Rough Group, Rough Subgroup and Their Properties

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Abstract. The theory of rough sets is an extension of the set theory, for the study of intelligent systems characterized by insufficient and incomplete information. Since proposed by Pawlak, rough sets have evoked a lot of research. Theoretic study has included algebra aspect of rough sets. In paper [1] the concept of rough group and rough subgroup was introduced, but with some deficiencies remaining. In this paper, we intend to make up for these shortages, improve definitions of rough group and rough subgroup, and prove their new properties.

1 Introduction

Pawlak proposed the rough set theory in 1982. In recent years, there has been a fast growing interest in this new emerging theory – ranging from work in pure theory, such as e.g. topological and algebraic foundations [4], [5], [6], [7], to diverse areas of applications.

In [2], based on Pawlak's definition of rough equality, as well as the works by Orlowska [8] and Banerjee and Chakraborty [9], the algebraic technique was used to give a deep mathematical meaning to the rough set theory. The concepts of topological Boolean algebra, rough algebra, quasi-Boolean algebra, topological quasi-Boolean algebra and topological rough algebra, etc., were introduced. Furthermore, it was proved that rough algebra is not a Boolean algebra, but a quasi-Boolean algebra. In [3], based on the lattice theoretical approach suggested by Iwinski [10], it was proved that the original family of rough sets $\Re^\circ$, as well as $\Re$ and the rough approximation space $\Re^*$ with corresponding operations, are stone algebras. However, for groups, rings and fields, little work has been done.

In [1], the concepts of rough group and rough subgroup have been introduced, but there are some parts that remain irrational. First, the definition of rough group based on G, if G is a rough set, $\forall x$, $y \in$G, $x * y \in$G, is vague. Second, in the definition of second property of rough group, there are some items which are unreasonable as concerning rough group G: (1) the range in which the association law holds is G; (2) the set e with the binary operation defined on U is trivial rough subgroup of G; (3) the intersection of two rough subgroups of a rough

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group is still a rough subgroup. The main work of this paper is to improve the parts that mentioned above, then, to prove some properties of rough group and rough subgroup. In section 2, a short overview of the work by Pawlak that relates to this paper is given; in section 3, new definitions of rough group and rough subgroup are introduced; in section 4, rough right cosets and rough left cosets are defined and some properties of rough cosets are proved; in section 5, the definition of rough invariant sets is given and some of its properties are proved; in section 6, the homomorphism and isomorphism of rough group are introduced; in section 7, some examples for rough group, rough subgroup and their properties are given; and in the last, further study need to be done in this field and the significance of this research work are given in the conclusion part.

2 The Basic Theory of Rough Sets

Definition 1. *Let U be a finite non-empty set called universe and R be a family equivalence relation on U. The pair (U, R) is called an approximation space, denoted by $K = (U, R)$.*

Definition 2. *Let U be a universe, C be a family of subsets of U, $C = \{X_1, X_2, \ldots, X_n\}$. C is called a classification of U if the following properties are satisfied:*

(1) $X_1 \cup X_2 \cup \ldots \cup X_n = U$;
(2) $X_i \cap X_j = (i \neq j)$.

Definition 3. *Let U be a universe and R be an equivalence relation on U. We denote the equivalence class of object x in R by $[x]_R$. The set $\{[x]_R | x \in U\}$ is called a classification of U induced by R.*

Definition 4. *Let (U, R) be an approximation space and X be a subset of U. The sets*

(1) $\overline{X} = \{x | [x]_R \cap X \neq \}$;
(2) $\underline{X} = \{x | [x]_R \subseteq X\}$;
(3) $BN(X) = \overline{X} - \underline{X}$

are called upper approximation, lower approximation, and boundary region of X in K, respectively.

Property 1. Let $X, Y \subset U$, where U is a universe. The following properties hold:

(1) $\underline{X} \subset X \subset \overline{X}$
(2) $= = , \underline{U} = \overline{U} = U$
(3) $\underline{X \cap Y} = \underline{X} \cap \underline{Y}$
(4) $\overline{X \cap Y} \subset \overline{X} \cap \overline{Y}$
(5) $\underline{X} \cup \underline{Y} \subset \underline{X \cup Y}$
(6) $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$
(7) $X \subset Y$ if and only if $\underline{X} \subset \underline{Y}, \overline{X} \subset \overline{Y}$

3 Rough Group and Rough Subgroup

Definition 5. *Let $K = (U, R)$ be an approximation space and $*$ be a binary operation defined on U. A subset G of universe U is called a rough group if the following properties are satisfied:*

(1) $\forall x, y \in G, x * y \in \overline{G}$*;*
(2) Association property holds in $\overline{G}$*;*
(3) $\exists e \in \overline{G}$ *such that* $\forall x \in G$, $x * e = e * x = x$*;* e *is called the rough identity element of rough group* G*;*
(4) $\forall x \in G, \exists y \in G$ *such that* $x * y = y * x = e$*;* y *is called the rough inverse element of* x *in* G*.*

Property 2. (1) There is one and only one identity element in rough group G.
(2) $\forall x \in G$, there is only one y such that $x * y = y * x = e$; we denote it by x^{-1}.

Property 3. (1) $(x^{-1})^{-1} = x$.
(2) $(x * y)^{-1} = y^{-1} * x^{-1}$.

Property 4. Elimination law holds in G, i.e. $\forall a, x, x', y, y' \in G$,

(1) if $a * x = a * x'$ then $x = x'$.
(2) if $y * a = y' * a$ then $y = y'$.

Definition 6. *A non-empty subset H of rough group G is called its rough subgroup, if it is a rough group itself with respect to operation $*$.*

There is only one guaranteed trivial rough subgroup of rough group G, i.e. G itself. A necessary and sufficient condition for $\{e\}$ to be a trivial rough subgroup of rough group G is $e \in G$.

Theorem 1. *A necessary and sufficient condition for a subset H of a rough group G to be a rough subgroup is that:*

(1) $\forall x, y \in H, x * y \in \overline{H}$*;*
(2) $\forall x \in H, x^{-1} \in H$*.*

Proof. The necessary condition is obvious. We prove only the sufficient condition. By (1) we have $\forall x, y \in H, x * y \in \overline{H}$, by (2) we have $\forall x \in H, x^{-1 \in H}$, by (1) and (2) we have $\forall x \in H, x * x^{-1} = e \in \overline{H}$, because association holds in $\overline{G}$, so it holds in $\overline{H}$. Hence the theorem is proved.

Another difference between rough group and group is the following:

Theorem 2. *Let H_1 and H_2 be two rough subgroups of the rough group G. A sufficient condition for intersection of two rough subgroups of a rough group to be a rough subgroup is* $\overline{H_1} \cap \overline{H_2} = \overline{H_1 \cap H_2}$*.*

Proof. Suppose H_1 and H_2 are two rough subgroups of G. It is obvious that $H_1 \cap H_2 \subset G$. Consider $x, y \in H_1 \cap H_2$. Because H_1 and H_2 are rough subgroups, we have $x * y \in \overline{H_1}$, $x * y \in \overline{H_2}$, and $x^{-1} \in H_1, x^{-1} \in H_2$, i.e. $x * y \in \overline{H_1} \cap \overline{H_2}$ and $x^{-1} \in H_1 \cap H_2$. Assuming $\overline{H_1} \cap \overline{H_2} = \overline{H_1 \cap H_2}$, we have $x * y \in \overline{H_1 \cap H_2}$ and $x^{-1} \in H_1 \cap H_2$. Thus $H_1 \cap H_2$ is a rough subgroup of G.

Definition 7. *A rough group is called a commutative rough group if for every $x, y \in G$, we have $x * y = y * x$.*

4 Rough Coset

Let (U, R) be a universe, $G \subset U$ be a rough group and H be a rough subgroup of G. Let us define a relationship of elements of rough group G as follows:

$\sim$: $a \sim b$ if and only if $a * b^{-1} \in H \cup \{e\}$.

Theorem 3. *"$\sim$" is a compatible relation over elements of rough group G.*

Proof. $\forall a \in G$, since G is a rough group, $a^{-1} \in G$. Since $a * a^{-1} = e$, we have $a \sim a$. Further, $\forall a, b \in G$, if $a \sim b$, then $a * b^{-1} \in H \cup \{e\}$ i.e. $a * b^{-1} \in H$ or $a * b^{-1} \in \{e\}$. If $a * b^{-1} \in H$, then, since H is a rough subgroup of G, we have $(a * b^{-1})^{-1} = b * a^{-1} \in H$, thus $b \sim a$. If $a * b^{-1} \in \{e\}$, then $a * b^{-1} = e$. That means $b * a^{-1} = (a * b^{-1})^{-1} = e^{-1} = e$, thus $b \sim a$. Hence, "$\sim$" is compatible.

Definition 8. *Compatible category defined by relation "$\sim$" is called rough right coset. Rough right coset that contains element a is denoted by $H * a$, i.e.*

$H * a = \{h * a | h \in H, a \in G, h * a \in G\} \cup \{a\}$.

Let (U, R) be an approximation space, $G \subset U$ be a rough group and H be its rough subgroup. Consider relation of elements of G defined as follows:

$\sim'$: $a \sim' b$ if and only if $a^{-1} * b \in H \cup \{e\}$.

Theorem 4. *"$\sim'$" is a compatible relation over elements of rough group G.*

Definition 9. *Compatible category defined by relation "$\sim'$" is called rough left coset. Rough left coset that contains element a is denoted by $a * H$, i.e.*

$a * H = \{a * h | h \in H, a \in G, a * h \in G\} \cup \{a\}$.

Remark: Generally speaking, the binary operation of rough group dissatisfies commutative law, so the compatible relations "$\sim$" and "$\sim'$" are different. As a result, the rough left and right cosets are also different.

Theorem 5. *The rough left cosets and rough right cosets are equal in number.*

Proof. Denote by S_1, S_2 the families of rough right and left cosets, respectively. Define $\varphi : S_1 \to S_2$ such that $\varphi(H * a) = a^{-1} * H$. We prove that φ is bijection.

1. If $H * a = H * b$ $(a \neq b)$, then $a * b^{-1} \in H$. Because H is a rough subgroup, we have $b * a^{-1} \in H$, that means $a^{-1} \in b^{-1} * H$, i.e. $a^{-1} * H = b^{-1} * H$. Hence, φ is a mapping.
2. Any element $a * H$ of S_2 is the image of $H * a^{-1}$ – the element of S_1. Hence, φ is onto mapping.
3. If $H * a \neq H * b$, then $a * b^{-1} \notin H$, i.e. $a^{-1} * H \neq b^{-1} * H$. Hence, φ is a one-to-one mapping.

Thus the rough left cosets and rough right cosets are equal in number.

Definition 10. *The number of both rough left cosets and rough right cosets is called index of subgroup H in G.*

5 Rough Invariant Subgroup

Definition 11. *A rough subgroup N of rough group G is called a rough invariant subgroup, if* $\forall a \in G, a * N = N * a$.

Theorem 6. *A necessary and sufficient condition for a rough subgroup N of rough group G to be a rough invariant subgroup is that* $\forall a \in G, a * N * a^{-1} = N$.

Proof. Suppose N is a rough invariant subgroup of G. By definition, $\forall a \in G$, we have $a * N = N * a$. Because G is a rough group, we have

$(a * N) * a^{-1} = (N * a) * a^{-1}$
$a * N * a^{-1} = N * (a * a^{-1})$
i.e. $a * N * a^{-1} = N$.

Suppose N is a rough subgroup of G and $\forall a \in G, a * N * a^{-1} = N$. Then

$(a * N * a^{-1}) * a = N * a$
i.e. $a * N = N * a$.

Thus N is a rough invariant subgroup of G.

Theorem 7. *A necessary and sufficient condition for a rough subgroup N of G to be a rough invariant subgroup is that $\forall a \in G$ and $n \in N$, $a * n * a^{-1} \in N$.*

Proof. Suppose N is a rough invariant subgroup of rough group G. We have

$\forall a \in G, a * N * a^{-1} = N$.

For any $n \in N$, we therefore have

$a * n * a^{-1} \in N$.

Suppose N is a rough subgroup of rough group G. Suppose $\forall a \in G$, $n \in N$, $a * n * a^{-1} \in N$. We have

$a * N * a^{-1} \subset N$

Because $a^{-1} \in G$, we further have

$a^{-1} * N * a \subset N$

It follows that

$a * (a^{-1} * N * a) * a^{-1} \subset a * N * a^{-1}$
i.e. $N \subset a * N * a^{-1}$

Since $a * N * a^{-1} \subset N$ and $N \subset a * N * a^{-1}$, we have $a * N * a^{-1} = N$. Thus N is a rough invariant subgroup.

6 Homomorphism and Isomorphism of Rough Group

Let $(U_1, R_1), (U_2, R_2)$ be two approximation spaces, and $*, \overline{*}$ be binary operations over universes U_1 and U_2, respectively.

Definition 12. *Let $G_1 \subset U_1, G_2 \subset U_2$. G_1, G_2 are called rough homomorphism sets if there exists a surjection $\varphi : \overline{G_1} \to \overline{G_2}$ such that*

$\forall x, y \in \overline{G_1}, \varphi(x * y) = \varphi(x) \overline{*} \varphi(y)$.

Theorem 8. *Let G_1 and G_2 be rough homomorphism sets. If $*$ satisfies commutative law, then $\overline{*}$ also satisfies it.*

Proof. Consider G_1, G_2, and φ such that $\forall x, y \in \overline{G_1}, \varphi(x * y) = \varphi(x)\overline{*}\varphi(y)$. For every $\varphi(x), \varphi(y) \in \overline{G_2}$, since φ is surjection, there exist $x, y \in \overline{G_1}$ such that $x \to \varphi(x), y \to \varphi(y)$. Thus $\varphi(x * y) = \varphi(x)\overline{*}\varphi(y)$, and $\varphi(y * x) = \varphi(y)\overline{*}\varphi(x)$. Now, assuming $x * y = y * x$, we obtain $\varphi(x)\overline{*}\varphi(y) = \varphi(y)\overline{*}\varphi(x)$. That means that $*$ satisfies commutative law.

Theorem 9. *Let $G_1 \subset U_1, G_2 \subset U_2$ be rough groups that are rough homomorphism and let $\overline{\varphi(G_1)} = \overline{G_2}$. Then $\varphi(G_1)$ is also a rough group.*

Proof. 1. $\forall x', y' \in \varphi(G_1)$, consider $x, y \in G_1$ such that $x \to x', y \to y'$. We have $\varphi(x * y) = \varphi(x)\overline{*}\varphi(y) \in \overline{G_2} = \overline{\varphi(G_1)}$, that is $x'\overline{*}y' \in \overline{\varphi(G_1)}$.
2. Since $e \in \overline{G_1}$, $\varphi(e) \in \overline{G_2}$ and $\forall \varphi(x) \in \varphi(G_1), \varphi(e)\overline{*}\varphi(x) = \varphi(x * e) = \varphi(x)$
3. G_1 is a rough group, so $\forall x, y, z \in G_1, x * (y * z) = (x * y) * z$. Hence:
$\varphi(x * (y * z)) = \varphi(x)\overline{*}\varphi(y * z) = \varphi(x)\overline{*}(\varphi(y)\overline{*}\varphi(z))$
$\varphi((x * y) * z)) = \varphi(x * y)\overline{*}\varphi(z) = (\varphi(x)\overline{*}\varphi(y))\overline{*}\varphi(z)$
i.e. $(\varphi(x)\overline{*}\varphi(y))\overline{*}\varphi(z)\varphi(x)\overline{*}(\varphi(y)\overline{*}\varphi(z))$.
4. $\forall x' \in \varphi(G_1)$, consider $x \in G_1$ such that $x \to x'$. Since G_1 is a rough group, $x^{-1} \in G_1$. Hence $\varphi(x^{-1}) \in \varphi(G_1)$ and $\varphi(x)\overline{*}\varphi(x^{-1}) = \varphi(x^{-1})\overline{*}\varphi(x) = \varphi(e)$. Therefore, we can put $(x')^{-1} = \varphi(x^{-1})$. Consequently, we can conclude that $\varphi(G_1)$ is a rough group.

Theorem 10. *Let $G_1 \subset U_1, G_2 \subset U_2$ be rough groups that are rough homomorphism. Let e and $\overline{e}$ be rough identity elements of G_1 and G_2 respectively. Then $\varphi(e) = \overline{e}$ and $\varphi(a^{-1}) = \varphi(a)^{-1}$.*

Definition 13. *Let $G_1 \subset U_1, G_2 \subset U_2$ be rough groups that are rough homomorphism. Let e and $\overline{e}$ be rough identity elements of G_1 and G_2 respectively. The set $\{x|\varphi(x) = \overline{e}, x \in G_1\}$ is called rough homomorphism kernel, denoted by N.*

Theorem 11. *Let $G_1 \subset U_1, G_2 \subset U_2$ be rough groups that are rough homomorphism. Rough homomorphism kernel N is a rough invariant subgroup of G_1.*

Proof. Let φ be onto mapping from $\overline{G_1}$ to $\overline{G_2}$. $\forall x, y \in N$ we have $\varphi(x) = \overline{e}, \varphi(y) = \overline{e}$. Thus $\varphi(x * y) = \varphi(x)\overline{*}\varphi(y) = \overline{e} \mathbin{\overline{*}} \overline{e} = \overline{e}$, i.e. $x * y \in N$. Moreover, $\forall x \in N$, we have $\varphi(x) = \overline{e}$. Because $\varphi(x^{-1}) = \varphi(y)^{-1} = \overline{e}^{-1} = \overline{e}$, we get $x^{-1} \in N$. We can conclude that N is a rough invariant subgroup of G_1.

Theorem 12. *Let $G_1 \subset U_1, G_2 \subset U_2$ be rough groups that are rough homomorphism. Let H_1, N_1 be rough subgroup and rough invariant subgroup of G_1, respectively. Then:*

(1) $\varphi(H_1)$ is rough subgroup of G_2 if $\varphi(\overline{H_1}) = \overline{\varphi(H_1)}$;
(2) $\varphi(N_1)$ is rough invariant subgroup of G_2 if $\varphi(G_1) = G_2$ & $\varphi(\overline{N_1}) = \overline{\varphi(N_1)}$.

Proof. (1):
Consider an onto mapping φ from $\overline{G_1}$ to $\overline{G_2}$ such that
$\forall x, y \in \overline{G_1}, \varphi(x * y) = \varphi(x)\overline{*}\varphi(y)$.
$\forall \varphi(x), \varphi(y) \in \varphi(H_1)$, by the definition of φ, there exists $x, y \in H_1$ such that
$x \to \varphi(x)$ and $\varphi(x)\overline{*}\varphi(y) = \varphi(x * y) \in \varphi(\overline{H_1})$.

Because $\varphi(\overline{H_1}) = \overline{\varphi(H_1)}$, we have

$\varphi(x)\overline{*}\varphi(y) \in \varphi(\overline{H_1})$.

Further, $\forall \varphi(x) \in \varphi(H_1)$, by the definition of φ, there exists $x \in H_1$ such that

$x \rightarrow \varphi(x), y \rightarrow \varphi(y)$.

Because H_1 is a rough subgroup of G_1, we have

$x^{-1} \in H_1$.

Thus

$\varphi(a)^{-1} = \varphi(a^{-1}) \in \varphi(H_1)$.

We can conclude that $\varphi(H_1)$ is a rough subgroup of G_2.

Proof. (2):

By (1), it is easy to see that $\varphi(N_1)$ is a rough subgroup of G_2 if

$\varphi(\overline{N_1}) = \overline{\varphi(N_1)}$.

$\forall \varphi(x) \in G_2$, because $\varphi(G_1) = G_2$, we have

$\varphi(x) \in \varphi(G_1)$.

Thus

$x \in G_1$, $x^{-1} \in G_1$ and $\varphi(x^{-1}) \in \varphi(G_1) = G_2$.

Because $\forall \varphi(x) \in G_2, \varphi(n) \in \varphi(N_1)$ there is

$\varphi(x)\overline{*}\varphi(n)\overline{*}\varphi(x^{-1}) = \varphi(x * n * x^{-1})$

and N_1 is rough invariant subgroup of G_1, we have

$x * n * x^{-1} \in N_1$.

Hence

$\varphi(x)\overline{*}\varphi(n)\overline{*}\varphi(x^{-1}) \in \varphi(N_1)$.

We can conclude that $\varphi(N_1)$ is a rough invariant subgroup of G_1.

Theorem 13. *Let $G_1 \subset U_1, G_2 \subset U_2$ be rough groups that are rough homomorphism. Let H_2, N_2 be rough subgroup and rough invariant subgroup of G_2 respectively. Then*

(1) H_1 which is the inverse image of H_2 is rough subgroup of G_1 if $\varphi(\overline{H_1}) = \overline{H_2}$
(2) N_1 which is the inverse image of N_2 is rough invariant subgroup of G_1 if $\varphi(G_1) = G_2$ & $\varphi(\overline{N_1}) = \overline{N_2}$.

Proof. (2):

Because H_1 is the inverse image of H_2, we have

$\varphi(H_1) = H_2$.

That is, $\forall x, y \in H_1$, we have

$\varphi(x), \varphi(y) \in H_2$.

Because H_2 is a rough subgroup of G_2, we have

$\varphi(x * y) = \varphi(x)\overline{*}\varphi(y) \in \overline{H_2} = \varphi(\overline{H_1})$.

Thus

$x * y \in \overline{H_1}$.

$\forall x \in H_1$, we have

$\varphi(x) \in H_2$.

Because H_2 is a rough subgroup of G_2, we have

$\varphi(x)^{-1} = \varphi(x^{-1}) \in H_2$.

Thus $x^{-1} \in H_1$.

Proof. (2):
By (1), it is easy to know that N_1 is a rough subgroup of G_2 if
$\varphi(\overline{N_1}) = \overline{\varphi(N_1)}$.
$\forall x \in G_1, n \in N_1$, we have
$\varphi(x) \in \varphi(G_1) = G_2, \varphi(x)^{-1} = \varphi(x^{-1}) \in \varphi(G_1) = G_2, \varphi(n) \in N_2$
Because N_2 is a rough invariant subgroup of G_2, we have
$\varphi(x)\overline{*}\varphi(n)\overline{*}\varphi(x^{-1}) = \varphi(x * n * x^{-1}) \in N_2$.
Thus
$x * n * x^{-1} \in N_1$.
Hence N_1 which is the inverse image of N_2 is a rough invariant subgroup of G_1 if $\varphi(G_1) = G_2$ and $\varphi(\overline{N_1}) = \overline{N_2}$.

7 Examples

Example 1. Let U be the set of all permutation of S_4 and $*$ be the multiplication operation of permutation. A classification of U is $U/R = \{E_1, E_2, E_3, E_4\}$, where
$E_1 = \{(1), (12), (13), (14), (23), (24), (34)\}$,
$E_2 = \{(123), (132), (124), (142), (134), (143), (234), (243)\}$,
$E_3 = \{(1234), (1243), (1324), (1342), (1423), (1432)\}$,
$E_4 = \{(12)(34), (13)(24), (14)(23)\}$,
Let $X_1 = \{(1), (12), (13)\}$, then
$\overline{X_1} = \{(1), (12), (13), (14), (23), (24), (34)\}$.
Because $(12) * (13) = (123) \notin \overline{X_1}$, we have X_1 is not a rough group.
Let $X_2 = \{(12), (123), (132)\}$, then
$\overline{X_2} = E_1 \cup E_2$.
Because

(1) $\forall x, y \in X_2, x * y \in \overline{X_2}$;
(2) $(12) * (12) = (1) \in \overline{X_2}$;
(3) Association property holds in $\overline{X_2}$;
(4) $(12)^{-1} = (12) \in X_2$, $(123)^{-1} = 132 \in X_2, (132)^{-1} = (123) \in X_2$.

Thus we have that X_2 is a rough group.
Let $X_3 = \{(1), (123), (132)\}$, then $\overline{X_3} = E_1 \cup E_2$.
Because

(1) $X_3 \subset X_2$;
(2) $\forall x, y \in X_3, x * y \in \overline{X_3}$;
(3) $(1)^{-1} = (1) \in X_3, (123)^{-1} = (132) \in X_3, (13)^{-1} = (13) \in X_3$;

Thus we have that X_3 is a rough subgroup of rough group X_2¡£

Example 2. Let $U = \{[0], [1], [2], [3], [4], [5], [6], [7], [8], [9]\}$ be a set of surplus class with respect to module 9 and $*$ be the plus of surplus class. A classification of U is $U/R = \{E_1, E_2, E_3\}$, where $E_1 = \{[0], [1], [2]\}$, $E_2 = \{[3], [4], [5]\}$, $E_3 = \{[6], [7], [8]\}$,
Let $X_1 = \{[2], [7], [8], [1]\}$, then $\overline{X_1} = E_1 \cup E_2$. Because $[2] * [1] = [3] \notin \overline{X_1}$, thus we have X_1 is not a rough group.

Let $X_2 = \{[2], [7], [5], [4]\}$, then $\overline{X_2} = E_1 \cup E_2 \cup E_3 = U$. Because

(1) $\forall x, y \in X_2, x * y \in \overline{X_2}$;
(2) Association property holds in $\overline{X_2}$;
(3) $[2] * [7] = 0 \in \overline{X_2}$;
(4) $[2]^{-1} = [7] \in X_2, [7]^{-1} = [2] \in X_2, [5]^{-1} = [4] \in X_2, [4]^{-1} = [5] \in X_2$;

We have thatX_2 is a rough group.

Let $X_3 = \{[2], [3], [6], [7]\}$, then $\overline{X_3} = E_1 \cup E_2 \cup E_3 = U$.
Because

(1) $\forall x, y \in X_3, x * y \in \overline{X_3}$;
(2) Association property holds in $\overline{X_3}$;
(3) $[0] \in \overline{X_3}$
(4) $[2]^{-1} = [7] \in X_3, [7]^{-1} = [2] \in X_3, [3]^{-1} = [6] \in X_3, [6]^{-1} = [3] \in X_3$;

We have that X_3 is a rough group.

Let $X_4 = X_2 \cap X_3 = \{[2], [7]\}$ then $\overline{X_4} = E_1 \cup E_3$
Because $[2] * [2] = [4] \notin X_4$, we have X_4 is not a rough subgroup of rough group X_2 or rough group X_3.

Example 3. Let U be the set of all permutation of S_4 and $*$ be the multiplication operation of permutation. A classification of U is $U/R = \{E_1, E_2, E_3, E_4, E_5\}$, where

$E_1 = \{(1), (123), (132)\}$,
$E_2 = \{(12), (13), (23), (14), (24), (34)\}$,
$E_3 = \{(124), (142), (134), (143), (234), (243)\}$,
$E_4 = \{(1234), (1243), (1324), (1342), (1423), (1432)\}$,
$E_5 = \{(12)(34), (13)(24), (14)(23)\}$,

Let $G = \{(12), (13), (123), (132)\}$, $N = (123), (132)$, then

$\overline{G} = (1), (123), (132), (12), (13), (23), (14), (24), (34)$ $\overline{N} = (1), (123), (132)$

It is easy to prove that G is a rough group and N is a rough subgroup of rough group G. Because

$(12) * N = (13), (23) = N * (12) = (23), (13)$
$(13) * N = (23), (12) = N * (13) = (12), (23)$
$(123) * N = (132), (1) = N * (123) = (132), (1)$
$(132) * N = (1), (123) = N * (132) = (1), (123)$

We have N is a rough invariant subgroup.

8 Conclusion

In this paper we have shown that the theory of rough sets can be applied to the algebra systems – groups. We have also improved some deficiencies of the approach proposed in [1], and proved some new properties of rough groups and rough subgroups. Following this, a lot of work should be still done continually, such as e.g. an extension of theory of rough sets to rough rings and rough fields. Further studies in this direction will enable to understand better the connections between relatively novel ideas of the theory of rough sets and already well-known algebraic approaches.

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