathscr{C}_{i_j} \in r(x)\}$;

Step 5: Output the reduct $\triangle^*$.

By Algorithm 4.1, we observe that constructing all attribute reducts of covering decision information systems by Algorithm 2.9 is NP hard problem, and it is enough to compute a reduct for covering decision information systems by Algorithm 4.1. Furthermore, if there exist two coverings $\mathscr{C}_i$ and $\mathscr{C}_j$ such that $\|\mathscr{C}_i\| = \|\mathscr{C}_j\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r(x) \in SR_k(U, \Delta, \mathscr{D})\}$, then we select $\|\mathscr{C}_i\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r(x) \in SR_k(U, \Delta, \mathscr{D})\}$ or $\|\mathscr{C}_j\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r(x) \in SR_k(U, \Delta, \mathscr{D})\}$.

Example 4.2. (Continuation from Example 2.10) (1) In Example 2.10(1), we derive $SR(U, \Delta, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_5\}, \{\mathscr{C}_2, \mathscr{C}_4\}\}$. By Algorithm 4.1, firstly, we obtain $SR_1(U, \Delta, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_5\}, \{\mathscr{C}_2, \mathscr{C}_4\}\}$ and $\|\mathscr{C}_1\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r(x) \in SR_1(U, \Delta, \mathscr{D})\}$. Secondly, we get $SR_2(U, \Delta, \mathscr{D}) = \{\{\mathscr{C}_2, \mathscr{C}_4\}\}$, and $\|\mathscr{C}_2\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r(x) \in SR_2(U, \Delta, \mathscr{D})\}$. Finally, we have a reduct $\triangle^* = \{\mathscr{C}_1, \mathscr{C}_2\}$ of $(U, \Delta, \mathscr{D})$.

(2) In Example 2.10(2), we derive $SR(U, \Delta, \mathscr{D}) = \{\{\mathscr{C}_2, \mathscr{C}_3\}, \{\mathscr{C}_1\}\}$. By Algorithm 4.1, firstly, we obtain $SR_1(U, \Delta, \mathscr{D}) = \{\{\mathscr{C}_2, \mathscr{C}_3\}, \{\mathscr{C}_1\}\}$ and $\|\mathscr{C}_1\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r(x) \in SR_1(U, \Delta, \mathscr{D})\}$. Secondly, we get $SR_2(U, \Delta, \mathscr{D}) = \{\{\mathscr{C}_2, \mathscr{C}_3\}\}$, and $\|\mathscr{C}_2\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r(x) \in SR_2(U, \Delta, \mathscr{D})\}$. Finally, we have a reduct $\triangle^* = \{\mathscr{C}_1, \mathscr{C}_2\}$ of $(U, \Delta, \mathscr{D})$.

Suppose $(U, \Delta^+, \mathscr{D})$ and $(U, \Delta, \mathscr{D})$ are covering decision information systems, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, and $\Delta^+ = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m, \mathscr{C}_{m+1}\}$, $\mathscr{D} = \{D_1, D_2, ..., D_k\}$, $R(U, \Delta^+, \mathscr{D}) = \{r^+(x)|x \in POS_{\cup\Delta^+}(\mathscr{D})\}$,

$$SR(U, \Delta^+, \mathscr{D}) = \{r^+(x) \in R(U, \Delta^+, \mathscr{D})|x \in POS_{\cup\Delta^+}(\mathscr{D}) \wedge (\forall y \in POS_{\cup\Delta^+}(\mathscr{D}), r^+(y) \nsubseteq r^+(x) \wedge r^+(y) \in R(U, \Delta^+, \mathscr{D}))\},$$

and $\|\mathscr{C}\|$ denotes the number of times for a covering $\mathscr{C}$ appeared in $SR(U, \Delta^+, \mathscr{D})$.

Algorithm 4.3. (Heuristic Algorithm of Computing a Reduct of $(U, \Delta^+, \mathscr{D})$)(IHAA)

Step 1: Input $(U, \Delta^+, \mathscr{D})$;

Step 2: Construct $POS_{\cup\Delta^+}(\mathscr{D})$;

Step 3: Compute $R(U, \Delta^+, \mathscr{D}) = \{r^+(x)|x \in POS_{\cup\Delta^+}(\mathscr{D})\}$, where

$$r^+(x) = \begin{cases} r(x) \cup \{\mathscr{C}_{m+1}\}, & \text{if } x \in \cup\mathscr{A}_{\mathscr{C}_{m+1}}; \\ r(x), & otherwise. \end{cases}$$

Step 4: Construct a reduct $\triangle^{*+} = \{\mathscr{C}_{i_1}, \mathscr{C}_{i_2}, ..., \mathscr{C}_{i_j}\}$, where $SR_1(U, \Delta^+, \mathscr{D}) = SR(U, \Delta^+, \mathscr{D})$, $\|\mathscr{C}_{i_1}\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^+(x) \in SR_1(U, \Delta^+, \mathscr{D})\}$; $SR_2(U, \Delta^+, \mathscr{D}) = \{r^+(x) \in SR(U, \Delta^+, \mathscr{D})|\mathscr{C}_{i_1} \notin r^+(x)\}$, $\|\mathscr{C}_{i_2}\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^+(x) \in SR_2(U, \Delta^+, \mathscr{D})\}$; $SR_3(U, \Delta^+, \mathscr{D}) = \{r^+(x) \in SR(U, \Delta^+, \mathscr{D})|\mathscr{C}_{i_1} \notin r^+(x) \vee \mathscr{C}_{i_2} \notin r^+(x)\}$, $\|\mathscr{C}_{i_3}\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^+(x) \in SR_3(U, \Delta^+, \mathscr{D})\}$; ... ; $SR_j(U, \Delta^+, \mathscr{D}) = \{r^+(x) \in SR(U, \Delta^+, \mathscr{D})|\mathscr{C}_{i_1} \notin r^+(x) \vee \mathscr{C}_{i_2} \notin r^+(x) \vee ... \vee \mathscr{C}_{i_{j-1}} \notin r^+(x)\}$, $\|\mathscr{C}_{i_j}\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^+(x) \in SR_j(U, \Delta^+, \mathscr{D})\}$, and $SR(U, \Delta^+, \mathscr{D}) = \{r^+(x)|\mathscr{C}_{i_1} \in r^+(x) \vee \mathscr{C}_{i_2} \in r^+(x) \vee ... \vee \mathscr{C}_{i_j} \in r^+(x)\}$;

Step 5: Output the reduct $\triangle^{*+}$.

In Algorithm 4.3, there are two situations for constructing reducts of dynamic covering decision information systems with immigration of attributes as follows: (1) if we have $POS_{\cup\Delta^+}(\mathscr{D}) = POS_{\cup\Delta}(\mathscr{D})$, then a reduct of $(U, \Delta, \mathscr{D})$ belongs to $\mathscr{R}(U, \Delta^+, \mathscr{D})$; (2) if we have $POS_{\cup\Delta^+}(\mathscr{D}) \neq POS_{\cup\Delta}(\mathscr{D})$, then we compute a reduct by Algorithm 4.3. Furthermore, if there are two coverings $\mathscr{C}_i$ and $\mathscr{C}_j$ such that $\|\mathscr{C}_i\| = \|\mathscr{C}_j\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^+(x) \in SR_k(U, \Delta^+, \mathscr{D})\}$, then we select $\|\mathscr{C}_i\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^+(x) \in SR_k(U, \Delta^+, \mathscr{D})\}$ or $\|\mathscr{C}_j\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^+(x) \in SR_k(U, \Delta^+, \mathscr{D})\}$.

Example 4.4. (Continuation from Example 3.8) (1) In Example 3.8 (1), we have $SR(U, \Delta^+, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_5\}, \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_5\}, \{\mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_6\}\}$. By Algorithm 4.3, firstly, we obtain $SR_1(U, \Delta^+, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_3, \mathscr{C}_5\}, \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_5\}, \{\mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_6\}\}$ and $\|\mathscr{C}_1\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^+(x) \in SR_1(U, \Delta^+, \mathscr{D})\}$. Secondly, we obtain $SR_2(U, \Delta^+, \mathscr{D}) = \{\{\mathscr{C}_2, \mathscr{C}_4, \mathscr{C}_6\}\}$, $\|\mathscr{C}_2\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^+(x) \in SR_2(U, \Delta^+, \mathscr{D})\}$. Finally, we get a reduct $\triangle^{*+} = \{\mathscr{C}_1, \mathscr{C}_2\}$ of $(U, \Delta^+, \mathscr{D})$.

(2) In Example 3.8 (2), we get $SR(U, \Delta^+, \mathscr{D}) = \{\{\mathscr{C}_1\}, \{\mathscr{C}_2, \mathscr{C}_3\}\}$. By Algorithm 4.3, firstly, we obtain $SR_1(U, \Delta^+, \mathscr{D}) = \{\{\mathscr{C}_1\}, \{\mathscr{C}_2, \mathscr{C}_3\}\}$ and $\|\mathscr{C}_1\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^+(x) \in SR_1(U, \Delta^+, \mathscr{D})\}$. Secondly, we get $SR_2(U, \Delta^+, \mathscr{D}) = \{\{\mathscr{C}_2, \mathscr{C}_3\}\}$, $\|\mathscr{C}_2\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^+(x) \in SR_2(U, \Delta^+, \mathscr{D})\}$. Finally, we have a reduct $\triangle^{*+} = \{\mathscr{C}_1, \mathscr{C}_2\}$ of $(U, \Delta^+, \mathscr{D})$.

Suppose $(U, \Delta^-, \mathscr{D})$ and $(U, \Delta, \mathscr{D})$ are covering decision information systems, where $U = \{x_1, x_2, ..., x_n\}$, $\Delta = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m\}$, and $\Delta^- = \{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_{m-1}\}$, $\mathscr{D} = \{D_1, D_2, ..., D_k\}$, $R(U, \Delta^-, \mathscr{D}) = \{r^-(x)|x \in POS_{\cup\Delta^-}(\mathscr{D})\}$,

$$SR(U, \Delta^-, \mathscr{D}) = \{r^-(x) \in R(U, \Delta^-, \mathscr{D})|x \in POS_{\cup\Delta^-}(\mathscr{D}) \wedge (\forall y \in POS_{\cup\Delta^+}(\mathscr{D}), r^-(y) \nsubseteq r^-(x) \wedge r^-(y) \in R(U, \Delta^-, \mathscr{D}))\},$$

and $\|\mathscr{C}\|$ denotes the number of times for a covering $\mathscr{C}$ appeared in $SR(U, \Delta^-, \mathscr{D})$.

Algorithm 4.5. (Heuristic Algorithm of Computing a Reduct of $(U, \Delta^-, \mathscr{D})$)(IHAD)

Step 1: Input $(U, \Delta^-, \mathscr{D})$;

Step 2: Construct $POS_{\cup\Delta^-}(\mathscr{D})$;

Step 3: Compute $R(U, \Delta^-, \mathscr{D}) = \{r^-(x)|x \in POS_{\cup\Delta^-}(\mathscr{D})\}$, where

$$r^-(x) = \begin{cases} r(x) \setminus \{\mathscr{C}_m\}, & \text{if } x \in \cup\mathscr{A}_{\mathscr{C}_m}; \\ r(x), & otherwise. \end{cases}$$

Step 4: Construct a reduct $\triangle^{*-} = \{\mathscr{C}_{i_1}, \mathscr{C}_{i_2}, ..., \mathscr{C}_{i_j}\}$, where $SR_1(U, \Delta^-, \mathscr{D}) = SR(U, \Delta^-, \mathscr{D})$, $\|\mathscr{C}_{i_1}\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^-(x) \in R_1(U, \Delta^-, \mathscr{D})\}$; $SR_2(U, \Delta^-, \mathscr{D}) = \{r^-(x) \in SR(U, \Delta^-, \mathscr{D})|\mathscr{C}_{i_1} \notin r^-(x)\}$, $\|\mathscr{C}_{i_2}\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^-(x) \in SR_2(U, \Delta^-, \mathscr{D})\}$; $SR_3(U, \Delta^-, \mathscr{D}) = \{r^-(x) \in SR(U, \Delta^-, \mathscr{D})|\mathscr{C}_{i_1} \notin r^-(x) \vee \mathscr{C}_{i_2} \notin r^-(x)\}$, $\|\mathscr{C}_{i_3}\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^-(x) \in SR_3(U, \Delta^-, \mathscr{D})\}$; ... ; $SR_j(U, \Delta^-, \mathscr{D}) = \{r^-(x) \in SR(U, \Delta^-, \mathscr{D})|\mathscr{C}_{i_1} \notin r^-(x) \vee \mathscr{C}_{i_2} \notin r^-(x) \vee ... \vee \mathscr{C}_{i_{j-1}} \notin r^-(x)\}$, $\|\mathscr{C}_{i_j}\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^-(x) \in SR_j(U, \Delta^-, \mathscr{D})\}$, and $SR(U, \Delta^-, \mathscr{D}) = \{r^-(x)|\mathscr{C}_{i_1} \in r^-(x) \vee \mathscr{C}_{i_2} \in r^-(x) \vee ... \vee \mathscr{C}_{i_j} \in r^-(x)\}$;

Step 5: Output the reduct $\triangle^{*-}$.

In Algorithm 4.5, we see that there are two situations for constructing reducts of dynamic covering decision information systems with emigration of attributes as follows: (1) if we have $POS_{\cup\Delta^-}(\mathscr{D}) = POS_{\cup\Delta}(\mathscr{D})$, then a reduct of $(U, \Delta, \mathscr{D})$ belongs to $\mathscr{R}(U, \Delta^-, \mathscr{D})$; (2) if we have $POS_{\cup\Delta^-}(\mathscr{D}) \neq POS_{\cup\Delta}(\mathscr{D})$, then we compute a reduct by Algorithm 4.5. Furthermore, if there exist two coverings $\mathscr{C}_i$ and $\mathscr{C}_j$ such that $\|\mathscr{C}_i\| = \|\mathscr{C}_j\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^-(x) \in SR_k(U, \Delta^-, \mathscr{D})\}$, then we select $\|\mathscr{C}_i\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^-(x) \in SR_k(U, \Delta^-, \mathscr{D})\}$ or $\|\mathscr{C}_j\| = max\{\|\mathscr{C}_i\||\mathscr{C}_i \in r^-(x) \in SR_k(U, \Delta^-, \mathscr{D})\}$.

Example 4.6. (Continuation from Example 3.16) (1) In Example 3.16(1), we derive $SR(U, \Delta^-, \mathscr{D})=\{\{\mathscr{C}_1, \mathscr{C}_3\}, \{\mathscr{C}_2, \mathscr{C}_4\}\}$. By Algorithm 4.5, firstly, we obtain $SR_1(U, \Delta^-, \mathscr{D}) = \{\{\mathscr{C}_1, \mathscr{C}_3\}, \{\mathscr{C}_2, \mathscr{C}_4\}\}$

Table 1
Data sets for experiments.

No.	Name	Samples	Conditional attributes	Decision attribute
1	Wine	178	13	1
2	Breast Cancer Wisconsin (wdbc)	569	30	1
3	Seismic-Bumps	2584	18	1
4	Abalone	4177	8	1
5	Car Evaluation	1728	6	1
6	Chess (King-Rook vs. King-Pawn)	3196	36	1
7	Optical Recognition of Handwritten Digits	5620	64	1
8	Letter Recognition	20,000	16	1

and $\|\mathscr{C}_1\| = max\{\|\mathscr{C}_i\| | \mathscr{C}_i \in r^-(x) \in SR_1(U, \Delta^-, \mathscr{D})\}$. Secondly, we have $SR_2(U, \Delta^-, \mathscr{D}) = \{\{\mathscr{C}_2, \mathscr{C}_4\}\}$, $\|\mathscr{C}_2\| = max\{\|\mathscr{C}_i\| | \mathscr{C}_i \in r^-(x) \in SR_2(U, \Delta^-, \mathscr{D})\}$. Finally, we get a reduct $\triangle^{*-} = \{\mathscr{C}_1, \mathscr{C}_2\}$ of $(U, \Delta^-, \mathscr{D})$.

(2) In Example 3.16(2), we have $SR(U, \Delta^-, \mathscr{D}) = \{\{\mathscr{C}_1\}, \{\mathscr{C}_2, \mathscr{C}_3\}\}$. By Algorithm 4.5, we firstly obtain $SR_1(U, \Delta^-, \mathscr{D}) = \{\{\mathscr{C}_1\}, \{\mathscr{C}_2, \mathscr{C}_3\}\}$ and $\|\mathscr{C}_1\| = max\{\|\mathscr{C}_i\| | \mathscr{C}_i \in r^-(x) \in SR_1(U, \Delta^-, \mathscr{D})\}$. Secondly, we get $SR_2(U, \Delta^-, \mathscr{D}) = \{\{\mathscr{C}_2, \mathscr{C}_3\}\}$, $\|\mathscr{C}_2\| = max\{\|\mathscr{C}_i\| | \mathscr{C}_i \in r^-(x) \in SR_2(U, \Delta^-, \mathscr{D})\}$. Finally, we obtain a reduct $\triangle^{*-} = \{\mathscr{C}_1, \mathscr{C}_2\}$ of $(U, \Delta^-, \mathscr{D})$.

5. Experimental analysis

In this section, we perform experiments to illustrate the effectiveness of Algorithms 4.1, 4.3 and 4.5 for computing attribute reducts of dynamic covering decision information systems with immigration and emigration of attributes.

To test Algorithms 4.1, 4.3 and 4.5, we converted eight data sets downloaded from UCI and depicted by Table 1 into covering decision information systems. Concretely, we derive a covering and a partition by a conditional attribute and a decision attribute, respectively. For the category attribute, we classify objects with the same attribute value into a block. For the numerical attribute, we classify two objects into a block if the Euclid Distance between them is less than 0.05 after normalization processing. Because the purpose of the experiment is to test the efficiency of Algorithms 4.1, 4.3 and 4.5 for attribute reduction in dynamic covering decision information systems, we do not discuss which is the best way to transform data sets into covering decision information systems. Especially, we see that $\{(U_i, \Delta_i, \mathscr{D}_i)|1 \leq i \leq 4\}$ are consistent covering decision information systems, and $\{(U_i, \Delta_i, \mathscr{D}_i)|5 \leq i \leq 8\}$ are inconsistent covering decision information systems. Moreover, we conducted all computations on a PC with a Intel(R) Dual-Core(TM) i7-7700K CPU @ 4.20 GHZ 4.20 GHZ and 32 GB memory, running 64-bit Windows 10; the software was 64-bit Matlab R2016a.

5.1. Compare effectiveness of computing attribute reducts using Algorithms 4.1 and 4.3

In this section, we construct attribute reducts of dynamic covering decision information systems when adding attributes with Algorithms 4.1 and 4.3, and compare their effectiveness of computing attribute reducts in dynamic covering decision information systems.

Firstly, we compare the times of computing a reduct using Algorithm 4.1 with those using Algorithm 4.3 in dynamic covering decision information systems when adding an attribute. Concretely, we perform each experiment ten times, and show all times of computing attribute reducts of dynamic covering decision information systems in Table 2. For instance, we have the computation times {0.5456, 0.5511, 0.5527, 0.5398, 0.5486, 0.5442, 0.5394, 0.5478, 0.5411, 0.5561} and {0.1510, 0.1491, 0.1476, 0.1486, 0.1480, 0.1520, 0.1477, 0.1476, 0.1494, 0.1828} for constructing reducts by Algorithms 4.1 and 4.3, respectively, in $(U_1, \Delta_1^+, \mathscr{D}_1)$. We see that the times of computing a reduct using Algorithm 4.1 are larger than those using Algorithm 4.3 in $(U_i, \Delta_i^+, \mathscr{D}_i)$. For example, we have the computation times $0.5456 \geq 0.1510$, $0.5511 \geq 0.1491$, $0.5527 \geq 0.1476$, $0.5398 \geq 0.1486$, $0.5486 \geq 0.1480$, $0.5442 \geq 0.1520$, $0.5394 \geq 0.1477$, $0.5478 \geq 0.1476$, $0.5411 \geq 0.1494$ and $0.5561 \geq 0.1828$ when computing a reduct in $(U_1, \Delta_1^+, \mathscr{D}_1)$.

Secondly, we employ Fig. 1 to illustrate the effectiveness of Algorithms 4.1 and 4.3. For example, Fig. 1(i) illustrates the times of computing a reduct with Algorithms 4.1 and 4.3 in $(U_i, \Delta_i^+, \mathscr{D}_i)$. In each figure, *NIHA* and *IHAA* mean Algorithms 4.1 and 4.3, respectively; *j* stands for the *j*th experiment on the dynamic covering decision information system $(U_i, \Delta_i^+, \mathscr{D}_i)$ in *X* Axis, where $j = 1, 2, \ldots, 10$, while the y-coordinate stands for the time to construct a reduct. Therefore, Algorithm 4.3 performs better than Algorithm 4.1 in dynamic covering decision information systems when adding attributes.

Thirdly, we show the average time $\bar{t}$ of ten experimental results in the 13th column of Table 2 and Fig. 2, which illustrates that Algorithm 4.3 executes faster than Algorithm 4.1 in dynamic covering decision information systems with immigrations of attributes. In Fig. 2, *NIHA* and *IHAA* mean Algorithms 4.1 and 4.3, respectively; *i* stands the experiment on the dynamic covering decision information system $(U_i, \Delta_i^+, \mathscr{D}_i)$ in *X* Axis, where $i = 1, 2, \ldots, 8$, while the y-coordinate stands

Table 2
Computational times using NIHA and IHAA in $\{(U_i, \Delta_i^+, \mathscr{D}_i)|1 \leq i \leq 8\}$.

No \ t(s)	Algo.	1	2	3	4	5	6	7	8	9	10	$\bar{t}$	SD
$(U_1, \Delta_1^+, \mathscr{D}_1)$	NIHA	0.5456	0.5511	0.5527	0.5398	0.5486	0.5442	0.5394	0.5478	0.5411	0.5561	0.5466	0.0057
	IHAA	0.1510	0.1491	0.1476	0.1486	0.1480	0.1520	0.1477	0.1476	0.1494	0.1828	0.1524	0.0108
$(U_2, \Delta_2^+, \mathscr{D}_2)$	NIHA	12.2361	12.1000	12.2665	12.1443	12.0938	12.0783	12.1472	12.1665	12.1751	12.1190	12.1527	0.0611
	IHAA	2.3138	2.3140	2.3219	2.3546	2.3145	2.3045	2.3005	2.2972	2.3155	2.3198	2.3156	0.0159
$(U_3, \Delta_3^+, \mathscr{D}_3)$	NIHA	15.7149	15.7251	15.7259	15.8077	15.6864	15.7181	15.7144	15.6278	15.7817	15.6703	15.7172	0.0513
	IHAA	0.4880	0.5055	0.4843	0.4937	0.5091	0.5254	0.4962	0.5098	0.5082	0.4931	0.5013	0.0125
$(U_4, \Delta_4^+, \mathscr{D}_4)$	NIHA	71.1000	71.1486	71.0970	71.1462	71.0236	71.5354	71.0191	70.6481	70.7136	71.0167	71.0448	0.2442
	IHAA	12.1369	12.1072	12.1357	12.1055	12.0998	12.0906	12.1626	12.1099	12.1588	12.0843	12.1191	0.0276
$(U_5, \Delta_5^+, \mathscr{D}_5)$	NIHA	0.5205	0.5167	0.5064	0.5094	0.5061	0.5006	0.5177	0.5030	0.5107	0.5005	0.5092	0.0072
	IHAA	0.1010	0.1051	0.1055	0.1004	0.1013	0.1017	0.1010	0.1004	0.1004	0.1004	0.1017	0.0019
$(U_6, \Delta_6^+, \mathscr{D}_6)$	NIHA	7.1708	7.1779	7.1221	7.1867	7.1616	7.2006	7.1731	7.1350	7.3827	7.1822	7.1893	0.0719
	IHAA	0.2026	0.2000	0.2028	0.2133	0.1942	0.2007	0.1973	0.1975	0.1999	0.1986	0.2007	0.0051
$(U_7, \Delta_7^+, \mathscr{D}_7)$	NIHA	42.1162	42.0302	42.1370	42.2014	42.1242	42.1460	42.2470	42.1718	42.1203	42.0872	42.1381	0.0599
	IHAA	1.1013	1.0864	1.0815	1.1246	1.1178	1.0869	1.0932	1.1178	1.1056	1.0937	1.1009	0.0151
$(U_8, \Delta_8^+, \mathscr{D}_8)$	NIHA	54.0630	54.1615	54.0686	54.1015	54.0985	54.0088	54.1642	53.9911	53.9212	54.1004	54.0679	0.0761
	IHAA	3.8415	3.8329	3.8280	3.8307	3.8462	3.8390	3.8444	3.8290	3.8315	3.8338	3.8357	0.0066

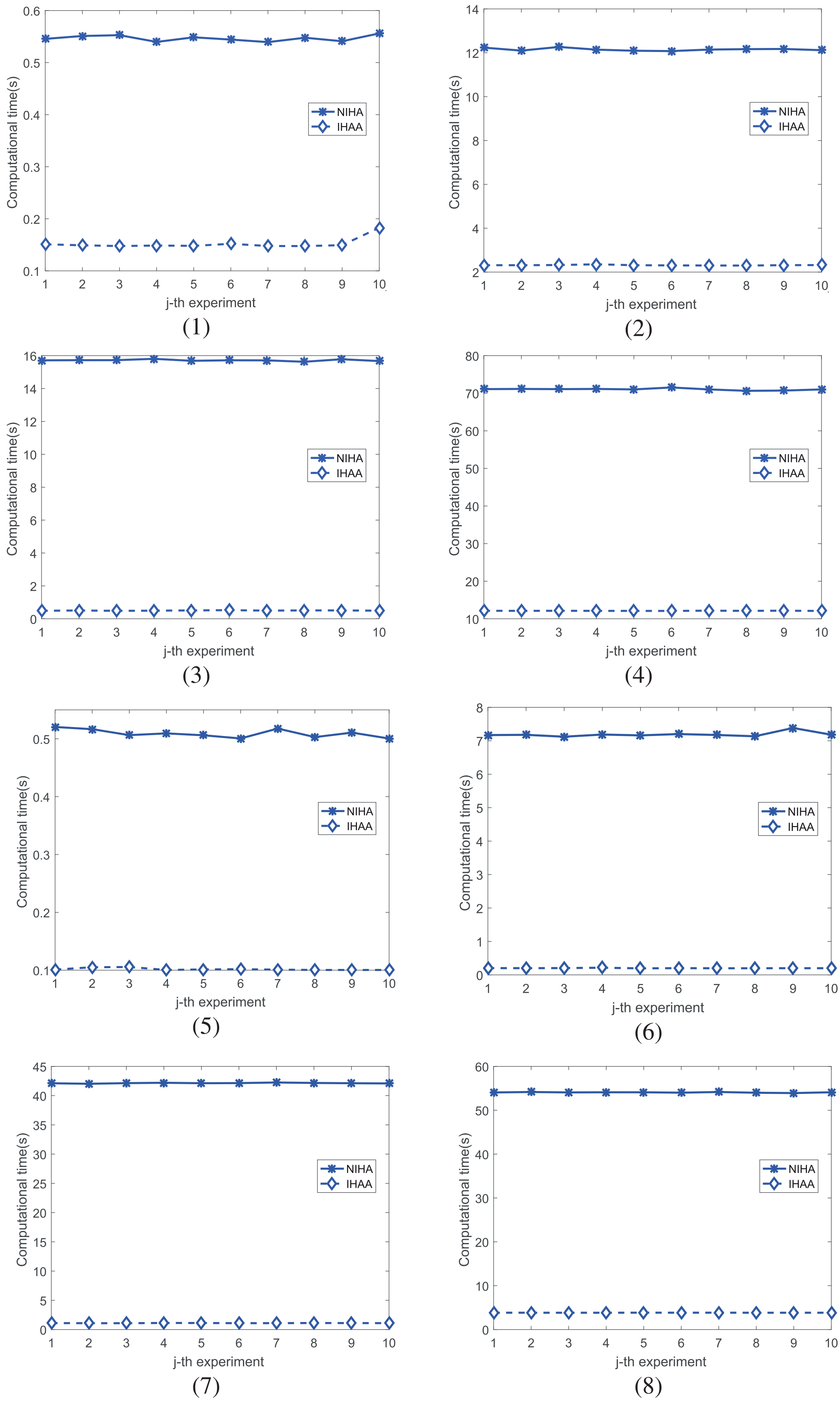

Fig. 1. Computational times using Algorithms 4.1 and 4.3 in $\{(U_i, \Delta_i^+, \mathscr{D}_i) | 1 \leq i \leq 8\}$.

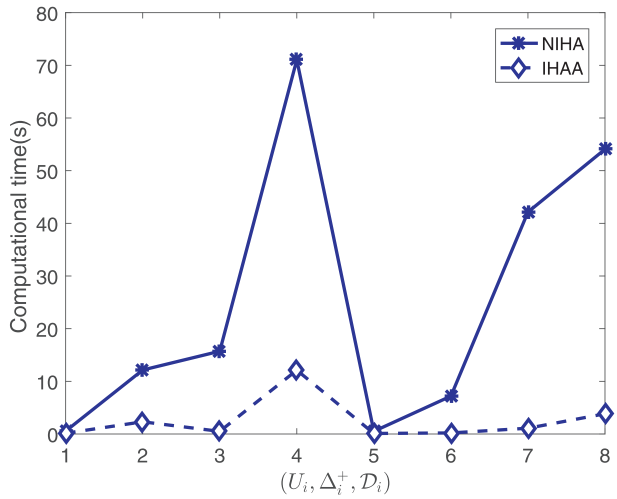

Fig. 2. Average computational times using Algorithms 4.1 and 4.3 in $\{(U_i, \Delta_i^+, \mathscr{D}_i) | 1 \leq i \leq 8\}$.

for the time to construct a reduct. From the last column of Table 2, we get the standard deviations(SD) of ten computational times using Algorithms 4.1 and 4.3 in the dynamic covering decision information system $(U_i, \Delta_i^+, \mathscr{D}_i)(1 \leq i \leq 8)$, which illustrate that Algorithms 4.1 and 4.3 are stable for computing attribute reducts of dynamic covering decision information systems with emigrations of attributes.

Remark. In this experiment, we take eight covering decision information systems induced by data sets of UCI as dynamic covering decision information systems, and get the original covering decision information system by deleting the last covering from a dynamic covering decision information system. For example, we transformed Wine data set into a dynamic covering decision information system $(U_1, \Delta_1^+, \mathscr{D}_1)$, and get the original covering decision information system $(U_1, \Delta_1, \mathscr{D}_1)$ by deleting $\mathscr{C}_{13}$ from $(U_1, \Delta_1^+, \mathscr{D}_1)$, where $|U_1| = 178$, $|\Delta_1^+| = 13$, $|\Delta_1| = 12$ and $|\mathscr{D}_1| = 1$.

5.2. *Compare effectiveness of computing attribute reducts using Algorithms 4.1 and 4.5*

In this section, we construct attribute reducts of dynamic covering decision information systems when deleting attributes with Algorithms 4.1 and 4.5, and compare their effectiveness of computing attribute reducts in dynamic covering decision information systems.

Firstly, we compare the times of computing a reduct using Algorithm 4.1 with those using Algorithm 4.5 in dynamic covering decision information systems when deleting an attribute. Concretely, we perform each experiment ten times and show all times of constructing attribute reducts of dynamic covering decision information systems in Table 3. For instance, we have the computation times {0.7177, 0.5120, 0.5026, 0.4946, 0.4931, 0.4897, 0.5035, 0.4916, 0.4909, 0.4861} and {0.1050, 0.0977, 0.1069, 0.1133, 0.0993, 0.0967, 0.0972, 0.0970, 0.0978, 0.0969} for computing reducts by Algorithms 4.1 and 4.5, respectively, in $(U_1, \Delta_1^-, \mathscr{D}_1)$. We observe that the times of computing a reduct using Algorithm 4.1 are larger than those using Algorithm 4.5 in $(U_i, \Delta_i^-, \mathscr{D}_i)$. For example, we see the computation times $0.1050 \leq 0.7177$, $0.0977 \leq 0.5120$, $0.1069 \leq 0.5026$, $0.1133 \leq 0.4946$, $0.0993 \leq 0.4931$, $0.0967 \leq 0.4897$, $0.0972 \leq 0.5035$, $0.0970 \leq 0.4916$, $0.0978 \leq 0.4909$, and $0.0969 \leq 0.4861$ when computing a reduct in $(U_1, \Delta_1^-, \mathscr{D}_1)$.

Secondly, we employ Fig. 3 to illustrate the effectiveness of Algorithms 4.1 and 4.5. For example, Fig. 3(i) illustrates the times of computing a reduct using Algorithms 4.1 and 4.5 in dynamic covering decision information systems $(U_i, \Delta_i^-, \mathscr{D}_i)$. In each figure, *NIHA* and *IHAD* mean Algorithms 4.1 and 4.5, respectively; j stands for the jth experiment in the dynamic covering decision information system $(U_i, \Delta_i^-, \mathscr{D}_i)$ in X Axis, where $j = 1, 2, ..., 10$, while the y-coordinate stands for the time to construct a reduct. Therefore, Algorithm 4.5 performs better than Algorithm 4.1 for computing reducts of dynamic covering decision information systems when deleting attributes.

Thirdly, we depict the average time $\bar{t}$ of ten experimental results for covering decision information systems in the 13th column of Table 3 and Fig. 4, which illustrates that Algorithm 4.5 executes faster than Algorithm 4.1 in dynamic covering decision information systems with emigrations of attributes. In Fig. 4, *NIHA* and *IHAA* mean Algorithms 4.1 and 4.3, respectively; i stands the experiment on the dynamic covering decision information system $(U_i, \Delta_i^-, \mathscr{D}_i)$ in X Axis, where $i = 1, 2, ..., 8$, while the y-coordinate stands for the time to construct a reduct. From the last column of Table 3, we get the standard deviations(SD) of ten computational times using Algorithms 4.1 and 4.5 in the dynamic covering decision information system $(U_i, \Delta_i^-, \mathscr{D}_i)$, which illustrate that Algorithms 4.1 and 4.5 are stable for computing attribute reducts of dynamic covering decision information systems with emigrations of attributes.

Remark. In this experiment, we take eight covering decision information systems induced by data sets of UCI as the original covering decision information systems and get a dynamic covering decision information system by deleting the last covering from the original covering decision information system. For example, we transformed Wine data set into a covering decision information

Table 3
Computational times using NIHA and IHAD in $\{(U_i, \Delta_i^-, \mathscr{D}_i) | 1 \leq i \leq 8\}$.

No ∖ t(s)	Algo.	1	2	3	4	5	6	7	8	9	10	$\bar{t}$	SD
$(U_1, \Delta_1^-, \mathscr{D}_1)$	NIHA	0.7177	0.5120	0.5026	0.4946	0.4931	0.4897	0.5035	0.4916	0.4909	0.4861	0.5182	0.0705
	IHAD	0.1050	0.0977	0.1069	0.1133	0.0993	0.0967	0.0972	0.0970	0.0978	0.0969	0.1008	0.0057
$(U_2, \Delta_2^-, \mathscr{D}_2)$	NIHA	11.7085	11.7356	11.6772	11.7660	11.6128	11.6953	11.7016	11.7953	11.6521	11.7422	11.7086	0.0540
	IHAD	1.8740	1.8793	1.8742	1.8628	1.8543	1.8580	1.9130	1.9525	1.8774	1.8667	1.8812	0.0299
$(U_3, \Delta_3^-, \mathscr{D}_3)$	NIHA	15.4702	15.3512	15.3561	15.4176	15.5611	15.6066	15.4625	15.3614	15.4634	15.3918	15.4442	0.0870
	IHAD	0.2330	0.2302	0.2298	0.2300	0.2323	0.2324	0.2350	0.2355	0.2366	0.2299	0.2325	0.0025
$(U_4, \Delta_4^-, \mathscr{D}_4)$	NIHA	59.2708	59.4960	59.6643	59.4157	59.0895	59.4361	59.3471	59.7387	59.4707	59.6875	59.4616	0.2003
	IHAD	0.7984	0.7838	0.7756	0.7955	0.7887	0.7749	0.7742	0.7895	0.8173	0.7774	0.7875	0.0136
$(U_5, \Delta_5^-, \mathscr{D}_5)$	NIHA	0.4130	0.4127	0.4153	0.4121	0.4109	0.4156	0.4092	0.4161	0.4122	0.4168	0.4134	0.0025
	IHAD	0.0127	0.0119	0.0121	0.0118	0.0122	0.0119	0.0123	0.0123	0.0125	0.0121	0.0122	0.0003
$(U_6, \Delta_6^-, \mathscr{D}_6)$	NIHA	7.0067	7.0072	6.9822	6.9787	6.9936	7.0719	7.0488	6.9692	6.9778	6.9946	7.0031	0.0331
	IHAD	0.0334	0.0335	0.0343	0.0440	0.0340	0.0334	0.0336	0.0331	0.0334	0.0339	0.0347	0.0033
$(U_7, \Delta_7^-, \mathscr{D}_7)$	NIHA	41.2621	41.2001	41.4330	41.2749	41.2382	41.3336	41.5394	41.5857	41.4748	41.4061	41.3748	0.1331
	IHAD	0.2605	0.2605	0.2662	0.2579	0.2581	0.2743	0.2704	0.2602	0.2596	0.2643	0.2632	0.0055
$(U_8, \Delta_8^-, \mathscr{D}_8)$	NIHA	50.4549	50.7105	50.5666	50.8127	50.6475	50.4567	50.4641	50.6006	50.6763	50.4885	50.5878	0.1237
	IHAD	0.4483	0.4325	0.4369	0.4220	0.4297	0.4242	0.4263	0.4312	0.4254	0.4304	0.4307	0.0076

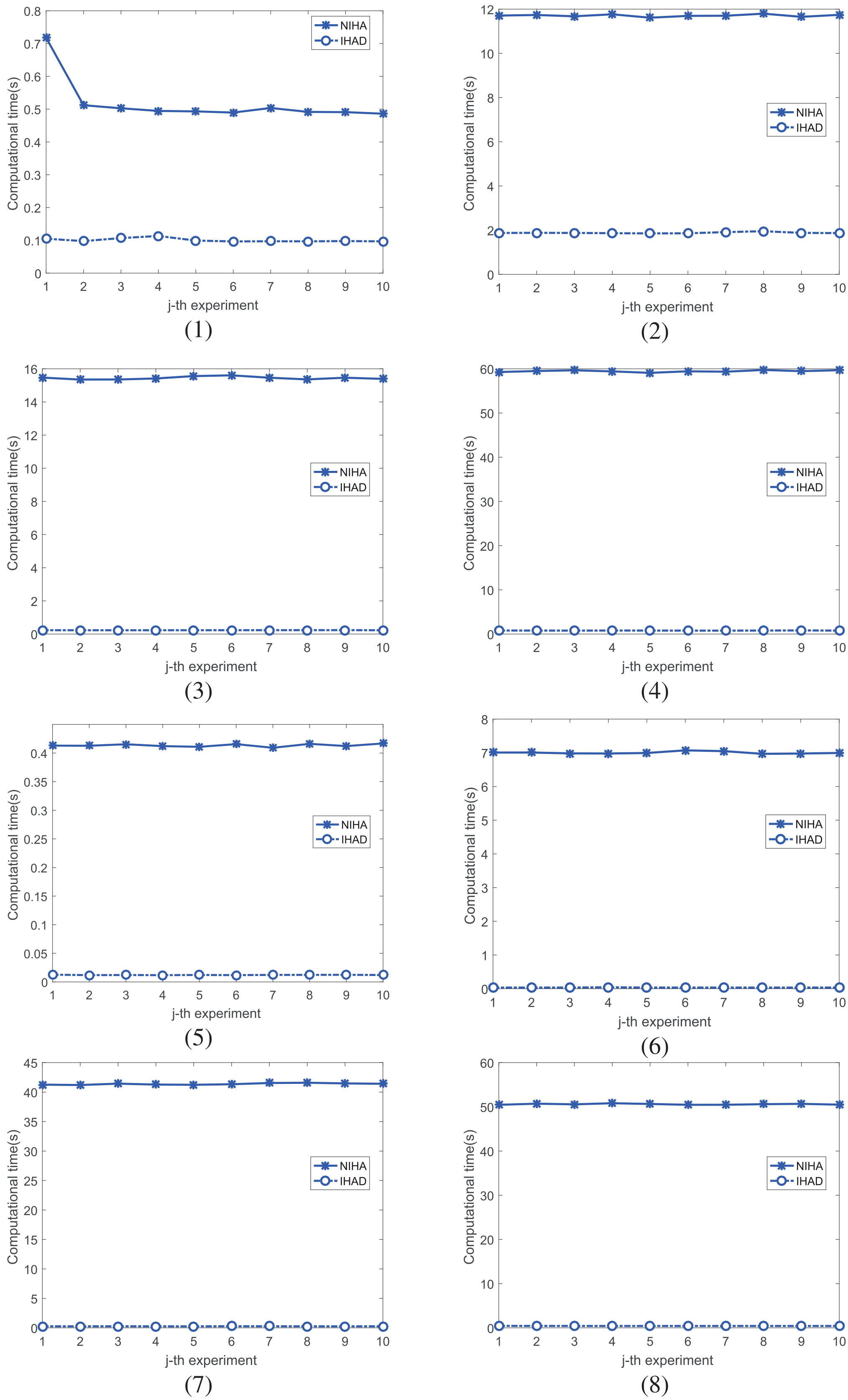

Fig. 3. Computational times using Algorithms 4.1 and 4.5 in $\{(U_i, \Delta_i^-, \mathscr{D}_i) | 1 \le i \le 8\}$.

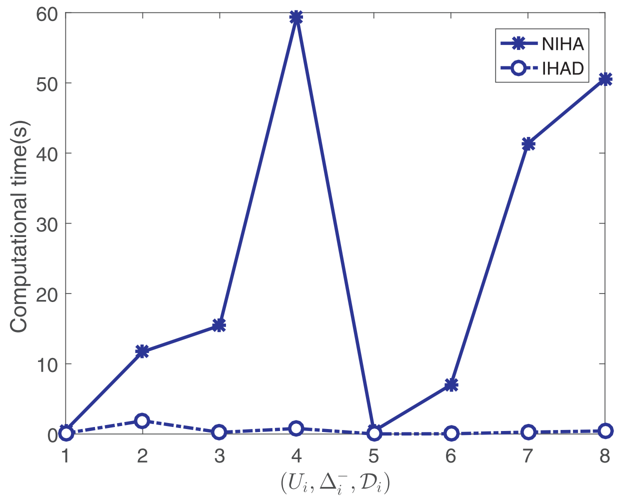

Fig. 4. Average computational times using Algorithms 4.1 and 4.5 in $\{(U_i, \Delta_i^-, \mathscr{D}_i)|1 \le i \le 8\}$.

system $(U_1, \Delta_1, \mathscr{D}_1)$, and get a dynamic covering decision information system $(U_1, \Delta_1^-, \mathscr{D}_1)$ by deleting $\mathscr{C}_{13}$ from $(U_1, \Delta_1, \mathscr{D}_1)$, where $|U_1| = 178$, $|\Delta_1| = 13$, $|\Delta_1^-| = 12$ and $|\mathscr{D}_1| = 1$.

6. Conclusions

Knowledge reduction of dynamic covering information systems is a significant challenge of covering-based rough sets. In this paper, firstly, we have analyzed the related families-based mechanisms of constructing attribute reducts of dynamic covering decision information systems with variations of attributes and employed examples to illustrate how to compute attribute reducts of dynamic covering decision information systems when varying attribute sets. Secondly, we have presented the related families-based heuristic algorithms for computing attribute reducts of dynamic covering decision information systems with attribute arriving and leaving and employed examples to demonstrate how to update attribute reducts with the heuristic algorithms. Finally, we have employed the experimental results to illustrate that the related families-based incremental approaches are effective and feasible for attribute reduction of dynamic covering decision information systems.

In the future, we will study knowledge reduction of dynamic covering decision information systems with variations of object sets. Especially, we will provide effective algorithms for knowledge reduction of dynamic covering decision information systems when object sets are varying with time.

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