Fuzzy Rough Set and Its Improvement

Duoqian Miao, Daoguo Li, Shidong Fan

***Abstract*—The fusion of rough set and fuzzy set has become one of the hot issues in the disposal of intelligence information in recent years. In this paper, fuzzy equivalence class on △-transitive compatible relation is introduced. A special △-transitivity, also referred to as *Tm*-fuzzy transitive compatible relation in information systems is created. On the basis, fuzzy rough set and its improvement are discussed respectively.**

***Index Terms*— △ -transitive compatible relation, Fuzzy equivalence class, fuzzy rough set.**

I. Introduction

Fuzzy set theory was proposed by Zadeh in 1965 and rough set theory was proposed by Pawlak in 1982 "Ref.[1] ." They are two kinds of approaches to the study of intelligent systems characterized by uncertain, imprecise and incomplete information. For the purpose of studying further, some researchers proposed the view of combing those two theories. As a result, the methods of the fuzzy rough set and the rough fuzzy set come into being successively "Ref.[2] and Ref.[3]." The fuzzy rough set is the keystone of this paper.

The paper is organized as follows: Section 2 provides some basic concepts in fuzzy rough set such as △-transitive compatible relation, fuzzy equivalence class , their acquisitions and so on. Fuzzy rough set and its improvement are explained in Section 3 and Section 4 respectively.

II. Fuzzy Compatible Relation and Fuzzy Equivalence Class

A. △-Transitive Compatible Relation

One extension of rough set is that the equivalence relation on universe U is generalized to △-Transitive compatible relation. Let R be the fuzzy relation on universe U, if the next three formulas

1) Reflexivity $\mu_R(u,u)=1\ (\forall u\in U)$;

2) Symmetry $\mu_R(u,v)=\mu_R(v,u)\ (\forall u,v\in U)$;

3) △-Transitivity

$$\mu_R(u,w)\ge \mu_R(u,v)\Delta\mu_R(v,w)\ (\forall u,v,w\in U),$$

are satisfied, R is called △-Transitive compatible relation on U, where △ is a triangle-module operator on unit closed interval [0,1] and $a\Delta b\le \min(a,b)$. The following three cases of △ is most common:

1) △=min(△-Transitive compatible relation is just fuzzy equivalence relation.);

2) △=×(× means general numeric multiplication.);

3) △=*Tm*(a *Tm* b=max(0, a+b-1)).

B. The Acquisition of △-Transitive Compatible Relation

Suppose R_1, R_2,..., R_n be n equivalence relations on universe *U*, let

$$\mu_R(u,v)=\sum_{i=1}^{n}\partial_i\mu_{R_i}(u,v). \qquad (1)$$

where $\sum_{i=1}^{n}\partial_i=1, \partial_i>0(\forall i)$.

The relation R defined in (1) is *Tm*- Transitive compatible relation on *U*. It is obviously that reflexivity and symmetry hold. The proof of its transitivity is shown as follow.

Proof: For any $u,v,w\in U$, $\mu_R(u,w)=\sum_{\mu_{R_i}(u,w)=1}\partial_i$, $\mu_R(u,v)=\sum_{\mu_{R_i}(u,v)=1}\partial_i$, $\mu_R(v,w)=\sum_{\mu_{R_i}(v,w)=1}\partial_i$. Among the three formulas, if ∂_i appears in the two of them, it must be in the other one. Hence, if $\sum\partial_i$ is sum of all the ∂_is which appear in the three formulas at the same time, $\mu_R(u,w)=\sum\partial_i+\alpha$, $\mu_R(u,v)=\sum\partial_i+\beta$, $\mu_R(v,w)=\sum\partial_i+\gamma$, in which α,β,γ is sums without common ∂_i. And

$$\mu_R(u,w)\ge\mu_R(u,v)+\mu_R(v,w)-1$$
$$\Leftrightarrow\sum\partial_i+\alpha\ge 2\sum\partial_i+\beta+\gamma-1$$
$$\Leftrightarrow\sum\partial_i\le 1-(\beta+\gamma)+\alpha$$
$$\Leftrightarrow\sum\partial_i\le\sum_{i=1}^{n}\partial_i-(\beta+\gamma)+\alpha.$$

The last formula must hold, so

$$\mu_R(u,w)\ge\mu_R(u,v)Tm\mu_R(v,w).$$

By the remark on the above, we can obtain a *Tm*-Transitive compatible relation from n equivalence relations on *U*. But it is still a problem that how to decompose a *Tm*-Transitive compatible relation into a weighted sum of n equivalence relations. When $R_i(i=1,2,\cdots,n)$ is a family of nested equivalence relations, namely $R_1\subseteq R_2\subseteq\cdots\subseteq R_n$,

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min- Transitive compatible relation on U can be obtained from formula (1). That is fuzzy equivalence relation.

C. The Acquisition of Compatible Relation in Information Systems

Edified by the view of previous section, we work out a method constructing compatible relation in information systems, which maybe become the great breakthrough of generalizing the classical rough set.

Suppose $I=(U,A,V,f)$ is an information system, $A=\{a_1,a_2,\cdots,a_n\}$ is a finite attribute sets, every attribute $a_i(i=1,2,\cdots,n)$ corresponds with an equivalence called indiscernibility relation. We can get *Tm*-Transitive compatible relation on U by (1), where ∂_i means the significance of a_i in A.

Method 1: Assign weight coefficient averagely,

$$\partial_i = 1/n\,(i=1,2,\cdots,n).$$

When

$$(u,v)\in\bigcap_{i=1}^{n} a_i,\ \mu_R(u,v)=1.$$

In general,

$$\mu_R(u,v)=card\{a_i : ua_iv\}/n.$$

Considering user's interests, we have the following method:

Method 2: Assign bigger weight coefficient to core and attributes user are interested in.

D. Fuzzy Equivalence Class

Suppose R is $\triangle$-Transitive compatible relation on U, the fuzzy equivalence class $[u]_R$ is defined as follows:

$$\mu_{[u]_R}(v)=\mu_R(u,v)\quad(\forall v\in U). \qquad (2)$$

When R is general equivalence relation, (2) just defined an equivalence class. In the most cases, $[u]_R$ is collection of elements of U adjacent to u, and it is a fuzzy set.

In 1988, Hohle proposed that a family of fuzzy sets U_i $(i=1,2,...,n)$ form fuzzy equivalence classes of U if and only if they satisfy the following axioms:

1) every U_i is normal, namely

$$core(U_i)=1\,(\forall i\in\{1,2,\cdots,n\});$$

2) $\mu_{U_i}(u)\Delta\mu_R(u,v)\le\mu_{U_i}(v)$;

3) $\mu_{U_i}(u)\Delta\mu_{U_i}(v)\le\mu_R(u,v)$.

In the axiom (1), all the U_i must be non-empty. In the axiom (2), elements adjacent to v should belong to the equivalence class of v. In the axiom (3), R includes the Cartesian product of any equivalence class and itself on $\triangle$. On the other hand, any two elements of U_i are correlate by R. Axiom (2) can also be described as

$$U_i\otimes R\subseteq U_i. \qquad (3)$$

Where $\mu_{U_i\otimes R}(v)=\sup_{u\in U}\mu_{U_i}(u)\Delta\mu_R(u,v)$ is matrix-vector product on $\triangle$. Actually ,U_i, a fuzzy set on U, is equivalent to a row vector; and R is equivalent to a matrix. The product of U and R on $\triangle$ is just shown as formula (3).

Obviously, the family of fuzzy equivalence classes $\{U_1,U_2,\cdots,U_n\}$ induced by R, satisfies the three axioms. For the reflexivity of R, when $U_i\subseteq U_i\otimes R$ is hold, the axiom (2) can be strengthened as $U_i\otimes R=U_i$, which is equivalent to the character fuzzy set of R on each U_i. The fuzzy equivalence classes $\{U_1,U_2,\cdots,U_n\}$ of $\triangle$ -Transitive compatible relation have following properties:

Proposition 1 :

If $U_i\ne U_j\Rightarrow\neg(\exists u\in U):\mu_{U_i}(u)=\mu_{U_j}(u)=1$.

Proof: Suppose $U_i=[u]_R, U_j=[v]_R$. If there exists $w\in U$ such that $\mu_{U_i}(w)=\mu_{U_j}(w)=1$, that is

$$\mu_R(u,w)=\mu_R(v,w)=1,\text{ then}$$

$$\mu_R(u,v)\ge\mu_R(u,w)\Delta\mu_R(v,w)=1\quad(1\Delta1=1).$$

So $\mu_R(u,v)=1$. For any $p\in U$,

$$\mu_{U_i}(p)=\mu_R(u,p)$$
$$\ge\mu_R(u,v)\Delta\mu_R(v,p)=\mu_R(v,p)=\mu_{U_j}(p)$$

Analogously, the reverse inequation is also holding. Thus, $\mu_{U_i}(p)=\mu_{U_j}(p)$, namely $U_i=U_j$.

From Proposition 1, for any two different fuzzy equivalence classes $U_i\ne U_j$, $core(U_i\cap U_j)=\Phi$. They are non-intersective. In other words, if $U_i\subseteq U_j$, then $U_i=U_j$. There doesn't exist any inclusion relation among U_i.

Proposition 2

$$\sup\left(\bigcup_{i=1}^{n}U_i\right)=U.$$

For any $u\in U$, $\mu_{[u]_R}=\mu_R(u,u)=1$,so Proposition 2 holds. Because of Proposition 1 and Proposition 2, R induces a weak fuzzy partition on U according to $\{U_1,U_2,\cdots,U_n\}$ generated from (2). But how is the $\triangle$-Transitive compatible relation R described by equivalence class U_i?

Suppose $\{U_1,U_2,\cdots,U_n\}$ is a group of fuzzy equivalence classes induced by R, R is fuzzy union of fuzzy Cartesian product $U_i\times U_i$ on the triangle module $\triangle$. That is

$$\mu_{\bigcup_{i=1,n}U_i\times U_i}=\max_{i=1,n}\mu_{U_i}(u)\Delta\mu_{U_i}(v)$$
$$=\max_{w\in U}\mu_R(u,w)\Delta\mu_R(v,w)$$
$$=\mu_R(u,v).$$

III. Fuzzy Rough Set

We can set about constructing fuzzy rough set from the weak fuzzy partition $\Phi=\{F_1,F_2,\cdots,F_n\}$ on U. Φ is from either fuzzy equivalence classes by $\triangle$-Transitive compatible relation or other approaches. In fuzzy rough set, fuzzy decisions(fuzzy events) F should be described by a group of fuzzy condition sets Φ . Let

$$M_i=\mu_{\overline{\Phi}(F)}(F_i)=\sup_{u\in U}\mu_{F_i}(u)\Delta\mu_F(u). \qquad (4)$$

$$m_i=\mu_{\underline{\Phi}(F)}(F_i)=\inf_{u\in U}(\mu_{F_i}(u)\to\mu_F(u)). \qquad (5)$$

where $a \to b = 1 - a\Delta(1-b) = \neg(a\Delta\neg b)$ $(\neg a = 1-a)$ is called multi-value implication or S-implication. The pair $(\underline{\Phi}(F), \overline{\Phi}(F))$ is called $\triangle$-fuzzy rough set and "$\triangle$=min" is our main attention.

When $\triangle$=min, Formula (1) and Formula (2) can be transformed to Formula (3) and Formula (4) respectively.

$$M_i = \mu_{\overline{\Phi}(F)}(F_i) = \sup_{u\in U} \mu_{F_i \cap F}(u). \quad (6)$$

$$m_i = \mu_{\underline{\Phi}(F)}(F_i) = \inf_{u\in U} \mu_{\neg F_i \cup F}(u). \quad (7)$$

In the above formulas, M_i *and* m_i mean possibility and inevitability of F_i contained in F separately.

When F is common subset of U, formula (6) and formula (7) are still correctly and can be transformed as follows:

$$M_i = \sup_{u\in F} \mu_{F_i}(u). \quad (8)$$

$$m_i = 1 - \sup_{u\in F} \mu_{F_i}(u). \quad (9)$$

When Φ induces the exact partition of U, formula (6) and formula (7) can be simplified as:

$$M_i = \sup_{u\in F_i} \mu_F(u). \quad (10)$$

$$m_i = \inf_{u\in F_i} \mu_F(u). \quad (11)$$

They are rough fuzzy sets of F employing partition Φ .When Φ induces the exact partition of U and F is non-fuzzy subset of U, formula (6) and formula (7) degenerate to Pawlak rough set $(\underline{\Phi}(F), \overline{\Phi}(F))$.

IV. Improvement of Fuzzy Rough Set

There are many forms of fuzzy rough set and every form provides some room of refinement. In this section, we will introduce a kind of improvement associated with some experiences in studying rough set.

The correlation intension between decision fuzzy event F and condition fuzzy events set $\Phi = \{F_1, F_2, \cdots, F_n\}$ is given by fuzzy rough set. Our aim is to find:

$$\underline{\Phi}(F) \text{ and } \overline{\Phi}(F) \in \Psi(U)$$

which meets $\mu_{\underline{\Phi}(F)}(u) \le \mu_F(u) \le \mu_{\overline{\Phi}(F)}(u)$, it is similar to $\mu_{\underline{R}(A)}(u) \le \mu_A(u) \le \mu_{\overline{R}(A)}(u)$ in exact set. $\underline{\Phi}(F)$ *and* $\overline{\Phi}(F)$ should be constructed just by $\Phi = \{F_1, F_2, \cdots, F_n\}$ and F. $\mu_{\overline{\Phi}(F)}(u) - \mu_{\underline{\Phi}(F)}(u)$ should be as small as possible. This is view of optimal approximation.

A. Inclusion Function in Fuzzy Set

Among the inclusion functions in fuzzy set, we adopt the definition hereinafter.

$\forall \hat{A}, \hat{B} \in \Psi(U)$, the degree of $\hat{A}$ contained in $\hat{B}$ is

$$I(\hat{A}, \hat{B}) = card(\hat{A} \cap \hat{B}) / card(\hat{A}),$$

where $card(\hat{A}) = \sum_{u\in U} \mu_{\hat{A}}(u)$.

Formally, the inclusion function can be formulated as follows.

Definition 1: Let $I : \Psi(U) \times \Psi(U) \to [0,1]$, if following conditions are satisfied, for $\forall \hat{A}, \hat{B} \in \Psi(U)$,

$$I(\hat{A}, \hat{B}) = 1 \textit{ iff } \hat{A} = \hat{B}$$

and

$$\textit{if } \hat{A} \cap \hat{B} = \phi \textit{ then } I(\hat{A}, \hat{B}) = I(\hat{B}, \hat{A}) = 0,$$

$I(\hat{A}, \hat{B})$ is called the inclusion function and its value is the degree of $\hat{A}$ contained in $\hat{B}$ or $\hat{B}$ containing $\hat{A}$.

B. λ-Approximate Fuzzy Set

Definition 2: Let U is a finite universe, $\Phi = \{F_1, F_2, \cdots, F_n\}$ is a group of fuzzy sets, $\sup \bigcup_{F_i \in \Phi} F_i = U$, $F \in \Psi(U)$ is a fuzzy set, I is a inclusion function and $\lambda \in (0,1]$. We say fuzzy set F in the approximate space (U, Φ, I) is λ-approximate if

$$\min\{I(\bigcap_i F_i, F), I(\bigcup_i F_i, F)\} \ge \lambda.$$

The coefficient $T_F = card(\bigcup_i F_i - \bigcap_i F_i) / card(U)$ (*card* means cardinality of sets) is called approximate tolerance degree.

Obviously, if $I(\cap F_i, F) = \lambda_L$, then for any $u \in U$, the reliability of $\mu_{\cap F_i}(u) \le \mu_F(u)$ is λ_L; if $I(F, \cup F_i) = \lambda_U$, then for any $u \in U$, the reliability of $\mu_F(u) \le \mu_{\cup F_i}(u)$ is λ_U; if $\lambda_L = \lambda_U = 1$, then for any $u \in U$,

$$\mu_{\cap F_i}(u) \le \mu_F(u) \le \mu_{\cup F_i}(u). \quad (12)$$

Formula (12) helps to approximate $\mu_F(u)$ in the case of $\mu_{\cap F_i}(u) > 0$ or $\mu_{\cup F_i}(u) < 1$. The tolerance degree T_F is a measure of characterizing upper and lower approximation. We hope T_F is very small.

C. Improved Fuzzy Rough Set

Suppose $F \in \Psi(U)$ is λ-approximate fuzzy set in space (U, Φ, I), the target is to find $\Phi^*, \Phi_* \subseteq \Phi$ such that

$$\min\left(I(\bigcap_{F_i\in\Phi} F_i, F), I(F, \bigcup_{F_i\in\Phi} F_i) \right) \ge \lambda$$

and

$$card(\bigcup_{F_i\in\Phi} F_i - \bigcap_{F_i\in\Phi} F_i)$$

is minimal. In other words, we want to find the intersection of the fuzzy sets $\Phi = \{F_1, F_2, \cdots, F_n\}$ as big as possible and the degree of the intersection contained in F is at least λ and to find the union of the fuzzy sets as small as possible and the degree of the union containing F is not less than λ.

Definition 3: Suppose $F \in \Psi(U)$ is λ-approximate fuzzy set in space (U, Φ, I), let

$$L(F) = \{A \subseteq \Phi : I(\bigcap_{F_i\in A} F_i, F) \ge \lambda\}$$

and

$$U(F)=\{B\subseteq\Phi : I(F,\bigcup_{F_i\in B}F_i)\geq\lambda\}.$$

If $A_*\in L(F)$, $B^*\in U(F)$ and

$$card\left(\bigcup_{F_i\in B^*}F_i-\bigcap_{F_i\in A_*}F_i\right)=\min_{A\in L(F),B\in U(F)} card\left(\bigcup_{F_i\in B}F_i-\bigcap_{F_i\in A}F_i\right).$$

We say $\underline{\Phi}_\lambda(F)=\bigcap_{F_i\in A_*}F_i$ and $\overline{\Phi}_\lambda(F)=\bigcup_{F_i\in B_*}F_i$ are upper and lower approximations of F in (U,Φ,I) and the pair $(\underline{\Phi}_\lambda(F),\overline{\Phi}_\lambda(F))$ is improved fuzzy rough set.

According to Definition 3, the reliability of $\min_{F_i\in A_*}\mu_{F_i}(u)\leq\mu_F(u)\leq\max_{F_i\in B^*}\mu_{F_i}(u)$ or $\underline{\Phi}_\lambda(F)\subseteq F\subseteq\overline{\Phi}_\lambda(F)$ is λ that provides evidences for estimating $\mu_F(u)$. The tolerance degree

$$T_{\Phi_\lambda}=card(\underline{\Phi}_\lambda(F)-\overline{\Phi}_\lambda(F))/card(U)$$

characterizes an approximate quality.

V. Conclusions

In this paper, we have shown the theory of fuzzy rough set. Fuzzy equivalence class on $\triangle$-transitive compatible relation is introduced, a special $\triangle$-transitivity, *Tm*-fuzzy transitive compatible relation in information systems is created and it may become the breakthrough of generalizing classical rough set theory , fuzzy rough set and its improvement on such a basis are discussed respectively.

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