Determining Thresholds in Three-Way Decisions: A Multi-object Optimization View

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Abstract. Determination of thresholds is recognized as a fundamental problem in decision-theoretic rough sets. Traditionally, thresholds are determined by observing Bayesian decision theory. Although the semantic seems to be enriched as compared to probabilistic rough sets, the functionality of risk is not comprehensively explored. In allusion to this situation, we develop a multi-object optimization view based model on determining thresholds. By generalizing the expected loss function to target function, this model claims that thresholds in three-way are radically constructed by pair-wise region-based target functions. By transferring the principle of pair-wise region-based target functions on multi-quantitative scenario, we present a finer-grained formulation for thresholds solving. Furthermore, we investigate the multi-layer of presented model. Finally, the optimistic and pessimistic multi-quantitative decision-theoretic rough set is defined to illustrate the value of presented model.

Keywords: Three-way decisions · Decision-theoretic rough sets · Multi-object optimization · Multi-quantitative

1 Introduction

Three-way decisions (TWD) [1], originated from rough set theory [2], has demonstrated the superiority in performing decision-making with uncertainty. Roughly speaking, three-way decisions managed to divide the universe into three non-overlapped regions and take actions of acceptance, rejection or non-commitment respectively. A rational explanation for taking different strategies is that the committed decision, whether acceptance or rejection, corresponds to the information with satisfied discernibility, whereas the non-commitment decision implies the information with flawed discrimination. By incorporating with three-way decisions, an increasing number of successful cases have been reported in different

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applications, including e-mail spam filtering [3], text semantic analysis [4,5] image recognition [6], and cognitive computing [7].

Thresholds play a pivot role in determining the region boundary, and the selections reflect the degree that concepts can be approximately defined by certain granular structure. In *decision-theoretic rough sets* (DTRS) [8], different combinations of thresholds correspond to different risk, and the minimization of risk is the decision principle. Studies on threshold solving can be categorized into two groups. In the first case, all loss coefficients are given but may uncertain. As suggested by [9], an analytical solution is invariantly expected. In the second case, however, only part of them are roughly known, while no specific expressions of the remaining are given. In this case, only numerical solutions can be expected [10,11]. Although both solutions can be interpreted from optimization view, it is not explicitly declared in terms of region construction. Consequently, the semantics of α, β, and γ are merely enriched by introducing more loss coefficients $\lambda_{\bullet\bullet}$. The problem thus becomes more obvious in the discussion of generalized double-quantitative rough sets [12], where the approximation operators are merely generated on the basis of parameters.

In this paper, we take a multi-object optimization (MOP) view for thresholds solving. By generalizing expected loss coefficients of DTRS to target functions, this paper provides a revisit to decision-theoretic rough sets. The confrontation within regions are thoroughly embodied in threshold solving. Accordingly, the semantics of thresholds in DTRS are more intuitive, and the construction of three-way structure is enriched simultaneously. By investigating the combinations of pair-wise region-based target functions, we present a finer-grained view for construction of thresholds. We further declare that under multi-quantitative scenario, more combinations of thresholds can be generated by considering diversity fusions of homogeneous region-based target function meanwhile. The formalized representation is constructive in flourishing approximate knowledge representation of multi-quantitative based rough sets.

The rest of the paper is organized as follows. Section 2 briefly reviews the basic concepts with regard to decision-theoretic rough sets. In Sect. 3, a multi-object optimization based model is investigated to solve the thresholds. By extending the syntax of optimizing principal presented in Sect. 3, proposed model is competent for thresholds solving for multi-quantitative scenario, as illustrated in Sect. 4. Finally, it is concluded in Sect. 5.

2 Preliminary

In this section, we present a review of some basic concepts with regard to decision-theoretic rough sets.

Definition 1 [2]**.** *$IS = \{U, A, V, f\}$ is an information system with quadruple, where U denotes a non-empty finite universe, $A = C \cup D$ be a set of attributes, V be the values of all attributes and is determined by the mapping function $f : U \times A \rightarrow V$.*

Under the equivalence relation R, a corresponding partition of U (U/R) can be generated.

Elements in the identical equivalence class constitute a basic information granule $[x]$. The affiliation of information granular $[x]$ to certain decision class X can be measured by conditional probability $P(X|[x])$. Although this measure can reflect the decision quality, the semantic of thresholds that support the three-way structure is vagueness. To address this issue, Yao [13] introduced loss coefficients $\lambda_{\bullet\bullet}$ to evaluate the effects of three-way decisions. Taking two classes classification problem as an example, there are totally six loss coefficients, as illustrated in Table 1.

Table 1. Loss coefficient matrix for two class classification problem

	X	$\neg X$
acceptance (a)	λ_{ap}	λ_{an}
non-commitment (n)	λ_{np}	λ_{nn}
rejection (r)	λ_{rp}	λ_{rn}

Consequently, the risk of all equivalence class $[x]$ in the process of decision making can be calculated as:

$$R = \sum_{[x]} R(a|[x]) + R(n|[x]) + R(r|[x]) \tag{1}$$

where $R(a|[x]), R(r|[x])$ and $R(n|[x])$ are defined as:

$R(a|[x]) = \lambda_{ap} \times P(X|[x]) + \lambda_{an} \times P(\neg X|[x]);$
$R(n|[x]) = \lambda_{np} \times P(X|[x]) + \lambda_{nn} \times P(\neg X|[x]);$
$R(r|[x]) = \lambda_{rp} \times P(X|[x]) + \lambda_{rn} \times P(\neg X|[x]).$

The risk minimization principle indicates that the affiliation of information granular $[x]$ with regard to class X is reasonable if the following three inequality are satisfied simultaneously.

$R(a|[x]) \leq R(n|[x]) \wedge R(a|[x]) \leq R(r|[x]) \Rightarrow decide\, [x] \subseteq POS(X);$
$R(n|[x]) \leq R(a|[x]) \wedge R(n|[x]) \leq R(r|[x]) \Rightarrow decide\, [x] \subseteq BND(X);$
$R(r|[x]) \leq R(a|[x]) \wedge R(r|[x]) \leq R(n|[x]) \Rightarrow decide\, [x] \subseteq NEG(X).$

Hence, we can make three-way decisions on the risk level. The DTRS model is thus defined as follows:

Definition 2 [13]. *Given relationship of loss coefficients* $\lambda_{ap} \leq \lambda_{np} \leq \lambda_{rp}$ *and* $\lambda_{rn} \leq \lambda_{nn} \leq \lambda_{an}$ *and condition* $(\lambda_{rp} - \lambda_{np})(\lambda_{an} - \lambda_{nn}) \geq (\lambda_{np} - \lambda_{ap})(\lambda_{nn} - \lambda_{rn})$, *three-way region with regard to* X *is defined as*

$POS(X) = \{[x]|P(X|[x]) \geq \alpha\};$
$BND(X) = \{[x]|\beta < P(X|[x]) < \alpha\};$
$NEG(X) = \{[x]|P(X|[x]) \leq \beta\}.$

where parameters α *and* β *are defined as:*

$\alpha = \frac{\lambda_{an} - \lambda_{nn}}{(\lambda_{an} - \lambda_{nn}) + (\lambda_{np} - \lambda_{ap})};\ \beta = \frac{\lambda_{nn} - \lambda_{rn}}{(\lambda_{nn} - \lambda_{rn}) + (\lambda_{rp} - \lambda_{np})};$

3 Multi-object Optimization View for Threshold Solving with Single Quantification

For a given information system, decision risks of positive region, negative region and defer region fluctuate as the selection of thresholds changes. It is reasonable to assume that each threshold is determined by pair-wise region-based target function, thus a multi-object optimization problem is formulated. To elaborate the solving mechanism, this section will limit the scope of target function on single granular structure.

3.1 Problem Formulation

Suppose the conditional probability $P(X|[x])$ is considered as evaluation criterion, we term target functions as:

Definition 3. *Target function T is an assemble of functions with T_P, T_B and T_N which describes the decision cost of positive region, boundary region and negative region induced by $P(X|[x])$ respectively.*

$$\begin{aligned} T_P &= (\lambda_{ap} - \lambda_{an})\, P\,(X\,|[x]\,) + \lambda_{an}; \\ T_B &= (\lambda_{np} - \lambda_{nn})\, P\,(X\,|[x]\,) + \lambda_{nn}; \\ T_N &= (\lambda_{rp} - \lambda_{rn})\, P\,(X\,|[x]\,) + \lambda_{rn}. \end{aligned}$$

All target functions suggest that region-based decision cost is linearly related to the conditional probability. Since three-way can be described by at most two parameters, we can formulate the thresholds solving problem as follows:

$$\underset{(\alpha^*,\beta^*)}{\arg\min}\, T|(\alpha,\beta,\gamma) = \{T_P|(\alpha,\beta,\gamma), T_B|(\alpha,\beta,\gamma), T_N|(\alpha,\beta,\gamma)\} \tag{2}$$

where $\alpha^* \geq \beta^*, \alpha^*, \beta^* \in \{\alpha, \beta, \gamma\}$ and $T|(\alpha,\beta,\gamma)$ denotes the target value given α,β,γ. α,β, and γ are three conditional probabilities that are to be optimized. The selection of α implies the relative boundary between positive region and boundary region, while β and γ suggest the relative boundary of negative region and boundary region, positive region and negative region respectively. To solve the problem formulated in Eq. (2), we define the following optimization model.

Definition 4. *Let $T_\alpha, T_\beta, T_\gamma$ be the decision cost induced merely by α, β and γ respectively, parameters α, β, γ can be solved by three pair-wise object optimization.*

$$\underset{(\alpha)}{\arg\min}\, T_\alpha = \begin{cases} T_P = (\lambda_{ap} - \lambda_{an}) \times \alpha + \lambda_{an} \\ T_B = (\lambda_{np} - \lambda_{nn}) \times \alpha + \lambda_{nn} \end{cases};$$

$$\underset{(\beta)}{\arg\min}\, T_\beta = \begin{cases} T_B = (\lambda_{np} - \lambda_{nn}) \times \beta + \lambda_{nn} \\ T_N = (\lambda_{rp} - \lambda_{rn}) \times \beta + \lambda_{rn} \end{cases};$$

$$\underset{(\gamma)}{\arg\min}\, T_\gamma = \begin{cases} T_P = (\lambda_{ap} - \lambda_{an}) \times \gamma + \lambda_{an} \\ T_N = (\lambda_{rp} - \lambda_{rn}) \times \gamma + \lambda_{rn} \end{cases}.$$

$$s.t. \quad 0 < \alpha < 1, 0 < \beta < 1, 0 < \gamma < 1$$

3.2 Problem Solving

It can be deduced from Definition 4 that solving for any parameter is similar. Without losing generality, we investigate the solving process for parameter α.

Let $T_P = T_B$, we have $P(X|[x]) = \frac{\lambda_{an}-\lambda_{nn}}{(\lambda_{an}-\lambda_{nn})+(\lambda_{np}-\lambda_{ap})}$. Then we have the following Theorem.

Theorem 1. T_α *achieves the minimum value if* $\alpha = \frac{\lambda_{an}-\lambda_{nn}}{(\lambda_{an}-\lambda_{nn})+(\lambda_{np}-\lambda_{ap})}$.

Proof. Given $\lambda_{ap} - \lambda_{an} > \lambda_{np} - \lambda_{nn}$, the slope of T_P is larger than that of T_B. If α is smaller than intersection of two target functions (see α_1 and α_0 in Fig. 1(a)), then the equivalence class with conditional probability in interval (α_1, α_0) will be determined to boundary region, which will have larger cost. If α is bigger than intersection of two target functions (see α_2 and α_0 in Fig. 1(a)), then the equivalence class with conditional probability in interval (α_0, α_2) will be determined to positive region, which will also have larger cost. Analogously, intersection α_0 corresponds to minimum cost given $\lambda_{ap} - \lambda_{an} < \lambda_{np} - \lambda_{nn}$, as illustrated in Fig. 1(b).

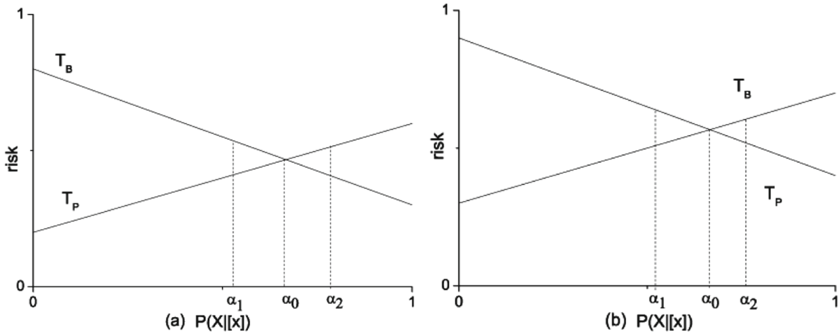

Fig. 1. Determination of parameter α given target function T_P and T_B

Based on Theorem 1, we have the following corollary holds.

Corollary 1. Let $\alpha = \frac{\lambda_{an}-\lambda_{nn}}{(\lambda_{an}-\lambda_{nn})+(\lambda_{np}-\lambda_{ap})}$, we have:

$$((\lambda_{ap} - \lambda_{an}) > (\lambda_{np} - \lambda_{nn})) \wedge (1 > P(X|[x]) > \alpha > 0) \Rightarrow T_P > T_B;$$
$$((\lambda_{ap} - \lambda_{an}) > (\lambda_{np} - \lambda_{nn})) \wedge (0 < P(X|[x]) < \alpha < 1) \Rightarrow T_P < T_B;$$
$$((\lambda_{ap} - \lambda_{an}) < (\lambda_{np} - \lambda_{nn})) \wedge (1 > P(X|[x]) > \alpha > 0) \Rightarrow T_P < T_B;$$
$$((\lambda_{ap} - \lambda_{an}) < (\lambda_{np} - \lambda_{nn})) \wedge (0 < P(X|[x]) < \alpha < 1) \Rightarrow T_P > T_B.$$

Proof. It is straightforward as Theorem 1 implies.

Analogously, we have the property with regard to β and γ according to Definition 4. For β, let $T_B = T_N$, we have $P(X|[x]) = \frac{\lambda_{nn}-\lambda_{rn}}{(\lambda_{nn}-\lambda_{rn})+(\lambda_{rp}-\lambda_{np})}$. Then we have Theorem 2 and Corollary 2 as follows:

Theorem 2. *T_β achieves the minimum value if* $\beta = \frac{\lambda_{nn}-\lambda_{rn}}{(\lambda_{nn}-\lambda_{rn})+(\lambda_{rp}-\lambda_{np})}$

Proof. It is similar to that of Theorem 1.

Corollary 2. Let $\beta = \frac{\lambda_{nn}-\lambda_{rn}}{(\lambda_{nn}-\lambda_{rn})+(\lambda_{rp}-\lambda_{np})}$, we have:

$$((\lambda_{rp}-\lambda_{rn}) > (\lambda_{np}-\lambda_{nn})) \wedge (1 > P(X|[x]) > \beta > 0) \Rightarrow T_N > T_B;$$
$$((\lambda_{rp}-\lambda_{rn}) > (\lambda_{np}-\lambda_{nn})) \wedge (0 < P(X|[x]) < \beta < 1) \Rightarrow T_N < T_B;$$
$$((\lambda_{rp}-\lambda_{rn}) < (\lambda_{np}-\lambda_{nn})) \wedge (1 > P(X|[x]) > \beta > 0) \Rightarrow T_N < T_B;$$
$$((\lambda_{rp}-\lambda_{rn}) < (\lambda_{np}-\lambda_{nn})) \wedge (0 < P(X|[x]) < \beta < 1) \Rightarrow T_N > T_B.$$

Proof. It is straightforward as Theorem 2 implies.

For γ, let $T_P = T_N$, we have $P(X|[x]) = \frac{\lambda_{an}-\lambda_{rn}}{(\lambda_{an}-\lambda_{rn})+(\lambda_{rp}-\lambda_{ap})}$, then we have Theorem 3 and Corollary 3.

Theorem 3. *T_γ achieves the minimum value if* $\gamma = \frac{\lambda_{an}-\lambda_{rn}}{(\lambda_{an}-\lambda_{rn})+(\lambda_{rp}-\lambda_{ap})}$.

Proof. It is similar to that of Theorem 1.

Corollary 3. Let $\gamma = \frac{\lambda_{an}-\lambda_{rn}}{(\lambda_{an}-\lambda_{rn})+(\lambda_{rp}-\lambda_{ap})}$, we have:

$$((\lambda_{ap}-\lambda_{an}) > (\lambda_{rp}-\lambda_{rn})) \wedge (1 > P(X|[x]) > \gamma > 0) \Rightarrow T_P > T_N;$$
$$((\lambda_{ap}-\lambda_{an}) > (\lambda_{rp}-\lambda_{rn})) \wedge (0 < P(X|[x]) < \gamma < 1) \Rightarrow T_P < T_N;$$
$$((\lambda_{ap}-\lambda_{an}) < (\lambda_{rp}-\lambda_{rn})) \wedge (1 > P(X|[x]) > \gamma > 0) \Rightarrow T_P < T_N;$$
$$((\lambda_{ap}-\lambda_{an}) < (\lambda_{rp}-\lambda_{rn})) \wedge (0 < P(X|[x]) < \gamma < 1) \Rightarrow T_P > T_N.$$

Proof. It is straightforward as Theorem 3 implies.

By simultaneously considering the relations of relative parameters and slope of target functions, we can determine the three-way structure as following theorems:

Theorem 4. *If $(\lambda_{rp}-\lambda_{rn}) > (\lambda_{np}-\lambda_{nn}) > (\lambda_{ap}-\lambda_{an})$ and $\beta \leq \gamma \leq \alpha$, then the following decision rules hold: (P) $P(X|[x]) \geq \alpha$, decide $x \in POS(X)$; (B) $\beta < P(X|[x]) < \alpha$, decide $x \in BND(X)$; (N) $P(X|[x]) \leq \beta$, decide $x \in NEG(X)$.*

Proof. Since the condition $(\lambda_{rp}-\lambda_{rn}) > (\lambda_{np}-\lambda_{nn}) > (\lambda_{ap}-\lambda_{an})$ is satisfied, we have $(\lambda_{rp}-\lambda_{rn}) > (\lambda_{np}-\lambda_{nn})$, $(\lambda_{np}-\lambda_{nn}) > (\lambda_{ap}-\lambda_{an})$, and $(\lambda_{rp}-\lambda_{rn}) > (\lambda_{ap}-\lambda_{an})$. Similarly, $\alpha \leq \gamma \leq \beta$ is equivalent to $\alpha \leq \gamma, \gamma \leq \beta$ and $\alpha \leq \beta$. According to Corollaries 1 and 3, if additional condition $P(X|[x]) \geq \alpha$ holds, decide $x \in POS(X)$. According to Corollaries 1 and 2, if additional condition $\beta < P(X|[x]) < \alpha$ holds, decide $x \in BND(X)$. According to Corollaries 2 and 3, if additional condition $P(X|[x]) \leq \beta$ holds, decide $x \in NEG(X)$.

Theorem 5. *If $(\lambda_{rp}-\lambda_{rn}) > (\lambda_{np}-\lambda_{nn}) > (\lambda_{ap}-\lambda_{an})$ and $\alpha \leq \gamma \leq \beta$, then the following decision rules hold: (P) $P(X|[x]) \leq \gamma$, decide $x \in NEG(X)$; (N) $P(X|[x]) \geq \gamma$, decide $x \in POS(X)$.*

Proof. It is similar to that of Theorem 4.

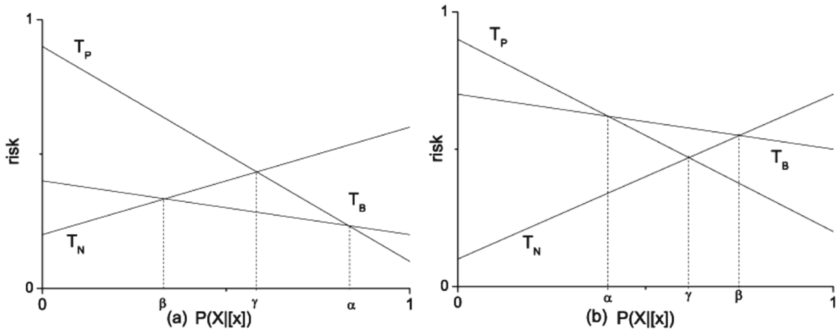

Fig. 2. Three-way structure for X given conditional probability $P(X|[x])$

Theorems 4 and 5 illustrate that three-way structure can be different given the relative relation of slope, as shown in Fig. 2. It reflects that the introduction of target function T_B do not necessarily give rise to three non-empty regions.

The three-way structure with regard to $\neg X$ is derivable from the conditional probability $P(X|[x])$. Since X and $\neg X$ is complementary with regard to 1, the slope of T_P, T_B and T_N is opposite. Consequently, the condition $(\lambda_{rp} - \lambda_{rn}) < (\lambda_{np} - \lambda_{nn}) < (\lambda_{ap} - \lambda_{an})$ is satisfied. Based on it, we investigate the relative relation of parameter α, β and γ.

Theorem 6. *If* $(\lambda_{rp} - \lambda_{rn}) < (\lambda_{np} - \lambda_{nn}) < (\lambda_{ap} - \lambda_{an})$ *and* $\alpha \leq \gamma \leq \beta$, *then the following decision rules hold: (P)* $P(X|[x]) \leq \alpha$, *decide* $x \in POS(X)$; *(B)* $\alpha < P(X|[x]) < \beta$, *decide* $x \in BND(X)$; *(N)* $P(X|[x]) \geq \beta$, *decide* $x \in NEG(X)$.

Proof. It is similar to that of Theorem 4, as illustrated in Fig. 3(a).

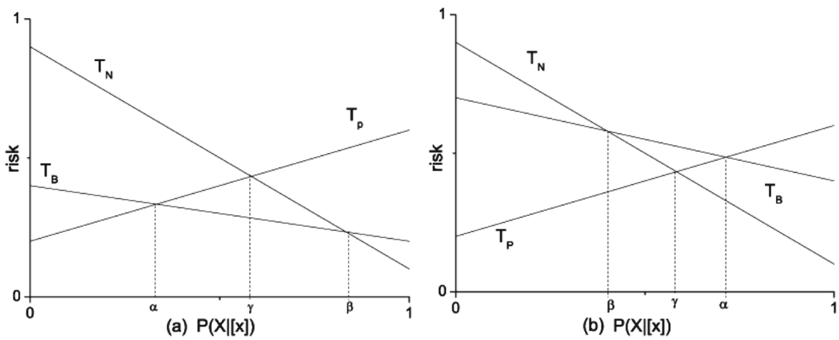

Fig. 3. Three-way structure for $\neg X$ given conditional probability $P(X|[x])$

Theorem 7. *If* $(\lambda_{rp} - \lambda_{rn}) < (\lambda_{np} - \lambda_{nn}) < (\lambda_{ap} - \lambda_{an})$ *and* $\beta \leq \gamma \leq \alpha$*, then the following decision rules hold: (P)* $P(X|[x]) \leq \gamma$*, decide* $x \in POS(X)$*; (N)* $P(X|[x]) \geq \gamma$*, decide* $x \in NEG(X)$*.*

Proof. It is similar to that of Theorem 4, as illustrated in Fig. 3(b).

4 Multi-object Optimization View for Threshold Solving with Multiple Quantification

The multi-view of target function is quite ubiquitous in complicated applications such as group decision-making and double-quantification. It signifies that for a specific object x, we may observe from different views, resulting in the appearance of x in at least two different granular structures. Although double-quantitative can define concepts with multi-view to some degree, some complicated concepts still cannot be defined. For example, consider the following requirements:

- To accept concept X, precision weighs more than grade, and precision should be at least 80%;
- The relative differences between precision and grade are limited to 10%, and percentages for cardinality of equivalent class in information system should be at least 5%;
- Both precision and grade contribute to the three-way decisions, but the evaluation metrics for different regions are different.

The aforementioned requirements cannot be resolved in existing double-quantitative rough set model since thresholds are not determined by target functions from homogeneous quantification. Regarding every target function as an atomic quantitative metric, this section intends to further examine the thresholds construction on multiple granulation.

4.1 Problem Formulation

Compared to single quantification, major difference is that the three regions with regard to concept X are implicitly determined by granular structures. We introduce three region integration functions f_P, f_B and f_N to induce the integrated region-based target function T_P, T_B, and T_N. Therefore, Eq. 2 is rewritten as:

$$\arg\min_{(\alpha^*,\beta^*)} T|(\alpha,\beta,\gamma) = \{f_P|(\alpha,\beta,\gamma), f_B|(\alpha,\beta,\gamma), f_N|(\alpha,\beta,\gamma)\} \tag{3}$$

Suppose there are two groups of target functions (T^i, T^j, with $T^i = \{T^i_P, T^i_B, T^i_N\}$, $T^j = \{T^j_P, T^j_B, T^j_N\}$), then parameters α, β and γ can be solved by transferring the principle of region confrontation, defined as:

Definition 5. *Let* (T_P^i, T_B^i, T_N^i) *and* (T_P^j, T_B^j, T_N^j) *represent two different groups of target functions that are determined by Definition 4 respectively, then* (α, β, γ) *can be computed as:*

$$\arg\min_{(\alpha)} T_\alpha = \begin{cases} f_P(T_P^i, T_P^j) \\ f_B(T_B^i, T_B^j) \end{cases};$$

$$\arg\min_{(\beta)} T_\beta = \begin{cases} f_B(T_B^i, T_B^j) \\ f_N(T_N^i, T_N^j) \end{cases};$$

$$\arg\min_{(\gamma)} T_\gamma = \begin{cases} f_P(T_P^i, T_P^j) \\ f_N(T_N^i, T_N^j) \end{cases}.$$

$$s.t. 0 < \alpha < 1, 0 < \beta < 1, 0 < \gamma < 1$$

where

$$T_P^i = \left(\lambda_{ap}^i - \lambda_{an}^i\right) \times \alpha + \lambda_{an}^i, \quad T_P^j = \left(\lambda_{ap}^j - \lambda_{an}^j\right) \times \alpha + \lambda_{an}^j$$
$$T_B^i = \left(\lambda_{np}^i - \lambda_{nn}^i\right) \times \beta + \lambda_{nn}^i, \quad T_B^j = \left(\lambda_{np}^j - \lambda_{nn}^j\right) \times \beta + \lambda_{nn}^j$$
$$T_N^i = \left(\lambda_{rp}^i - \lambda_{rn}^i\right) \times \gamma + \lambda_{rn}^i, \quad T_N^j = \left(\lambda_{rp}^j - \lambda_{rn}^j\right) \times \gamma + \lambda_{rn}^j$$

To elaborate the structure of integrated region, we consider the trivial case, namely, the output of region integration function is one of the integrated target functions:

$$f_P(T_P^i, T_P^j) \in \{T_P^i, T_P^j\};\ f_B(T_B^i, T_B^j) \in \{T_B^i, T_B^j\};\ f_N(T_N^i, T_N^j) \in \{T_N^i, T_N^j\}.$$

It can be inferred from Definition 5 that for each parameter α, β, γ, there are four candidate combinations. Figure 4 illustrates the candidate α and β in multi-quantification space for trivial cases, and combinations of the three-way structure in this scenario can be at most sixteen cases.

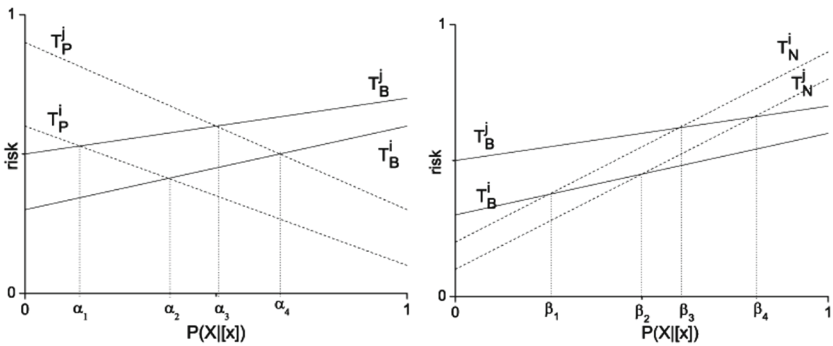

Fig. 4. Thresholds solving in multi-quantification: α (left) and β (right)

By allowing integrations on two different region integration functions, the multi-objective optimization based model can be further generalized as:

Definition 6. *Let* (f_P^i, f_B^i, f_N^i) *and* (f_P^j, f_B^j, f_N^j) *represent two different groups of integrated target functions, then* (α, β, γ) *can be computed as:*

$$\arg\min_{(\alpha)} T_\alpha = \begin{cases} f_P(f_P^i, f_P^j) \\ f_B(f_B^i, f_B^j) \end{cases};$$

$$\arg\min_{(\beta)} T_\beta = \begin{cases} f_B(f_B^i, f_B^j) \\ f_N(f_N^i, f_N^j) \end{cases};$$

$$\arg\min_{(\gamma)} T_\gamma = \begin{cases} f_P(f_P^i, f_P^j) \\ f_N(f_N^i, f_N^j) \end{cases}.$$

$$s.t. \quad 0 < \alpha < 1, 0 < \beta < 1, 0 < \gamma < 1$$

4.2 Problem Solving

Definitions 5 and 6 are applicable in explaining thresholds construction of double-quantitative rough sets [14]. Although the incorporation of decision-theoretic has been discussed, it still deserves to be improved. For example, in GMDq-DTRS [12], thresholds are approximated by performing operations on pair-wise three-way thresholds. The results can be regarded as adding additional requirements on Definition 5 that $i = j$ holds for both α and β but with heterogeneous trivial selections, whereas the remaining cases are not covered. Indeed, the trivial cases correspond to the semantic that certain thresholds are completely determined by certain target functions, whereas the non-trivial cases reflect the relevant degree to target functions for certain thresholds. For example, one may define that an equivalent class with 90% precision but with 10 objects should be deferred, whereas an equivalent class with 80% precision but with 50 objects should be accepted. The reason is that the acceptance of the latter may yield to a more robust decision than acceptance of the former.

Figure 5 systematically illustrates the multi-object optimization perspective for solving thresholds (α^*, β^*) of three-way decisions in granulation space. From finest to coarsest, there are three levels. In the first level (Sq-TWD), only pair-wise region confrontation is required to generate region boundary. The three-way structure is totally determined by the very group of target function, which means the three-way can not be further optimized given the target function T. Thresholds solving for single-quantitative three-way decisions is completed in this level. However, given another group of target function, the level is upgraded to the second (Dq-TWD), where the three-way is determined by both target functions (T^i, T^j) and integrated target functions (f_P, f_B, f_N). The result of integrated target functions can be either trivial or non-trivial, and how to define the integrated target function is an open issue. Take α for example, the f_P can generate trivial or non-trivial results, and similarly for f_B and f_N. Consequently, the number of solution structure for each parameter is four. As an example, we enumerate the cases for α as follows:

$$(trivial_P, trivial_B), (trivial_P, non-trivial_B),$$
$$(non-trivial_P, trivial_B), (non-trivial_P, non-trivial_B).$$

where $trivial_P \in \{T_P^i, T_P^j\}, trivial_B \in \{T_B^i, T_B^j\}$.

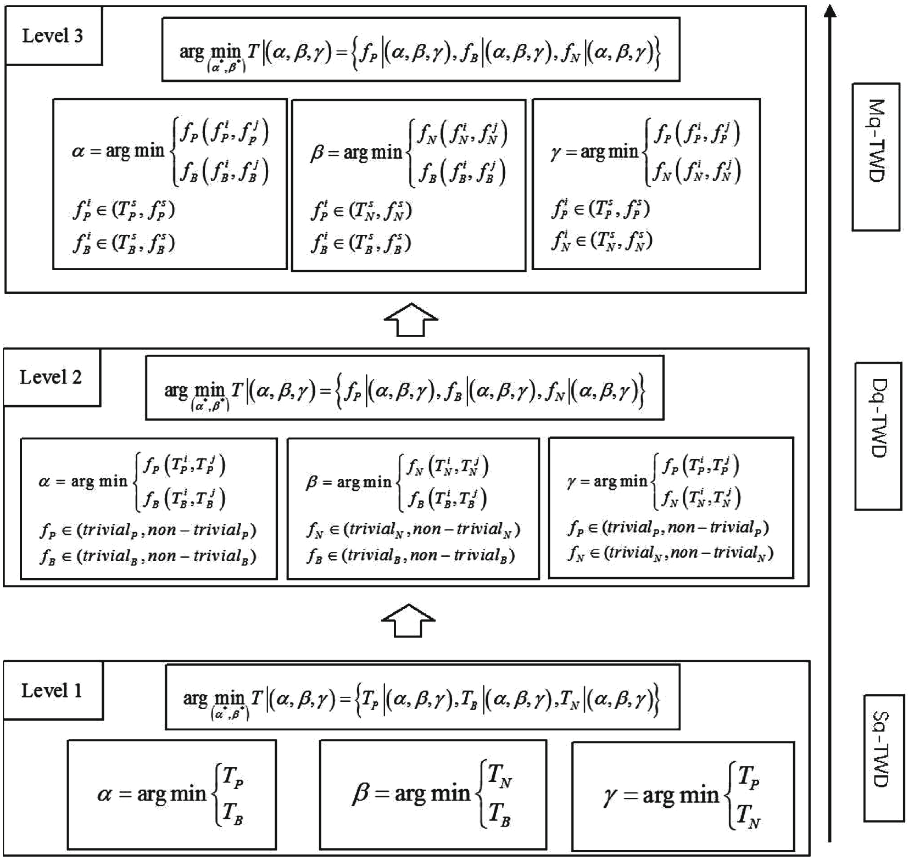

Fig. 5. Levels for threshold solving of three-way decisions from multi-objective optimization view

Thresholds of double-quantitative rough set with decision-theoretic rough set are solved in this level. In the third level (Mq-TWD), there are more than two groups of target functions, which indicates that integrated target functions may be iteratively used. The trivial output is defined as the result which is identical to either input. In terms of input type, there are also four cases for f_P^i as follows:

$$(T_P^s, T_P^t), (T_P^s, f_P^t), (f_P^s, T_P^t), (f_P^s, f_P^t).$$

It corresponds the general case for multi-granulation rough sets. For thresholds with uncertainty like [16,17], we argue that they are the extensions of the exact solution, and thus are not particularly treated as a level.

4.3 Examples

Three-way structure of multi-quantitative rough set is not as intuitive as single-quantitative because of uncertainty in the selection of integrated target functions f_P, f_B, and f_N. Suppose the solutions for all integrated target functions are trivial, by introducing the idea of optimistic and pessimistic defined in [15], we can define optimistic multi-quantitative decision-theoretic rough set and pessimistic multi-quantitative decision-theoretic rough set respectively as:

Definition 7. *Given information system $IS = (U, A, V, f)$, if $(\lambda_{rp} - \lambda_{rn}) < (\lambda_{np} - \lambda_{nn}) < (\lambda_{ap} - \lambda_{an})$ and $\alpha \geq \gamma \geq \beta$, then optimistic multi-quantitative rough set with regard to concept X are defined as:*

$$\underline{X} = \{x|P(X|[x]) \geq \alpha\}$$
$$\overline{X} = \{x|P(X|[x]) \geq \beta\}$$

where $\alpha = \underset{(\alpha)}{\arg\min}(f_P(T_P^i, T_P^j) = f_B(T_B^i, T_B^j)), \forall i, j$

$$\beta = \underset{(\beta)}{\arg\max}(f_N(T_N^i, T_N^j) = f_B(T_B^i, T_B^j)), \forall i, j.$$

The boundary region of optimistic multi-quantification rough set is the smallest. Specifically, $\alpha = \alpha_1, \beta = \beta_4$ if target functions T_i and T_j are as shown in Fig. 4.

Definition 8. *Given information system $IS = (U, A, V, f)$, if $(\lambda_{rp} - \lambda_{rn}) < (\lambda_{np} - \lambda_{nn}) < (\lambda_{ap} - \lambda_{an})$ and $\alpha \geq \gamma \geq \beta$, then pessimistic multi-quantitative rough sets with regard to concept X are defined as:*

$$\underline{X} = \{x|P(X|[x]) \geq \alpha\}$$
$$\overline{X} = \{x|P(X|[x]) \geq \beta\}$$

where $\alpha = \underset{(\alpha)}{\arg\max}(f_P(T_P^i, T_P^j) = f_B(T_B^i, T_B^j)), \forall i, j$

$$\beta = \underset{(\beta)}{\arg\min}(f_N(T_N^i, T_N^j) = f_B(T_B^i, T_B^j)), \forall i, j.$$

The boundary region of pessimistic multi-quantitative rough set is the smallest. Specifically, $\alpha = \alpha_4, \beta = \beta_1$ if target functions T_i and T_j are as shown in Fig. 4.

For other concept descriptions, there are varying methods to develop three-way structure. One feasible solution is to interpret the problem as learning the weights among pair-wise target functions, and for this part we intend to elaborate the details in our future work. Hence, we argue that our work present a finer-grained threshold construction, since we can not only enrich the meaning of three-way in single quantification but also applicable in describing complicated concept.

5 Conclusion

This paper presents a novel threshold solving model from the perspective of multi-object optimization for three-way decisions. From the view of region-based target function, theories on determining three-way thresholds are significantly enriched. Multi-object optimization on target function is demonstrated to generate finer-grained thresholds as compared to discussions on double-quantitative decision-theoretic rough set. In the next step, we will not only theoretically examine the properties of multi-quantitative rough set, but also practically investigate efficient algorithms for knowledge reduction.

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