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Isomorphic Relationship Between L-three-way Concept Lattices

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Abstract
The three-way decision (3WD) creates a new perspective for decision-making by adding a third option in addition to acceptance and rejection. The decision cost, caused by the yes-or-no decision pattern, is avoided. The 3WD is a human-cognition-inspired problem-solving pattern which offers new theories, models, and tools for cognitive analytics. Formal concept analysis, as a method proposed to mine hidden patterns in data, can only deal with binary-valued data when it appeared. To process more types of data, **L**-concept analysis, where **L** represents a truth-value structure, is presented with the generation of various **L**-two-way (**L**2W) and **L**-three-way (**L**3W) concept lattices. The aim in this study is to explore the relationship between various **L**3W concept lattices that have not been represented by any existing theorems. To fulfill this goal, first, the relationship between **L**2W concept lattices is examined, and then, the relationship between **L**3W concept lattices is analysed. Finally, the relationship between the **L**2W and **L**3W concepts is revealed. The results show that the eight types of **L**2W concept lattices form two isomorphic groups. The four types of **L**-object-induced three-way concept lattices, as well as the four types of **L**-attribute-induced three-way concept lattices, are isomorphic respectively. In addition, the equivalent relationship between **L**3W concepts and **L**2W concepts provides a way to construct **L**3W concept lattices based on **L**2W concept lattices.

Keywords **L**-two-way concept · **L**-two-way concept lattice · **L**-three-way concept · **L**-three-way concept lattice

Introduction

In human cognition, concepts are the fundamental units that humans utilise to understand the world and solve problems. Obtaining concepts has always been a fundamental problem in artificial intelligence. A concept is a granule in the view of granular computing. Commonly used granular computing models are rough sets [1, 2], fuzzy sets [3], formal concept analysis (FCA) [4, 5], cloud model [6], etc. The three-way decision (3WD), which is another granular computing model, was first proposed by Yao [7] to overcome the shortcomings of binary decision-making patterns. 3WD explores thinking, problem-solving, and information processing in threes (namely, three parts or items); it offers new theories, models, and tools for cognitive analytics. Rapid growth of 3WD has been observed both in its theory and application [8–15].

Proposed by Wille [5], FCA provides a way to generate concepts from formal contexts by introducing a pair of concept-forming operators that form a Galois connection between the power set of the object universe and the power set of the attribute universe. The formal context is the basic notion of FCA, and it is a triple constituted by a set of objects, a set of attributes, and a binary relation characterising whether an object has an attribute. Imitating the dynamic process of human cognition, Mi et al. [16] proposed the semi-supervised concept learning method for dynamic semi-supervised learning by employing concept spaces. Li et al. [17] introduced a cognitive mechanism of forming concepts based on the principles from philosophy and cognitive psychology. Other applications of FCA can be found in knowledge reduction [18–21], decision-making [22], online social networks [23], medical diagnosis [24, 25], image processing [26], etc.

FCA can only handle binary-valued data [27–33]. To improve data processing, Burusco and Fuentes-González [34] generalised the concept-forming operators

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Table 1 Brief introduction to symbol abbreviations

Abbreviation	Original meaning	Abbreviation	Original meaning
L2W	**L**-two-way	$\mathbf{L}^{?}$-2W	$\mathbf{L}^{?}$-two-way
LO2W	**L**-object-induced two-way	$\mathbf{L}^{?}$-O2W	$\mathbf{L}^{?}$-object-induced two-way
LA2W	**L**-attribute-induced two-way	$\mathbf{L}^{?}$-A2W	$\mathbf{L}^{?}$-attribute-induced two-way
L3W	**L**-three-way	$\mathbf{L}^{?}$-3W	$\mathbf{L}^{?}$-three-way
LO3W	**L**-object-induced three-way	$\mathbf{L}^{?}$-O3W	$\mathbf{L}^{?}$-object-induced three-way
LA3W	**L**-attribute-induced three-way	$\mathbf{L}^{?}$-A3W	$\mathbf{L}^{?}$-attribute-induced three-way

Here, '?' represents a type of concept-forming operator or a combination of concept-forming operators, such as $*$, $\bar{*}$, $\overline{\Box}$, $\Diamond$, and $\overline{\Box}\Diamond$

to fuzzy cases to obtain fuzzy concepts or **L**-concepts. They adopted a complete lattice as a truth-value set and defined the concept-forming operators based on the t-conorm. Subsequently, Belohlavek [35] used a residuated lattice as a truth-value set and defined the concept-forming operators based on residual implications. Such generalisations are called fuzzy concept analysis or **L**-concept analysis (**L** CA), where **L** represents the truth-value structure, similar to a residuated lattice. Fan, Zhang, and Xu [36] summarised three types of fuzzy concepts (namely, fuzzy formal concept [35], fuzzy object-oriented formal concept [37], and fuzzy property-oriented formal concept [37]) and examined a new form of fuzzy concept (namely, dual fuzzy formal concept). Other methods have been proposed to study **L** CA, such as the variable threshold concept [38], *L*-fuzzy *W*-concept [39], *L*-fuzzy biconcept [40], one-sided formal concept [41], and three-way fuzzy concept [24, 42].

Three-way concept analysis (3WCA) [43–45] is another generalisation of FCA, and it combines FCA with 3WD. A three-way concept considers both attributes shared by objects and attributes not shared by objects. Various three-way concepts have been developed by considering different semantic meanings, such as OE-concept, AE-concept, OEO-concept, AEP-concept, OEP-concept, and OED-concept [43, 46, 47]. Li et al. [48] explored axiomatic approaches to characterise three-way concepts using multi-granularity. The aforementioned studies are for complete formal contexts, namely, symbolic data without missing values. There has been some excellent work on 3WCA with incomplete contexts (i.e. some information between objects and attributes is unknown) [18, 49–53]. However, those studies are not considered here as this study is only focused on complete contexts.

As the method of generalising classical operations to fuzzy cases is not unique, different methods have been proposed to investigate **L**-concepts. This study is focused only on the one introduced by Bězlohlávek [35]. Based on his idea, one can form eight types of **L**-two-way (**L**2W) concepts. Moreover, different **L**-three-way (**L**3W) concepts have been developed by applying the idea of 3WD. Although each type of **L**-concept has its semantic meaning and application, it is unnecessary to construct **L**-concept lattices separately (particularly considering that forming a concept lattice is an NP-hard problem). In addition, it is not necessary to respectively investigate the properties of different concept lattices. However, for the relationship between **L**-concept lattices, no universal representation theorems exist to date; this is the primary goal of this study. First, the isomorphic relationship between **L**2W concept lattices is explored by defining various **L**2W concepts. Then, the isomorphic relationship between **L**3W concept lattices is examined in the same manner. Finally, the relationship between the **L**2W concepts and **L**3W concepts is revealed. The results provide a way to construct **L**3W concept lattices based on **L**2W concept lattices.

The remainder of this paper is organised as follows. Section 2 presents a brief review of related notions, including residuated lattice, **L**-sets, and concept-forming operators in FCA. Sections 3 and 4 detail the main work in this study: investigating the relationships between **L**2W concept lattices and between **L**3W concept lattices. Section 5 reveals the relationship between **L**2W concepts and **L**3W concepts. Section 6 concludes this paper. For convenience of reading, Table 1 lists the abbreviations of the symbols.

Preliminaries

This section presents the notions and properties related to the residuated lattice, **L**-set, and formal context.

A complete residuated lattice, **L**, is an algebra $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$, where $(L, \vee, \wedge, 0, 1)$ is a complete lattice with the greatest element, 1, and the least element, 0, $(L, \otimes, 1)$ is a commutative monoid, and $(\otimes, \rightarrow)$ is an adjoint pair, satisfying $a \otimes b \leqslant c \Leftrightarrow a \leqslant b \rightarrow c$ for all $a, b, c \in L$ ($\leq$ is the order of L). For a complete residuated lattice, the following properties are valid.

Lemma 1 [54] *Let* $\mathbf{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ *be a complete residuated lattice. Then, for* $a, b, b_1, b_2, a_i \in L$ ($i \in \Lambda$, *where* Λ *is an index set*), *the following hold*:

1. $a \leqslant b \Leftrightarrow a \rightarrow b = 1$;
2. $b_1 \leqslant b_2 \Rightarrow a \otimes b_1 \leqslant a \otimes b_2$;
3. $b_1 \leqslant b_2 \Rightarrow a \rightarrow b_1 \leqslant a \rightarrow b_2, b_2 \rightarrow a \leqslant b_1 \rightarrow a$;
4. $\vee_{i\in\Lambda} a_i \rightarrow b = \wedge_{i\in\Lambda}(a_i \rightarrow b), \wedge_{i\in\Lambda} a_i \rightarrow b \geqslant \vee_{i\in\Lambda}(a_i \rightarrow b)$;
5. $a \leq (a \rightarrow b) \rightarrow b$.

For $a \in L$, a unary operator, referred to as the pre-complement operator, is defined as $\neg a = a \rightarrow 0$. If, for any $a \in L$, $\neg\neg a = a$, then $\mathbf{L}$ is a complete regular residuated lattice.

Lemma 2 [54, 55] *Let* $\mathbf{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ *be a complete regular residuated lattice. Then, for* $a, b, a_i \in L$ $(i \in \Lambda)$, *the following hold*:

1. $a \rightarrow \neg b = b \rightarrow \neg a, \neg a \rightarrow b = \neg b \rightarrow a, \neg a \rightarrow \neg b = b \rightarrow a$;
2. $a \otimes b = \neg(a \rightarrow \neg b), a \rightarrow b = \neg(a \otimes \neg b)$;
3. $\neg(\wedge_{i\in\Lambda} a_i) = \vee_{i\in\Lambda}(\neg a_i)$.

Let U be a finite universe. A mapping, $\tilde{A} : U \longrightarrow L$, is an $\mathbf{L}$-set on U. The family of all $\mathbf{L}$-sets on U is denoted by L^U. For two $\mathbf{L}$-sets, $\tilde{A}, \tilde{B} \in L^U$, the order is as follows:

$$\tilde{A} \subseteq \tilde{B} \Leftrightarrow \tilde{A}(x) \leq \tilde{B}(x), \forall x \in U. \tag{1}$$

The basic operations are defined pointwise as follows: *for* $x \in U$,

$$\begin{aligned}
&(\tilde{A} \cap \tilde{B})(x) = \tilde{A}(x) \wedge \tilde{B}(x),\\
&(\tilde{A} \cup \tilde{B})(x) = \tilde{A}(x) \vee \tilde{B}(x),\\
&\tilde{A}^c(x) = \neg\tilde{A}(x) = \tilde{A}(x) \rightarrow 0.
\end{aligned} \tag{2}$$

Given two pairs of $\mathbf{L}$-*sets*, $(\tilde{A}_1, \tilde{A}_2)$ *and* $(\tilde{B}_1, \tilde{B}_2)$, *we define*

$$\begin{aligned}
&(\tilde{A}_1, \tilde{A}_2) \cap (\tilde{B}_1, \tilde{B}_2) = (\tilde{A}_1 \cap \tilde{B}_1, \tilde{A}_2 \cap \tilde{B}_2),\\
&(\tilde{A}_1, \tilde{A}_2) \cup (\tilde{B}_1, \tilde{B}_2) = (\tilde{A}_1 \cup \tilde{B}_1, \tilde{A}_2 \cup \tilde{B}_2),\\
&(\tilde{A}_1, \tilde{A}_2)^c = (\tilde{A}_1^c, \tilde{A}_2^c).
\end{aligned} \tag{3}$$

The pairs of $\mathbf{L}$-*sets are ordered in the following way*:

$$(\tilde{A}_1, \tilde{A}_2) \subseteq (\tilde{B}_1, \tilde{B}_2) \Leftrightarrow \tilde{A}_1 \subseteq \tilde{B}_1,\ \tilde{A}_2 \subseteq \tilde{B}_2. \tag{4}$$

Let OB and AT be two finite universes of objects and attributes, *respectively*, *and R be a binary relation from OB to AT*. *For* $o \in OB$ *and* $a \in AT$, *oRa represents object o with attribute a*, *and* oR^ca *represents object o without attribute a*. *The triple*, (OB, AT, R), *is called a formal context* [5]. *Based on the connections between objects and attributes*, *one can define eight types of concept-forming operators* [12, 56]: *for* $O \subseteq OB$,

$$\begin{aligned}
O^{*} &= \{a \in AT \mid \forall o \in O\,(oRa)\}\\
&= \{a \in AT \mid O \subseteq [a]_R\},
\end{aligned} \tag{5}$$

$$\begin{aligned}
O^{\bar{*}} &= \{a \in AT \mid \forall o \in O\,(\neg(oRa))\}\\
&= \{a \in AT \mid O \subseteq [a]_{R^c}\},
\end{aligned} \tag{6}$$

$$\begin{aligned}
O^{\overline{\Box}} &= \{a \in AT \mid \forall o \in O^c\,(oRa)\}\\
&= \{a \in AT \mid O^c \subseteq [a]_R\},
\end{aligned} \tag{7}$$

$$\begin{aligned}
O^{\Box} &= \{a \in AT \mid \forall o \in O^c\,(\neg(oRa))\}\\
&= \{a \in AT \mid O^c \subseteq [a]_{R^c}\},
\end{aligned} \tag{8}$$

$$\begin{aligned}
O^{\Diamond} &= \{a \in AT \mid \exists o \in O\,(oRa)\}\\
&= \{a \in AT \mid O \cap [a]_R \neq \emptyset\},
\end{aligned} \tag{9}$$

$$\begin{aligned}
O^{\overline{\Diamond}} &= \{a \in AT \mid \exists o \in O\,(\neg(oRa))\}\\
&= \{a \in AT \mid O \cap [a]_{R^c} \neq \emptyset\},
\end{aligned} \tag{10}$$

$$\begin{aligned}
O^{\bar{\#}} &= \{a \in AT \mid \exists o \in O^c\,(oRa)\}\\
&= \{a \in AT \mid O^c \cap [a]_R \neq \emptyset\},
\end{aligned} \tag{11}$$

$$\begin{aligned}
O^{\#} &= \{a \in AT \mid \exists o \in O^c\,(\neg(oRa))\}\\
&= \{a \in AT \mid O^c \cap [a]_{R^c} \neq \emptyset\},
\end{aligned} \tag{12}$$

where $[a]_R$ *is a set of objects with attribute a*, *and* $[a]_{R^c}$ *is a set of objects without attribute a*. *In a similar way*, *one can define the eight types of operators for attribute set* $A \subseteq AT$. *Using these concept-forming operators*, *one can formulate different formal concepts or two-way concepts* (*see* [12, 56]).

Relationship Between L2W Concept Lattices

In this section, first, the crisp-set-based concept-forming operators are generalised to fuzzy cases, and then, various $\mathbf{L}$2W concepts are introduced based on different fuzzy-set-based concept-forming operators. Finally, the isomorphic relationship between $\mathbf{L}$2W concept lattices is revealed.

L2W Operators

Let $\tilde{R}$ be an $\mathbf{L}$-relation from OB to AT, i.e. $\tilde{R} : OB \times AT \longrightarrow L$, then $\mathbf{K} = (OB, AT, \tilde{R}, L)$ is an $\mathbf{L}$-(formal) context, and $\mathbf{K}^c = (OB, AT, \tilde{R}^c, L)$ is a dual $\mathbf{L}$-context of $\mathbf{K}$, where $\tilde{R}^c$ is the complement of $\tilde{R}$. Generalising Eqs. (5)–(12) from a fuzzy set setting can yield eight types of fuzzy-set-based concept-forming operators: for $\tilde{O} \in L^{OB}$, $\tilde{A} \in L^{AT}$, $o \in OB$, and $a \in AT$,

$$\tilde{O}^{*}(a) = \bigwedge_{o \in OB} (\tilde{O}(o) \to \tilde{R}(o, a)), \tag{13}$$

$$\tilde{A}^{*}(o) = \bigwedge_{a \in AT} (\tilde{A}(a) \to \tilde{R}(o, a)), \tag{14}$$

$$\tilde{O}^{\bar{*}}(a) = \bigwedge_{o \in OB} (\tilde{O}(o) \to \tilde{R}^{c}(o, a)), \tag{15}$$

$$\tilde{A}^{\bar{*}}(o) = \bigwedge_{a \in AT} (\tilde{A}(a) \to \tilde{R}^{c}(o, a)), \tag{16}$$

$$\tilde{O}^{\overline{\square}}(a) = \bigwedge_{o \in OB} (\tilde{O}^{c}(o) \to \tilde{R}(o, a)), \tag{17}$$

$$\tilde{A}^{\overline{\square}}(o) = \bigwedge_{a \in AT} (\tilde{A}^{c}(a) \to \tilde{R}(o, a)), \tag{18}$$

$$\tilde{O}^{\square}(a) = \bigwedge_{o \in OB} (\tilde{O}^{c}(o) \to \tilde{R}^{c}(o, a)), \tag{19}$$

$$\tilde{A}^{\square}(o) = \bigwedge_{a \in AT} (\tilde{A}^{c}(a) \to \tilde{R}^{c}(o, a)), \tag{20}$$

$$\tilde{O}^{\Diamond}(a) = \bigvee_{o \in OB} (\tilde{O}(o) \otimes \tilde{R}(o, a)), \tag{21}$$

$$\tilde{A}^{\Diamond}(o) = \bigvee_{a \in AT} (\tilde{A}(a) \otimes \tilde{R}(o, a)), \tag{22}$$

$$\tilde{O}^{\overline{\Diamond}}(a) = \bigvee_{o \in OB} (\tilde{O}(o) \otimes \tilde{R}^{c}(o, a)), \tag{23}$$

$$\tilde{A}^{\overline{\Diamond}}(o) = \bigvee_{a \in AT} (\tilde{A}(a) \otimes \tilde{R}^{c}(o, a)), \tag{24}$$

$$\tilde{O}^{\overline{\#}}(a) = \bigvee_{o \in OB} (\tilde{O}^{c}(o) \otimes \tilde{R}(o, a)), \tag{25}$$

$$\tilde{A}^{\overline{\#}}(o) = \bigvee_{a \in AT} (\tilde{A}^{c}(a) \otimes \tilde{R}(o, a)), \tag{26}$$

$$\tilde{O}^{\#}(a) = \bigvee_{o \in OB} (\tilde{O}^{c}(o) \otimes \tilde{R}^{c}(o, a)), \tag{27}$$

$$\tilde{A}^{\#}(o) = \bigvee_{a \in AT} (\tilde{A}^{c}(a) \otimes \tilde{R}^{c}(o, a)). \tag{28}$$

For $\tilde{O} \in L^{OB}$ and $a \in AT$, $\tilde{O}^{*}(a)$ and $\tilde{O}^{\overline{\square}}(a)$ characterise the subsethood degree of $\tilde{O}$ to $[a]_{\tilde{R}}$ and $\tilde{O}^{c}$ to $[a]_{\tilde{R}}$, respectively, where $[a]_{\tilde{R}}(o) = \tilde{R}(o, a)$ for $o \in OB$. $\tilde{O}^{\bar{*}}(a)$ and $\tilde{O}^{\square}(a)$ characterise the subsethood degree of $\tilde{O}$ to $[a]_{\tilde{R}^{c}}$ and $\tilde{O}^{c}$ to $[a]_{\tilde{R}^{c}}$, respectively, where $[a]_{\tilde{R}^{c}}(o) = \tilde{R}^{c}(o, a)$ for $o \in OB$. $\tilde{O}^{\Diamond}(a)$ and $\tilde{O}^{\overline{\#}}(a)$ characterise the degree to which the intersections of $\tilde{O}$ with $[a]_{\tilde{R}}$ and $\tilde{O}^{c}$ with $[a]_{\tilde{R}}$ are not empty. $\tilde{O}^{\overline{\Diamond}}(a)$ and $\tilde{O}^{\#}(a)$ characterise the degree to which the intersections of $\tilde{O}$ with $[a]_{\tilde{R}^{c}}$ and $\tilde{O}^{c}$ with $[a]_{\tilde{R}^{c}}$ are not empty. There are similar interpretations for $\tilde{A}^{?}(o)$, where $\tilde{A} \in L^{AT}$, $o \in OB$, and $? = *, \bar{*}, \overline{\square}, \square, \Diamond, \overline{\Diamond}, \overline{\#}$, and $\#$. If $\mathbf{L}$ is a complete regular lattice, then Eqs. (17)–(20) are equivalently reformulated as follows (by using Lemma 2(1)):

$$\tilde{O}^{\overline{\square}}(a) = \bigwedge_{o \in OB} (\tilde{R}^{c}(o, a) \to \tilde{O}(o)), \tag{29}$$

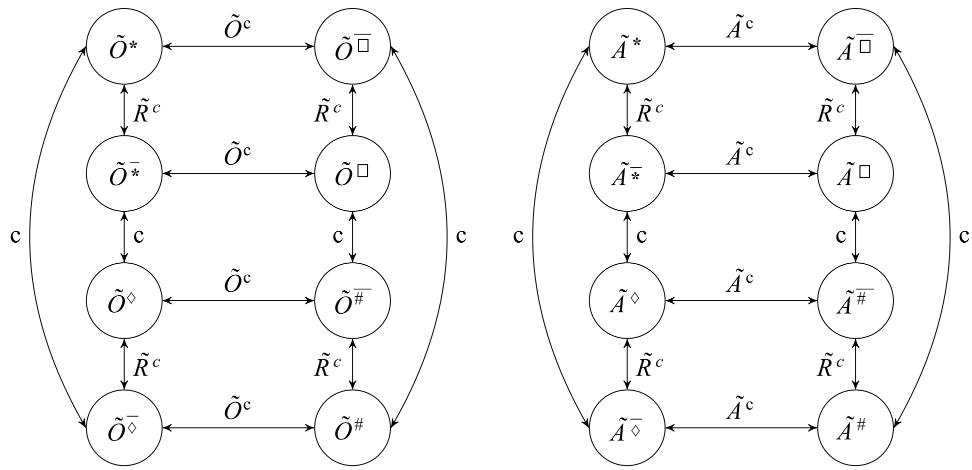
$$\tilde{A}^{\overline{\square}}(o) = \bigwedge_{a \in AT} (\tilde{R}^{c}(o, a) \to \tilde{A}(a)), \tag{30}$$

$$\tilde{O}^{\square}(a) = \bigwedge_{o \in OB} (\tilde{R}(o, a) \to \tilde{O}(o)), \tag{31}$$

$$\tilde{A}^{\square}(o) = \bigwedge_{a \in AT} (\tilde{R}(o, a) \to \tilde{A}(a)). \tag{32}$$

Note that the operators in Eqs. (13)–(16), (21), (22), (27), (28), (31), and (32) have already appeared in [35–37]. However, this does not affect our study because we mainly explore the relationship between different $\mathbf{L}$-concept lattices. To distinguish between the operators defined on fuzzy object sets and fuzzy attribute sets, we call $\tilde{O}^{*}$, $\tilde{O}^{\bar{*}}$, $\tilde{O}^{\overline{\square}}$, $\tilde{O}^{\square}$, $\tilde{O}^{\overline{\Diamond}}$, $\tilde{O}^{\Diamond}$, $\tilde{O}^{\#}$, $\tilde{O}^{\overline{\#}}$ as $\mathbf{L}$-object-induced two-way ($\mathbf{L}$O2W) operators and $\tilde{A}^{*}$, $\tilde{A}^{\bar{*}}$, $\tilde{A}^{\overline{\square}}$, $\tilde{A}^{\square}$, $\tilde{A}^{\overline{\Diamond}}$, $\tilde{A}^{\Diamond}$, $\tilde{A}^{\#}$, $\tilde{A}^{\overline{\#}}$ as $\mathbf{L}$-attribute-induced two-way ($\mathbf{L}$A2W) operators for $\tilde{O} \in L^{OB}$ and $\tilde{A} \in L^{AT}$. The $\mathbf{L}$O2W and $\mathbf{L}$A2W operators are both called $\mathbf{L}$2W operators. In the following, the $\mathbf{L}^{?}$-two-way ($\mathbf{L}^{?}$-2W) operator indicates a particular operator where $? = *, \bar{*}, \overline{\square}, \square, \Diamond, \overline{\Diamond}, \overline{\#}$, and $\#$. For example, the $\mathbf{L}^{*}$-2W operator is $\mathbf{L}^{*}$-object-induced two-way ($\mathbf{L}^{*}$-O2W) operator $\tilde{O}^{*}$ or $\mathbf{L}^{*}$-attribute-induced two-way ($\mathbf{L}^{*}$-A2W) operator $\tilde{A}^{*}$, or both.

Example 1 Suppose $OB = \{o_1, o_2, \cdots, o_5\}$, and each o_i represents a real estate. An agent scores each estate based on the following four factors: a_1—the location is good, a_2—the price is reasonable, a_3—the room layout is comfortable, and a_4—the residential environment is pleasant; thus, $AT = \{a_1, a_2, a_3, a_4\}$. Let $L = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ be the score set. Relation $\tilde{R} : OB \times AT \longrightarrow L$ represents the scores of each estate with



(a) Relationship between **LO2W** operators. **(b)** Relationship between **LA2W** operators.

Fig. 1 Relationship between **L**2W operators

$$\tilde{R} = \begin{array}{c} \\ o_1 \\ o_2 \\ o_3 \\ o_4 \\ o_5 \end{array} \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{pmatrix} 0.20 & 1 & 0.20 & 0.80 \\ 0.80 & 0.60 & 1 & 0.40 \\ 0.20 & 0.80 & 0.40 & 1 \\ 0.40 & 0.40 & 1 & 0 \\ 1 & 0.80 & 0.60 & 0.20 \end{pmatrix} \end{array}.$$

We define $\lambda \otimes \mu = \max\{\lambda + \mu - 1, 0\}$ and $\lambda \rightarrow \mu = \min\{1 - \lambda + \mu, 1\}$ for $\lambda, \mu \in L$. Then, $\mathbf{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is a complete regular residuated lattice, and $\mathbf{K} = (OB, AT, \tilde{R}, L)$ is an **L**-context.

Now, suppose there is an intentional buyer, and he rates each factor as 0.4, 1, 0.6, and 0.8. Based on this information, we must determine which estate the agent should recommend.

To solve this problem, first, let $\tilde{A} = \frac{0.4}{a_1} + \frac{1}{a_2} + \frac{0.6}{a_3} + \frac{0.8}{a_4}$; then, we compute according to Eq. (13). The result is an **L**-set:

$$\tilde{A}^* = \frac{0.6}{o_1} + \frac{0.6}{o_2} + \frac{0.8}{o_3} + \frac{0.2}{o_4} + \frac{0.4}{o_5},$$

which means that the agent should recommend estate o_3 to this buyer.

To establish the relationship between **L**2W operators, we assume that **L** is a complete regular residuated lattice. Thus, for an **L**-set $\tilde{O}$, $(\tilde{O}^c)^c = \tilde{O}$ always holds. Moreover, $\mathbf{L}^{\bar{*}}$-, $\mathbf{L}^{\square}$-, $\mathbf{L}^{\Diamond}$-, and $\mathbf{L}^{\#}$-2W operators defined in **K** are $\mathbf{L}^{*}$-, $\mathbf{L}^{\overline{\square}}$-, $\mathbf{L}^{\overline{\Diamond}}$-, and $\mathbf{L}^{\overline{\#}}$-2W operators defined in the dual **L**-context, $\mathbf{K}^c$. Figure 1 shows the fundamental relationship between the eight types of fuzzy concept-forming operators. (Figure 1a and 1b is the same, except that the former is for **L**O2W operators and the latter for **L**A2W operators.) The operators, connected by a double-arrowed line, are converted into each other by taking the operation attached to the line. For example, replacing $\tilde{O}$ with $\tilde{O}^c$ in $\tilde{O}^*$ gives $\tilde{O}^{\overline{\square}}$; thus, $\tilde{O}^{\overline{\square}} = (\tilde{O}^c)^*$. In the following, the parentheses are omitted for simplicity. For a better understanding, Table 2 presents the equivalences

Table 2 Equivalences of **L**O2W operators

	$*$	$\overline{\square}$	$\overline{\Diamond}$	$\#$	$\bar{*}$	$\square$	$\Diamond$	$\overline{\#}$
$*$	–	$\tilde{O}^* = \tilde{O}^{c\overline{\square}}$	$\tilde{O}^* = \tilde{O}^{\overline{\Diamond}c}$	$\tilde{O}^* = \tilde{O}^{c\#c}$	$\tilde{O}^* = \tilde{O}^{\bar{*}}_{\tilde{R}^c}$	$\tilde{O}^* = \tilde{O}^{c\square}_{\tilde{R}^c}$	$\tilde{O}^* = \tilde{O}^{\Diamond c}_{\tilde{R}^c}$	$\tilde{O}^* = \tilde{O}^{c\overline{\#}c}_{\tilde{R}^c}$
$\overline{\square}$	$\tilde{O}^{\overline{\square}} = \tilde{O}^{c*}$	–	$\tilde{O}^{\overline{\square}} = \tilde{O}^{c\overline{\Diamond}c}$	$\tilde{O}^{\overline{\square}} = \tilde{O}^{\#c}$	$\tilde{O}^{\overline{\square}} = \tilde{O}^{c\bar{*}}_{\tilde{R}^c}$	$\tilde{O}^{\overline{\square}} = \tilde{O}^{\square}_{\tilde{R}^c}$	$\tilde{O}^{\overline{\square}} = \tilde{O}^{c\Diamond c}_{\tilde{R}^c}$	$\tilde{O}^{\overline{\square}} = \tilde{O}^{\overline{\#}c}_{\tilde{R}^c}$
$\overline{\Diamond}$	$\tilde{O}^{\overline{\Diamond}} = \tilde{O}^{*c}$	$\tilde{O}^{\overline{\Diamond}} = \tilde{O}^{c\overline{\square}c}$	–	$\tilde{O}^{\overline{\Diamond}} = \tilde{O}^{c\#}$	$\tilde{O}^{\overline{\Diamond}} = \tilde{O}^{\bar{*}c}_{\tilde{R}^c}$	$\tilde{O}^{\overline{\Diamond}} = \tilde{O}^{c\square c}_{\tilde{R}^c}$	$\tilde{O}^{\overline{\Diamond}} = \tilde{O}^{\Diamond}_{\tilde{R}^c}$	$\tilde{O}^{\overline{\Diamond}} = \tilde{O}^{c\overline{\#}}_{\tilde{R}^c}$
$\#$	$\tilde{O}^{\#} = \tilde{O}^{c*c}$	$\tilde{O}^{\#} = \tilde{O}^{\overline{\square}c}$	$\tilde{O}^{\#} = \tilde{O}^{c\overline{\Diamond}}$	–	$\tilde{O}^{\#} = \tilde{O}^{c\bar{*}c}_{\tilde{R}^c}$	$\tilde{O}^{\#} = \tilde{O}^{\square c}_{\tilde{R}^c}$	$\tilde{O}^{\#} = \tilde{O}^{c\Diamond}_{\tilde{R}^c}$	$\tilde{O}^{\#} = \tilde{O}^{\overline{\#}}_{\tilde{R}^c}$
$\bar{*}$	$\tilde{O}^{\bar{*}} = \tilde{O}^{*}_{\tilde{R}^c}$	$\tilde{O}^{\bar{*}} = \tilde{O}^{c\overline{\square}}_{\tilde{R}^c}$	$\tilde{O}^{\bar{*}} = \tilde{O}^{\overline{\Diamond}c}_{\tilde{R}^c}$	$\tilde{O}^{\bar{*}} = \tilde{O}^{c\#c}_{\tilde{R}^c}$	–	$\tilde{O}^{\bar{*}} = \tilde{O}^{c\square}$	$\tilde{O}^{\bar{*}} = \tilde{O}^{\Diamond c}$	$\tilde{O}^{\bar{*}} = \tilde{O}^{c\overline{\#}c}$
$\square$	$\tilde{O}^{\square} = \tilde{O}^{c*}_{\tilde{R}^c}$	$\tilde{O}^{\square} = \tilde{O}^{\overline{\square}}_{\tilde{R}^c}$	$\tilde{O}^{\square} = \tilde{O}^{c\overline{\Diamond}c}_{\tilde{R}^c}$	$\tilde{O}^{\square} = \tilde{O}^{\#c}_{\tilde{R}^c}$	$\tilde{O}^{\square} = \tilde{O}^{c\bar{*}}$	–	$\tilde{O}^{\square} = \tilde{O}^{c\Diamond c}$	$\tilde{O}^{\square} = \tilde{O}^{\overline{\#}c}$
$\Diamond$	$\tilde{O}^{\Diamond} = \tilde{O}^{*c}_{\tilde{R}^c}$	$\tilde{O}^{\Diamond} = \tilde{O}^{c\overline{\square}c}_{\tilde{R}^c}$	$\tilde{O}^{\Diamond} = \tilde{O}^{\overline{\Diamond}}_{\tilde{R}^c}$	$\tilde{O}^{\Diamond} = \tilde{O}^{c\#}_{\tilde{R}^c}$	$\tilde{O}^{\Diamond} = \tilde{O}^{\bar{*}c}$	$\tilde{O}^{\Diamond} = \tilde{O}^{c\square c}$	–	$\tilde{O}^{\Diamond} = \tilde{O}^{c\overline{\#}}$
$\overline{\#}$	$\tilde{O}^{\overline{\#}} = \tilde{O}^{c*c}_{\tilde{R}^c}$	$\tilde{O}^{\overline{\#}} = \tilde{O}^{\overline{\square}c}_{\tilde{R}^c}$	$\tilde{O}^{\overline{\#}} = \tilde{O}^{c\overline{\Diamond}}_{\tilde{R}^c}$	$\tilde{O}^{\overline{\#}} = \tilde{O}^{\#}_{\tilde{R}^c}$	$\tilde{O}^{\overline{\#}} = \tilde{O}^{c\bar{*}c}$	$\tilde{O}^{\overline{\#}} = \tilde{O}^{\square c}$	$\tilde{O}^{\overline{\#}} = \tilde{O}^{c\Diamond}$	–

Table 3 Equivalences of double compound operations of **L**2W operators

	$*$	$\overline{\Box}$	$\overline{\Diamond}$	$\#$	$\bar{*}$	$\Box$	$\Diamond$	$\bar{\#}$
$*$	$\tilde{O}^{**}=\tilde{O}^{\overline{\Diamond}\overline{\Box}}$	$\tilde{O}^{*\overline{\Box}}=\tilde{O}^{\overline{\Diamond}*}$	$\tilde{O}^{*\overline{\Diamond}}=\tilde{O}^{\overline{\Diamond}\#}$	$\tilde{O}^{*\#}=\tilde{O}^{\overline{\Diamond\Diamond}}$	$\tilde{O}^{*\bar{*}}=\tilde{O}^{\overline{\Diamond}\Box}$	$\tilde{O}^{*\Box}=\tilde{O}^{\overline{\Diamond}\bar{*}}$	$\tilde{O}^{*\Diamond}=\tilde{O}^{\overline{\Diamond}\bar{\#}}$	$\tilde{O}^{*\bar{\#}}=\tilde{O}^{\overline{\Diamond}\Diamond}$
$\overline{\Box}$	$\tilde{O}^{\overline{\Box}*}=\tilde{O}^{\#\overline{\Box}}$	$\tilde{O}^{\overline{\Box\Box}}=\tilde{O}^{\#*}$	$\tilde{O}^{\overline{\Box}\overline{\Diamond}}=\tilde{O}^{\#\#}$	$\tilde{O}^{\overline{\Box}\#}=\tilde{O}^{\#\overline{\Diamond}}$	$\tilde{O}^{\overline{\Box}\bar{*}}=\tilde{O}^{\#\Box}$	$\tilde{O}^{\overline{\Box}\Box}=\tilde{O}^{\#\bar{*}}$	$\tilde{O}^{\overline{\Box}\Diamond}=\tilde{O}^{\#\bar{\#}}$	$\tilde{O}^{\overline{\Box}\bar{\#}}=\tilde{O}^{\#\Diamond}$
$\overline{\Diamond}$	$\tilde{O}^{\overline{\Diamond}*}=\tilde{O}^{*\overline{\Box}}$	$\tilde{O}^{\overline{\Diamond}\overline{\Box}}=\tilde{O}^{**}$	$\tilde{O}^{\overline{\Diamond\Diamond}}=\tilde{O}^{*\#}$	$\tilde{O}^{\overline{\Diamond}\#}=\tilde{O}^{*\overline{\Diamond}}$	$\tilde{O}^{\overline{\Diamond}\bar{*}}=\tilde{O}^{*\Box}$	$\tilde{O}^{\overline{\Diamond}\Box}=\tilde{O}^{*\bar{*}}$	$\tilde{O}^{\overline{\Diamond}\Diamond}=\tilde{O}^{*\bar{\#}}$	$\tilde{O}^{\overline{\Diamond}\bar{\#}}=\tilde{O}^{*\Diamond}$
$\#$	$\tilde{O}^{\#*}=\tilde{O}^{\overline{\Box\Box}}$	$\tilde{O}^{\#\overline{\Box}}=\tilde{O}^{\overline{\Box}*}$	$\tilde{O}^{\#\overline{\Diamond}}=\tilde{O}^{\overline{\Box}\#}$	$\tilde{O}^{\#\#}=\tilde{O}^{\overline{\Box}\overline{\Diamond}}$	$\tilde{O}^{\#\bar{*}}=\tilde{O}^{\overline{\Box}\Box}$	$\tilde{O}^{\#\Box}=\tilde{O}^{\overline{\Box}\bar{*}}$	$\tilde{O}^{\#\Diamond}=\tilde{O}^{\overline{\Box}\bar{\#}}$	$\tilde{O}^{\#\bar{\#}}=\tilde{O}^{\overline{\Box}\Diamond}$
$\bar{*}$	$\tilde{O}^{\bar{*}*}=\tilde{O}^{\Diamond\overline{\Box}}$	$\tilde{O}^{\bar{*}\overline{\Box}}=\tilde{O}^{\Diamond*}$	$\tilde{O}^{\bar{*}\overline{\Diamond}}=\tilde{O}^{\Diamond\#}$	$\tilde{O}^{\bar{*}\#}=\tilde{O}^{\Diamond\overline{\Diamond}}$	$\tilde{O}^{\bar{*}\bar{*}}=\tilde{O}^{\Diamond\Box}$	$\tilde{O}^{\bar{*}\Box}=\tilde{O}^{\Diamond\bar{*}}$	$\tilde{O}^{\bar{*}\Diamond}=\tilde{O}^{\Diamond\bar{\#}}$	$\tilde{O}^{\bar{*}\bar{\#}}=\tilde{O}^{\Diamond\Diamond}$
$\Box$	$\tilde{O}^{\Box*}=\tilde{O}^{\bar{\#}\overline{\Box}}$	$\tilde{O}^{\Box\overline{\Box}}=\tilde{O}^{\bar{\#}*}$	$\tilde{O}^{\Box\overline{\Diamond}}=\tilde{O}^{\bar{\#}\#}$	$\tilde{O}^{\Box\#}=\tilde{O}^{\bar{\#}\overline{\Diamond}}$	$\tilde{O}^{\Box\bar{*}}=\tilde{O}^{\bar{\#}\Box}$	$\tilde{O}^{\Box\Box}=\tilde{O}^{\bar{\#}\bar{*}}$	$\tilde{O}^{\Box\Diamond}=\tilde{O}^{\overline{\#\#}}$	$\tilde{O}^{\Box\bar{\#}}=\tilde{O}^{\bar{\#}\Diamond}$
$\Diamond$	$\tilde{O}^{\Diamond*}=\tilde{O}^{\bar{*}\overline{\Box}}$	$\tilde{O}^{\Diamond\overline{\Box}}=\tilde{O}^{\bar{*}*}$	$\tilde{O}^{\Diamond\overline{\Diamond}}=\tilde{O}^{\bar{*}\#}$	$\tilde{O}^{\Diamond\#}=\tilde{O}^{\bar{*}\overline{\Diamond}}$	$\tilde{O}^{\Diamond\bar{*}}=\tilde{O}^{\bar{*}\Box}$	$\tilde{O}^{\Diamond\Box}=\tilde{O}^{\bar{*}\bar{*}}$	$\tilde{O}^{\Diamond\Diamond}=\tilde{O}^{\bar{*}\bar{\#}}$	$\tilde{O}^{\Diamond\bar{\#}}=\tilde{O}^{\bar{*}\Diamond}$
$\bar{\#}$	$\tilde{O}^{\bar{\#}*}=\tilde{O}^{\Box\overline{\Box}}$	$\tilde{O}^{\bar{\#}\overline{\Box}}=\tilde{O}^{\Box*}$	$\tilde{O}^{\bar{\#}\overline{\Diamond}}=\tilde{O}^{\Box\#}$	$\tilde{O}^{\bar{\#}\#}=\tilde{O}^{\Box\overline{\Diamond}}$	$\tilde{O}^{\bar{\#}\bar{*}}=\tilde{O}^{\Box\Box}$	$\tilde{O}^{\bar{\#}\Box}=\tilde{O}^{\Box\bar{*}}$	$\tilde{O}^{\bar{\#}\Diamond}=\tilde{O}^{\Box\bar{\#}}$	$\tilde{O}^{\overline{\#\#}}=\tilde{O}^{\Box\Diamond}$

illustrated in Fig. 1a as equations. (Figure 1b is similarly translated.) One can refer to Appendix I for the verification of these equations. $\tilde{O}^{?}_{\tilde{R}^c}$ means that operator ? is defined based on $\tilde{R}^c$ for $\tilde{O}$, where $? = *, \bar{*}, \overline{\Box}, \Box, \Diamond, \overline{\Diamond}, \bar{\#}$, and $\#$.

Table 3 shows the connections between the double compound operations of **L**2W operators. It lists all the possible combinations. Thus, there is some overlap. For example, $\tilde{O}^{**} = \tilde{O}^{\overline{\Diamond}\overline{\Box}}$ and $\tilde{O}^{\overline{\Diamond}\overline{\Box}} = \tilde{O}^{**}$ are the same. However, for convenience of reference, the same equations have not been deleted. The following equations present the equivalent relationship between the triple compound operations, namely, applying three **L**2W operators successively. For $\tilde{O} \in L^{OB}$,

$$
\begin{aligned}
&\tilde{O}^{***} = \tilde{O}^{\overline{\Diamond}\overline{\Box}\overline{\Diamond}c},\ \tilde{O}^{\Diamond\Diamond\Diamond} = \tilde{O}^{\bar{*}\bar{\#}\bar{*}c},\\
&\tilde{O}^{\Box\Box\Box} = \tilde{O}^{\bar{\#}\bar{*}\bar{\#}c},\ \tilde{O}^{\#\#\#} = \tilde{O}^{\overline{\Box}\overline{\Diamond}\overline{\Box}c},\\
&\tilde{O}^{\bar{*}\bar{*}\bar{*}} = \tilde{O}^{\Diamond\Box\Diamond c},\ \tilde{O}^{\overline{\Diamond\Diamond\Diamond}} = \tilde{O}^{*\#*c},\\
&\tilde{O}^{\overline{\Box\Box\Box}} = \tilde{O}^{\#*\#c},\ \tilde{O}^{\overline{\#\#\#}} = \tilde{O}^{\Box\Diamond\Box c},\\
&\tilde{O}^{***} = \tilde{O}^{c\overline{\Box}\overline{\Diamond}\overline{\Box}},\ \tilde{O}^{\Diamond\Diamond\Diamond} = \tilde{O}^{c\bar{\#}\bar{*}\bar{\#}},\\
&\tilde{O}^{\Box\Box\Box} = \tilde{O}^{c\bar{*}\bar{\#}\bar{*}},\ \tilde{O}^{\#\#\#} = \tilde{O}^{c\overline{\Diamond}\overline{\Box}\overline{\Diamond}},\\
&\tilde{O}^{\bar{*}\bar{*}\bar{*}} = \tilde{O}^{c\Box\Diamond\Box},\ \tilde{O}^{\overline{\Diamond\Diamond\Diamond}} = \tilde{O}^{c\#*\#},\\
&\tilde{O}^{\overline{\Box\Box\Box}} = \tilde{O}^{c*\#*},\ \tilde{O}^{\overline{\#\#\#}} = \tilde{O}^{c\Diamond\Box\Diamond}.
\end{aligned}
\tag{33}
$$

Using the order and basic operations of the fuzzy sets (see Eqs. (1) and (2)), we can prove the following properties of $\mathbf{L}^*$-2W operators.

Proposition 1 *For $\tilde{O}, \tilde{O}_i, \tilde{O}_j \in L^{OB}$ and $\tilde{A}, \tilde{A}_i, \tilde{A}_j \in L^{AT}$ ($i \in \Lambda$ and $j = 1, 2$), and $A \in L^{AT}$, the following properties hold:*

1. *If $\tilde{O}_1 \subseteq \tilde{O}_2$, then $\tilde{O}_2^* \subseteq \tilde{O}_1^*$, if $\tilde{A}_1 \subseteq \tilde{A}_2$, then $\tilde{A}_2^* \subseteq \tilde{A}_1^*$;*
2. $\tilde{O} \subseteq \tilde{O}^{**}$, $\tilde{A} \subseteq \tilde{A}^{**}$;
3. $\tilde{O}^* = \tilde{O}^{***}$, $\tilde{A}^* = \tilde{A}^{***}$;
4. $\left(\bigcup_{i\in\Lambda} \tilde{O}_i\right)^* = \bigcap_{i\in\Lambda} \tilde{O}_i^*$, $\left(\bigcup_{i\in\Lambda} \tilde{A}_i\right)^* = \bigcap_{i\in\Lambda} \tilde{A}_i^*$;
5. $\left(\bigcap_{i\in\Lambda} \tilde{O}_i\right)^* \supseteq \bigcup_{i\in\Lambda} \tilde{O}_i^*$, $\left(\bigcap_{i\in\Lambda} \tilde{A}_i\right)^* \supseteq \bigcup_{i\in\Lambda} \tilde{A}_i^*$;
6. $\tilde{O} \subseteq \tilde{A}^* \Leftrightarrow \tilde{A} \subseteq \tilde{O}^*$.

Proof Given are the proof of the properties of fuzzy object sets; we can similarly prove the properties of fuzzy attribute sets.

1. To prove the monotonicity, we assume $\tilde{O}_1 \subseteq \tilde{O}_2$, which means that $\tilde{O}_1(o) \le \tilde{O}_2(o)$ for each $o \in OB$. Then, from Lemma 1(3), it follows that $\tilde{O}_2^*(a) = \bigwedge_{o\in OB}\left(\tilde{O}_2(o) \to \tilde{R}(o,a)\right) \le \bigwedge_{o\in OB}\left(\tilde{O}_1(o) \to \tilde{R}(o,a)\right) = \tilde{O}_1^*(a)$, $\forall a \in AT$, which means $\tilde{O}_2^* \subseteq \tilde{O}_1^*$.
2. For $\tilde{O} \in L^{OB}$ and $a \in AT$, according to Lemma 1(5), it holds that $\tilde{O}^{**}(o) = \bigwedge_{a\in AT}\left(\tilde{O}^*(a) \to \tilde{R}(o,a)\right)$ $= \bigwedge_{a\in AT}\left(\bigwedge_{o'\in OB}\left(\tilde{O}(o') \to \tilde{R}(o',a)\right) \to \tilde{R}(o,a)\right)$ $\ge \bigwedge_{a\in AT}\left(\left(\tilde{O}(o) \to \tilde{R}(o,a)\right) \to \tilde{R}(o,a)\right)$ $\ge \bigwedge_{a\in AT} \tilde{O}(o) = \tilde{O}(o)$, which means $\tilde{O} \subseteq \tilde{O}^{**}$.
3. From Items (1) and (2), we obtain $\tilde{O}^{***} \subseteq \tilde{O}^*$. Substituting $\tilde{A} = \tilde{O}^*$ follows $\tilde{A} \subseteq \tilde{A}^{**}$; thus, $\tilde{O}^* \subseteq \tilde{O}^{***}$.
4. For $\tilde{O}_i \in L^{OB}$ ($i \in \Lambda$) and $a \in AT$, Lemma 1(4) supports that $\left(\bigcup_{i\in\Lambda} \tilde{O}_i\right)^*(a) = \bigwedge_{o\in OB}\left(\left(\bigcup_{i\in\Lambda} \tilde{O}_i\right)(o) \to \tilde{R}(o,a)\right)$ $= \bigwedge_{o\in OB}\left(\bigvee_{i\in\Lambda} \tilde{O}_i(o) \to \tilde{R}(o,a)\right)$ $= \bigwedge_{o\in OB}\bigwedge_{i\in\Lambda}\left(\tilde{O}_i(o) \to \tilde{R}(o,a)\right)$ $= \bigwedge_{i\in\Lambda}\bigwedge_{o\in OB}\left(\tilde{O}_i(o) \to \tilde{R}(o,a)\right)$ $= \bigwedge_{i\in\Lambda} \tilde{O}_i^*(a) = \left(\bigcap_{i\in\Lambda} \tilde{O}_i^*\right)(a)$.
5. For $\tilde{O}_i \in L^{OB}$ ($i \in \Lambda$) and $a \in AT$, Lemma 1(4) indicates that $\left(\bigcap_{i\in\Lambda} \tilde{O}_i\right)^*(a) = \bigwedge_{o\in OB}\left(\left(\bigcap_{i\in\Lambda} \tilde{O}_i\right)(o) \to \tilde{R}(o,a)\right)$ $= \bigwedge_{o\in OB}\left(\bigwedge_{i\in\Lambda} \tilde{O}_i(o) \to \tilde{R}(o,a)\right)$ $\ge \bigwedge_{o\in OB}\bigvee_{i\in\Lambda}\left(\tilde{O}_i(o) \to \tilde{R}(o,a)\right)$ $\ge \bigvee_{i\in\Lambda}\bigwedge_{o\in OB}\left(\tilde{O}_i(o) \to \tilde{R}(o,a)\right)$ $= \bigvee_{i\in\Lambda} \tilde{O}_i^*(a) = \left(\bigcup_{i\in\Lambda} \tilde{O}_i^*\right)(a)$.
6. For $\tilde{O} \in L^{OB}$ and $\tilde{A} \in L^{AT}$, we assume that $\tilde{O} \subseteq \tilde{A}^*$. Based on Lemma 1(5), we verify that $\tilde{O}^*(a) = \bigwedge_{o\in OB}\left(\tilde{O}(o) \to \tilde{R}(o,a)\right) \ge \bigwedge_{o\in OB}\left(\tilde{A}^*(o) \to \tilde{R}(o,a)\right) = \bigwedge_{o\in OB}\left(\bigwedge_{a'\in AT}\left(\tilde{A}(a') \to \tilde{R}(o,a')\right) \to \tilde{R}(o,a)\right) \ge \bigwedge_{o\in OB}\left(\left(\tilde{A}(a) \to \tilde{R}(o,a)\right) \to \tilde{R}(o,a)\right) \ge \tilde{A}(a)$, which indicates $\tilde{A} \subseteq \tilde{O}^*$ by the arbitrariness of a. The converse is similarly proved.

The properties in Item (1) show the monotonicity of the $\mathbf{L}^*$-2W operator. The properties in Items (2) and (3) show that two $\mathbf{L}^*$-2W operator applications will enlarge an $\mathbf{L}$-set, and three applications of the $\mathbf{L}^*$-2W operator result in the same outcome as one application. The properties in Items (4) and (5) show the distributivity of the $\mathbf{L}^*$-2W operator over union and intersection. Items (1) and (6) of Proposition 1 show that $(*, *)$ forms an antitone Galois connection[1,2] between $(L^{OB}, \subseteq)$ and $(L^{AT}, \subseteq)$. The properties of other $\mathbf{L}$2W operators are easily obtained based on the properties of $\mathbf{L}^*$-2W operators and the relationship between $\mathbf{L}$2W operators.

Proposition 2 *For $\tilde{O}, \tilde{O}_i, \tilde{O}_j \in L^{OB}$ ($i \in \Lambda$ and $j = 1, 2$), and $\tilde{A} \in L^{AT}$, the following properties hold*:

1. *If* $\tilde{O}_1 \subseteq \tilde{O}_2$, *then* $\tilde{O}_2^{\bar{*}} \subseteq \tilde{O}_1^{\bar{*}}$, $\tilde{O}_1^{\overline{\Box}} \subseteq \tilde{O}_2^{\overline{\Box}}$, $\tilde{O}_1^{\Box} \subseteq \tilde{O}_2^{\Box}$, $\tilde{O}_1^{\overline{\Diamond}} \subseteq \tilde{O}_2^{\overline{\Diamond}}$, $\tilde{O}_1^{\Diamond} \subseteq \tilde{O}_2^{\Diamond}$, $\tilde{O}_2^{\#} \subseteq \tilde{O}_1^{\#}$, $\tilde{O}_2^{\bar{\#}} \subseteq \tilde{O}_1^{\bar{\#}}$;
2. $\tilde{O} \subseteq \tilde{O}^{\bar{*}\bar{*}}$, $\tilde{O}^{\overline{\Box}\overline{\Diamond}} \subseteq \tilde{O} \subseteq \tilde{O}^{\overline{\Diamond}\overline{\Box}}$, $\tilde{O}^{\Box\Diamond} \subseteq \tilde{O} \subseteq \tilde{O}^{\Diamond\Box}$, $\tilde{O} \supseteq \tilde{O}^{\#\#}$, $\tilde{O} \supseteq \tilde{O}^{\overline{\#\#}}$;
3. $\tilde{O}^{\bar{*}} = \tilde{O}^{\bar{*}\bar{*}\bar{*}}$, $\tilde{O}^{\overline{\Diamond}} = \tilde{O}^{\overline{\Diamond}\overline{\Box}\overline{\Diamond}}$, $\tilde{O}^{\Diamond} = \tilde{O}^{\Diamond\Box\Diamond}$, $\tilde{O}^{\overline{\Box}} = \tilde{O}^{\overline{\Box}\overline{\Diamond}\overline{\Box}}$, $\tilde{O}^{\Box} = \tilde{O}^{\Box\Diamond\Box}$, $\tilde{O}^{\#} = \tilde{O}^{\#\#\#}$, $\tilde{O}^{\bar{\#}} = \tilde{O}^{\overline{\#\#\#}}$;
4. $\left(\bigcup_{i\in\Lambda} \tilde{O}_i\right)^{\bar{*}} = \bigcap_{i\in\Lambda} \tilde{O}_i^{\bar{*}}$, $\left(\bigcap_{i\in\Lambda} \tilde{O}_i\right)^{\overline{\Box}} = \bigcap_{i\in\Lambda}$, $\tilde{O}_i)^{\Box} = \bigcap_{i\in\Lambda} \tilde{O}_i^{\Box}$, $\left(\bigcup_{i\in\Lambda} \tilde{O}_i\right)^{\overline{\Diamond}} = \bigcup_{i\in\Lambda} \tilde{O}_i^{\overline{\Diamond}}$, $\left(\bigcup_{i\in\Lambda} \tilde{O}_i\right)^{\Diamond} = \bigcup_{i\in\Lambda} \tilde{O}_i^{\Diamond}$, $\left(\bigcap_{i\in\Lambda} \tilde{O}_i\right)^{\#} = \bigcup_{i\in\Lambda} \tilde{O}_i^{\#}$, $\left(\bigcap_{i\in\Lambda} \tilde{O}_i\right)^{\bar{\#}} = \bigcup_{i\in\Lambda} \tilde{O}_i^{\bar{\#}}$;
5. $\left(\bigcap_{i\in\Lambda} \tilde{O}_i\right)^{\bar{*}} \supseteq \bigcup_{i\in\Lambda} \tilde{O}_i^{\bar{*}}$, $\left(\bigcup_{i\in\Lambda} \tilde{O}_i\right)^{\overline{\Box}} \supseteq \bigcup_{i\in\Lambda} \tilde{O}_i^{\overline{\Box}}$, $\left(\bigcup_{i\in\Lambda} \tilde{O}_i\right)^{\Box} \supseteq \bigcup_{i\in\Lambda} \tilde{O}_i^{\Box}$, $\left(\bigcap_{i\in\Lambda} \tilde{O}_i\right)^{\overline{\Diamond}} \subseteq \bigcap_{i\in\Lambda} \tilde{O}_i^{\overline{\Diamond}}$, $\left(\bigcap_{i\in\Lambda} \tilde{O}_i\right)^{\Diamond} \subseteq \bigcap_{i\in\Lambda} \tilde{O}_i^{\Diamond}$, $\left(\bigcup_{i\in\Lambda} \tilde{O}_i\right)^{\#} \subseteq \bigcap_{i\in\Lambda} \tilde{O}_i^{\#}$, $\left(\bigcup_{i\in\Lambda} \tilde{O}_i\right)^{\bar{\#}} \subseteq \bigcap_{i\in\Lambda} \tilde{O}_i^{\bar{\#}}$;
6. $\tilde{O} \subseteq \tilde{A}^{\bar{*}} \Leftrightarrow \tilde{A} \subseteq \tilde{O}^{\bar{*}}$, $\tilde{O} \subseteq \tilde{A}^{\overline{\Box}} \Leftrightarrow \tilde{A} \supseteq \tilde{O}^{\overline{\Diamond}}$, $\tilde{O} \subseteq \tilde{A}^{\Box} \Leftrightarrow \tilde{A} \supseteq \tilde{O}^{\Diamond}$, $\tilde{O} \supseteq \tilde{A}^{\overline{\Diamond}} \Leftrightarrow \tilde{A} \subseteq \tilde{O}^{\overline{\Box}}$, $\tilde{O} \supseteq \tilde{A}^{\Diamond} \Leftrightarrow \tilde{A} \subseteq \tilde{O}^{\Box}$, $\tilde{A} \supseteq \tilde{O}^{\#}$, $\tilde{O} \supseteq \tilde{A}^{\bar{\#}} \Leftrightarrow \tilde{A} \supseteq \tilde{O}^{\bar{\#}}$.

Note that the properties in Propositions 2(1)–(5) also hold for fuzzy attribute sets. From Proposition 2(1) and (5), one can see that $(\bar{*}, \bar{*})$ forms an antitone Galois connection between $(L^{OB}, \subseteq)$ and $(L^{AT}, \subseteq)$, $(\overline{\Diamond}, \overline{\Box})$ and $(\Diamond, \Box)$ are two isotone Galois connections[3] between $(L^{OB}, \subseteq)$ and $(L^{AT}, \subseteq)$, and $(\overline{\Box}, \overline{\Diamond})$ and $(\Box, \Diamond)$ are two isotone Galois connections between $(L^{AT}, \subseteq)$ and $(L^{OB}, \subseteq)$.

L2W Concepts

Different $\mathbf{L}$2W operators determine various $\mathbf{L}$2W concepts.

Definition 1 Let $\mathbf{K} = (OB, AT, \tilde{R}, L)$ be an $\mathbf{L}$-context. A pair of $\mathbf{L}$-sets, $\langle \tilde{O}, \tilde{A} \rangle$ with $\tilde{O} \in L^{OB}$ and $\tilde{A} \in L^{AT}$, is an

1. $\mathbf{L}^*$-2W concept if $\tilde{O}^* = \tilde{A}$ and $\tilde{A}^* = \tilde{O}$;
2. $\mathbf{L}^{\bar{*}}$-2W concept if $\tilde{O}^{\bar{*}} = \tilde{A}$ and $\tilde{A}^{\bar{*}} = \tilde{O}$;
3. $\mathbf{L}^{\Diamond\Box}$-2W concept if $\tilde{O}^{\Diamond} = \tilde{A}$ and $\tilde{A}^{\Box} = \tilde{O}$;
4. $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$-2W concept if $\tilde{O}^{\overline{\Diamond}} = \tilde{A}$ and $\tilde{A}^{\overline{\Box}} = \tilde{O}$;
5. $\mathbf{L}^{\Box\Diamond}$-2W concept if $\tilde{O}^{\Box} = \tilde{A}$ and $\tilde{A}^{\Diamond} = \tilde{O}$;
6. $\mathbf{L}^{\overline{\Box}\overline{\Diamond}}$-2W concept if $\tilde{O}^{\overline{\Box}} = \tilde{A}$ and $\tilde{A}^{\overline{\Diamond}} = \tilde{O}$;
7. $\mathbf{L}^{\#}$-2W concept if $\tilde{O}^{\#} = \tilde{A}$ and $\tilde{A}^{\#} = \tilde{O}$;
8. $\mathbf{L}^{\bar{\#}}$-2W concept if $\tilde{O}^{\bar{\#}} = \tilde{A}$ and $\tilde{A}^{\bar{\#}} = \tilde{O}$.

Definition 1 shows that the $\mathbf{L}$2W concept is a pair of $\mathbf{L}$-sets that mutually determine each other. Note that $\mathbf{L}^*$-, $\mathbf{L}^{\Diamond\Box}$-, $\mathbf{L}^{\Box\Diamond}$-, and $\mathbf{L}^{\#}$-2W concepts have been proposed [35–37]. We define another four types of $\mathbf{L}$2W concepts, namely, $\mathbf{L}^{\bar{*}}$-, $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$-, $\mathbf{L}^{\overline{\Box}\overline{\Diamond}}$-, and $\mathbf{L}^{\bar{\#}}$-2W concepts.

Example 2 (Continued from Example 1) Let $\tilde{O}_1 = \frac{0.8}{o_1} + \frac{0.8}{o_2} + \frac{1}{o_3} + \frac{0.6}{o_4} + \frac{0.8}{o_5}$, $\tilde{A}_1 = \frac{0.2}{a_1} + \frac{0.8}{a_2} + \frac{0.4}{a_3} + \frac{0.4}{a_4}$ and $\tilde{O}_2 = \frac{0.2}{o_1} + \frac{0.6}{o_2} + \frac{0.2}{o_3} + \frac{0.6}{o_4} + \frac{0.4}{o_5}$, $\tilde{A}_2 = \frac{0.6}{a_1} + \frac{0.8}{a_2} + \frac{0.4}{a_3} + \frac{0.8}{a_4}$. It can be easily verified that $\langle \tilde{O}_1, \tilde{A}_1 \rangle$ is an $\mathbf{L}^*$-2W concept and $\langle \tilde{O}_2, \tilde{A}_2 \rangle$ is an $\mathbf{L}^{\bar{*}}$-2W concept.

From a fuzzy object set or a fuzzy attribute set, we obtain various $\mathbf{L}$2W concepts.

Proposition 3 *For $\tilde{O} \subseteq L^{OB}$ and $\tilde{A} \subseteq L^{AT}$, the following are valid*:

1. $\langle \tilde{O}^{**}, \tilde{O}^* \rangle$ *and* $\langle \tilde{A}^*, \tilde{A}^{**} \rangle$ *are* $\mathbf{L}^*$-2*W concepts*;
2. $\langle \tilde{O}^{\bar{*}\bar{*}}, \tilde{O}^{\bar{*}} \rangle$ *and* $\langle \tilde{A}^{\bar{*}}, \tilde{A}^{\bar{*}\bar{*}} \rangle$ *are* $\mathbf{L}^{\bar{*}}$-2*W concepts*;
3. $\langle \tilde{O}^{\overline{\Diamond}\overline{\Box}}, \tilde{O}^{\overline{\Diamond}} \rangle$ *and* $\langle \tilde{A}^{\overline{\Box}}, \tilde{A}^{\overline{\Box}\overline{\Diamond}} \rangle$ *are* $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$-2*W concepts*;
4. $\langle \tilde{O}^{\Diamond\Box}, \tilde{O}^{\Diamond} \rangle$ *and* $\langle \tilde{A}^{\Box}, \tilde{A}^{\Box\Diamond} \rangle$ *are* $\mathbf{L}^{\Diamond\Box}$-2*W concepts*;
5. $\langle \tilde{O}^{\overline{\Box}\overline{\Diamond}}, \tilde{O}^{\overline{\Box}} \rangle$ *and* $\langle \tilde{A}^{\overline{\Diamond}}, \tilde{A}^{\overline{\Diamond}\overline{\Box}} \rangle$ *are* $\mathbf{L}^{\overline{\Box}\overline{\Diamond}}$-2*W concepts*;
6. $\langle \tilde{O}^{\Box\Diamond}, \tilde{O}^{\Box} \rangle$ *and* $\langle \tilde{A}^{\Diamond}, \tilde{A}^{\Diamond\Box} \rangle$ *are* $\mathbf{L}^{\Box\Diamond}$-2*W concepts*;
7. $\langle \tilde{O}^{\#\#}, \tilde{O}^{\#} \rangle$ *and* $\langle \tilde{A}^{\#}, \tilde{A}^{\#\#} \rangle$ *are* $\mathbf{L}^{\#}$-2*W concepts*;
8. $\langle \tilde{O}^{\overline{\#\#}}, \tilde{O}^{\bar{\#}} \rangle$ *and* $\langle \tilde{A}^{\bar{\#}}, \tilde{A}^{\overline{\#\#}} \rangle$ *are* $\mathbf{L}^{\bar{\#}}$-2*W concepts*.

Proof These assertions are obvious according to Propositions 2(3) and 1(3).

For an $\mathbf{L}$-context, $\mathbf{K} = (OB, AT, \tilde{R}, L)$, we denote

[1] Given two ordered sets, $(P, \leq_P)$ and $(Q, \leq_Q)$, an **(antitone) Galois connection** between $(P, \leq_P)$ and $(Q, \leq_Q)$ consists of two antitone mappings, $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$, such that $x \leq_P g(y) \Leftrightarrow y \leq_Q f(x)$ for all $(x, y) \in P \times Q$.

[2] It is noteworthy that in [35], an $\mathbf{L}$-Galois connection is defined based on the subsethood degree. Here, we refer to the fuzzy set order defined in Eq. (1).

[3] Given two ordered sets, $(P, \leq_P)$ and $(Q, \leq_Q)$, an **isotone Galois connection** between $(P, \leq_P)$ and $(Q, \leq_Q)$ consists of two isotone mappings, $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$, such that $x \leq_P g(y) \Leftrightarrow f(x) \leq_Q y$ for all $(x, y) \in P \times Q$.

$\mathbf{C}^{*}_{\mathbf{L}}(\mathbf{K}) = \{\langle \tilde{O}, \tilde{A}\rangle \mid \tilde{O}^{*} = \tilde{A}, \tilde{A}^{*} = \tilde{O}\}$,
$\mathbf{C}^{\bar{*}}_{\mathbf{L}}(\mathbf{K}) = \{\langle \tilde{O}, \tilde{A}\rangle \mid \tilde{O}^{\bar{*}} = \tilde{A}, \tilde{A}^{\bar{*}} = \tilde{O}\}$,
$\mathbf{C}^{\Diamond\Box}_{\mathbf{L}}(\mathbf{K}) = \{\langle \tilde{O}, \tilde{A}\rangle \mid \tilde{O}^{\Diamond} = \tilde{A}, \tilde{A}^{\Box} = \tilde{O}\}$,
$\mathbf{C}^{\overline{\Diamond}\overline{\Box}}_{\mathbf{L}}(\mathbf{K}) = \{\langle \tilde{O}, \tilde{A}\rangle \mid \tilde{O}^{\overline{\Diamond}} = \tilde{A}, \tilde{A}^{\overline{\Box}} = \tilde{O}\}$,
$\mathbf{C}^{\Box\Diamond}_{\mathbf{L}}(\mathbf{K}) = \{\langle \tilde{O}, \tilde{A}\rangle \mid \tilde{O}^{\Box} = \tilde{A}, \tilde{A}^{\Diamond} = \tilde{O}\}$,
$\mathbf{C}^{\overline{\Box}\overline{\Diamond}}_{\mathbf{L}}(\mathbf{K}) = \{\langle \tilde{O}, \tilde{A}\rangle \mid \tilde{O}^{\overline{\Box}} = \tilde{A}, \tilde{A}^{\overline{\Diamond}} = \tilde{O}\}$,
$\mathbf{C}^{\#}_{\mathbf{L}}(\mathbf{K}) = \{\langle \tilde{O}, \tilde{A}\rangle \mid \tilde{O}^{\#} = \tilde{A}, \tilde{A}^{\#} = \tilde{O}\}$,
$\mathbf{C}^{\bar{\#}}_{\mathbf{L}}(\mathbf{K}) = \{\langle \tilde{O}, \tilde{A}\rangle \mid \tilde{O}^{\bar{\#}} = \tilde{A}, \tilde{A}^{\bar{\#}} = \tilde{O}\}$

as the sets of $\mathbf{L}^{*}$-, $\mathbf{L}^{\bar{*}}$-, $\mathbf{L}^{\Diamond\Box}$-, $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$-, $\mathbf{L}^{\Box\Diamond}$-, $\mathbf{L}^{\overline{\Box}\overline{\Diamond}}$-, $\mathbf{L}^{\#}$-, and $\mathbf{L}^{\bar{\#}}$-2W concepts, respectively.

According to Definition 1 and the regularity of $\mathbf{L}$, the following statements hold: an $\mathbf{L}^{\bar{*}}$-2W concept in $\mathbf{K}$ is equivalent to an $\mathbf{L}^{*}$-2W concept in $\mathbf{K}^{c}$; an $\mathbf{L}^{\Diamond\Box}$-2W concept in $\mathbf{K}$ is equivalent to an $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$-2W concept in $\mathbf{K}^{c}$; an $\mathbf{L}^{\Box\Diamond}$-2W concept in $\mathbf{K}$ is equivalent to an $\mathbf{L}^{\overline{\Box}\overline{\Diamond}}$-2W concept in $\mathbf{K}^{c}$; an $\mathbf{L}^{\bar{\#}}$-2W concept in $\mathbf{K}$ is equivalent to an $\mathbf{L}^{\#}$-2W concept in $\mathbf{K}^{c}$. The converse is also valid. In summation, the following equations hold:

$$\mathbf{C}^{\bar{*}}_{\mathbf{L}}(\mathbf{K}) = \mathbf{C}^{*}_{\mathbf{L}}(\mathbf{K}^{c}),\ \mathbf{C}^{\overline{\Diamond}\overline{\Box}}_{\mathbf{L}}(\mathbf{K}) = \mathbf{C}^{\Diamond\Box}_{\mathbf{L}}(\mathbf{K}^{c}),$$
$$\mathbf{C}^{\overline{\Box}\overline{\Diamond}}_{\mathbf{L}}(\mathbf{K}) = \mathbf{C}^{\Box\Diamond}_{\mathbf{L}}(\mathbf{K}^{c}),\ \mathbf{C}^{\bar{\#}}_{\mathbf{L}}(\mathbf{K}) = \mathbf{C}^{\#}_{\mathbf{L}}(\mathbf{K}^{c}),$$
$$\mathbf{C}^{*}_{\mathbf{L}}(\mathbf{K}) = \mathbf{C}^{\bar{*}}_{\mathbf{L}}(\mathbf{K}^{c}),\ \mathbf{C}^{\Diamond\Box}_{\mathbf{L}}(\mathbf{K}) = \mathbf{C}^{\overline{\Diamond}\overline{\Box}}_{\mathbf{L}}(\mathbf{K}^{c}),$$
$$\mathbf{C}^{\Box\Diamond}_{\mathbf{L}}(\mathbf{K}) = \mathbf{C}^{\overline{\Box}\overline{\Diamond}}_{\mathbf{L}}(\mathbf{K}^{c}),\ \mathbf{C}^{\#}_{\mathbf{L}}(\mathbf{K}) = \mathbf{C}^{\bar{\#}}_{\mathbf{L}}(\mathbf{K}^{c}).$$

The equivalence between $\mathbf{L}$2W operators naturally yields the following correspondence of the $\mathbf{L}$2W concepts.

Theorem 1 *For $\tilde{O} \in L^{OB}$ and $\tilde{A} \in L^{AT}$, the following statements are equivalent*:

1. $\langle \tilde{O}, \tilde{A}\rangle \in \mathbf{C}^{*}_{\mathbf{L}}(\mathbf{K})$;
2. $\langle \tilde{O}, \tilde{A}^{c}\rangle \in \mathbf{C}^{\overline{\Diamond}\overline{\Box}}_{\mathbf{L}}(\mathbf{K})$;
3. $\langle \tilde{O}^{c}, \tilde{A}\rangle \in \mathbf{C}^{\overline{\Box}\overline{\Diamond}}_{\mathbf{L}}(\mathbf{K})$;
4. $\langle \tilde{O}^{c}, \tilde{A}^{c}\rangle \in \mathbf{C}^{\#}_{\mathbf{L}}(\mathbf{K})$.

Proof For $\tilde{O} \subseteq L^{OB}$ and $\tilde{A} \subseteq L^{AT}$, the following holds:

$$\begin{aligned}\langle \tilde{O}^{c}, \tilde{A}\rangle \in \mathbf{C}^{\overline{\Box}\overline{\Diamond}}_{\mathbf{L}}(\mathbf{K}) &\Leftrightarrow (\tilde{O}^{c})^{\overline{\Box}} = \tilde{A},\ \tilde{A}^{\overline{\Diamond}} = \tilde{O}^{c}\\ &\Leftrightarrow (\tilde{O}^{c})^{\#c} = \tilde{A},\ (\tilde{A}^{c})^{\#} = \tilde{O}^{c}\\ &\Leftrightarrow (\tilde{O}^{c})^{\#} = \tilde{A}^{c},\ (\tilde{A}^{c})^{\#} = \tilde{O}^{c}\\ &\Leftrightarrow \langle \tilde{O}^{c},\ \tilde{A}^{c}\rangle \in \mathbf{C}^{\#}_{\mathbf{L}}(\mathbf{K}).\end{aligned}$$

Similarly, one can prove the other equivalences.

As a direct consequence of Theorem 1, we obtain the equivalent relationship between the other four types of $\mathbf{L}$2W concepts.

Corollary 1 *For $\tilde{O} \subseteq L^{OB}$ and $\tilde{A} \subseteq L^{AT}$, the following statements are equivalent*:

1. $\langle \tilde{O}, \tilde{A}\rangle \in \mathbf{C}^{\bar{*}}_{\mathbf{L}}(\mathbf{K})$;
2. $\langle \tilde{O}, \tilde{A}^{c}\rangle \in \mathbf{C}^{\Diamond\Box}_{\mathbf{L}}(\mathbf{K})$;
3. $\langle \tilde{O}^{c}, \tilde{A}\rangle \in \mathbf{C}^{\Box\Diamond}_{\mathbf{L}}(\mathbf{K})$;
4. $\langle \tilde{O}^{c}, \tilde{A}^{c}\rangle \in \mathbf{C}^{\bar{\#}}_{\mathbf{L}}(\mathbf{K})$.

Theorem 1 and Corollary 1 show the equivalent relationship between the $\mathbf{L}$2W concepts. These can be divided into two groups. From each type of $\mathbf{L}$2W concept, one can obtain the other three types, within the same group, by computing only the complements of the extent and (or) the intent. Referring to Example 2, we can observe that $\langle \tilde{O}_1, \tilde{A}_1^{c}\rangle$ is an $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$-2W concept and $\langle \tilde{O}_2, \tilde{A}_2^{c}\rangle$ is an $\mathbf{L}^{\Diamond\Box}$-2W concept.

L2W Concept Lattices

This section proves that each type of concept forms a complete lattice, and these complete lattices form two isomorphism groups. The following are the definitions of order, infimum, and supremum of each type of $\mathbf{L}$2W concept.

Definition 2 Let $\mathbf{K} = (OB, AT, \tilde{R}, L)$ be an $\mathbf{L}$-context, $\tilde{O}_1, \tilde{O}_2 \in L^{OB}$, and $\tilde{A}_1, \tilde{A}_2 \in L^{AT}$.

1. For $\langle \tilde{O}_1, \tilde{A}_1\rangle, \langle \tilde{O}_2, \tilde{A}_2\rangle \in \mathbf{C}^{*}_{\mathbf{L}}(\mathbf{K})$, $\langle \tilde{O}_1, \tilde{A}_1\rangle \leq_{*} \langle \tilde{O}_2, \tilde{A}_2\rangle$ iff $\tilde{O}_1 \subseteq \tilde{O}_2$ (or $\tilde{A}_2 \subseteq \tilde{A}_1$), and
$$\langle \tilde{O}_1, \tilde{A}_1\rangle \wedge_{*} \langle \tilde{O}_2, \tilde{A}_2\rangle = \langle \tilde{O}_1 \cap \tilde{O}_2, (\tilde{A}_1 \cup \tilde{A}_2)^{**}\rangle,$$
$$\langle \tilde{O}_1, \tilde{A}_1\rangle \vee_{*} \langle \tilde{O}_2, \tilde{A}_2\rangle = \langle (\tilde{O}_1 \cup \tilde{O}_2)^{**}, \tilde{A}_1 \cap \tilde{A}_2\rangle.$$
2. For $\langle \tilde{O}_1, \tilde{A}_1\rangle, \langle \tilde{O}_2, \tilde{A}_2\rangle \in \mathbf{C}^{\bar{*}}_{\mathbf{L}}(\mathbf{K})$, $\langle \tilde{O}_1, \tilde{A}_1\rangle \leq_{\bar{*}} \langle \tilde{O}_2, \tilde{A}_2\rangle$ iff $\tilde{O}_1 \subseteq \tilde{O}_2$ (or $\tilde{A}_2 \subseteq \tilde{A}_1$), and
$$\langle \tilde{O}_1, \tilde{A}_1\rangle \wedge_{\bar{*}} \langle \tilde{O}_2, \tilde{A}_2\rangle = \langle \tilde{O}_1 \cap \tilde{O}_2, (\tilde{A}_1 \cup \tilde{A}_2)^{\overline{**}}\rangle,$$
$$\langle \tilde{O}_1, \tilde{A}_1\rangle \vee_{\bar{*}} \langle \tilde{O}_2, \tilde{A}_2\rangle = \langle (\tilde{O}_1 \cup \tilde{O}_2)^{\overline{**}}, \tilde{A}_1 \cap \tilde{A}_2\rangle.$$
3. For $\langle \tilde{O}_1, \tilde{A}_1\rangle, \langle \tilde{O}_2, \tilde{A}_2\rangle \in \mathbf{C}^{\overline{\Diamond}\overline{\Box}}_{\mathbf{L}}(\mathbf{K})$, $\langle \tilde{O}_1, \tilde{A}_1\rangle \leq_{\overline{\Diamond}\overline{\Box}} \langle \tilde{O}_2, \tilde{A}_2\rangle$ iff $\tilde{O}_1 \subseteq \tilde{O}_2$ (or $\tilde{A}_1 \subseteq \tilde{A}_2$), and
$$\langle \tilde{O}_1, \tilde{A}_1\rangle \wedge_{\overline{\Diamond}\overline{\Box}} \langle \tilde{O}_2, \tilde{A}_2\rangle = \langle \tilde{O}_1 \cap \tilde{O}_2, (\tilde{A}_1 \cap \tilde{A}_2)^{\overline{\Box}\overline{\Diamond}}\rangle,$$
$$\langle \tilde{O}_1, \tilde{A}_1\rangle \vee_{\overline{\Diamond}\overline{\Box}} \langle \tilde{O}_2, \tilde{A}_2\rangle = \langle (\tilde{O}_1 \cup \tilde{O}_2)^{\overline{\Diamond}\overline{\Box}}, \tilde{A}_1 \cup \tilde{A}_2\rangle.$$
4. For $\langle \tilde{O}_1, \tilde{A}_1\rangle, \langle \tilde{O}_2, \tilde{A}_2\rangle \in \mathbf{C}^{\Diamond\Box}_{\mathbf{L}}(\mathbf{K})$, $\langle \tilde{O}_1, \tilde{A}_1\rangle \leq_{\Diamond\Box} \langle \tilde{O}_2, \tilde{A}_2\rangle$ iff $\tilde{O}_1 \subseteq \tilde{O}_2$ (or $\tilde{A}_1 \subseteq \tilde{A}_2$), and

$$\langle \tilde{O}_1, \tilde{A}_1 \rangle \wedge_{\Diamond\Box} \langle \tilde{O}_2, \tilde{A}_2 \rangle = \langle \tilde{O}_1 \cap \tilde{O}_2, (\tilde{A}_1 \cap \tilde{A}_2)^{\Box\Diamond} \rangle,$$
$$\langle \tilde{O}_1, \tilde{A}_1 \rangle \vee_{\Diamond\Box} \langle \tilde{O}_2, \tilde{A}_2 \rangle = \langle (\tilde{O}_1 \cup \tilde{O}_2)^{\Diamond\Box}, \tilde{A}_1 \cup \tilde{A}_2 \rangle.$$

5. For $\langle \tilde{O}_1, \tilde{A}_1 \rangle, \langle \tilde{O}_2, \tilde{A}_2 \rangle \in \mathbf{C}_{\mathbf{L}}^{\overline{\Box}\overline{\Diamond}}(\mathbf{K})$, $\langle \tilde{O}_1, \tilde{A}_1 \rangle \leq_{\overline{\Box}\overline{\Diamond}} \langle \tilde{O}_2, \tilde{A}_2 \rangle$ iff $\tilde{O}_2 \subseteq \tilde{O}_1$ (or $\tilde{A}_2 \subseteq \tilde{A}_1$), and
$$\langle \tilde{O}_1, \tilde{A}_1 \rangle \wedge_{\overline{\Box}\overline{\Diamond}} \langle \tilde{O}_2, \tilde{A}_2 \rangle = \langle \tilde{O}_1 \cup \tilde{O}_2, (\tilde{A}_1 \cup \tilde{A}_2)^{\overline{\Diamond}\overline{\Box}} \rangle,$$
$$\langle \tilde{O}_1, \tilde{A}_1 \rangle \vee_{\overline{\Box}\overline{\Diamond}} \langle \tilde{O}_2, \tilde{A}_2 \rangle = \langle (\tilde{O}_1 \cap \tilde{O}_2)^{\overline{\Box}\overline{\Diamond}}, \tilde{A}_1 \cap \tilde{A}_2 \rangle.$$

6. For $\langle \tilde{O}_1, \tilde{A}_1 \rangle, \langle \tilde{O}_2, \tilde{A}_2 \rangle \in \mathbf{C}_{\mathbf{L}}^{\Box\Diamond}(\mathbf{K})$, $\langle \tilde{O}_1, \tilde{A}_1 \rangle \leq_{\Box\Diamond} \langle \tilde{O}_2, \tilde{A}_2 \rangle$ iff $\tilde{O}_2 \subseteq \tilde{O}_1$ (or $\tilde{A}_2 \subseteq \tilde{A}_1$), and
$$\langle \tilde{O}_1, \tilde{A}_1 \rangle \wedge_{\Box\Diamond} \langle \tilde{O}_2, \tilde{A}_2 \rangle = \langle \tilde{O}_1 \cup \tilde{O}_2, (\tilde{A}_1 \cup \tilde{A}_2)^{\Diamond\Box} \rangle,$$
$$\langle \tilde{O}_1, \tilde{A}_1 \rangle \vee_{\Box\Diamond} \langle \tilde{O}_2, \tilde{A}_2 \rangle = \langle (\tilde{O}_1 \cap \tilde{O}_2)^{\Box\Diamond}, \tilde{A}_1 \cap \tilde{A}_2 \rangle.$$

7. For $\langle \tilde{O}_1, \tilde{A}_1 \rangle, \langle \tilde{O}_2, \tilde{A}_2 \rangle \in \mathbf{C}_{\mathbf{L}}^{\#}(\mathbf{K})$, $\langle \tilde{O}_1, \tilde{A}_1 \rangle \leq_{\#} \langle \tilde{O}_2, \tilde{A}_2 \rangle$ iff $\tilde{O}_2 \subseteq \tilde{O}_1$ (or $\tilde{A}_1 \subseteq \tilde{A}_2$), and
$$\langle \tilde{O}_1, \tilde{A}_1 \rangle \wedge_{\#} \langle \tilde{O}_2, \tilde{A}_2 \rangle = \langle \tilde{O}_1 \cup \tilde{O}_2, (\tilde{A}_1 \cap \tilde{A}_2)^{\#\#} \rangle,$$
$$\langle \tilde{O}_1, \tilde{A}_1 \rangle \vee_{\#} \langle \tilde{O}_2, \tilde{A}_2 \rangle = \langle (\tilde{O}_1 \cap \tilde{O}_2)^{\#\#}, \tilde{A}_1 \cup \tilde{A}_2 \rangle.$$

8. For $\langle \tilde{O}_1, \tilde{A}_1 \rangle, \langle \tilde{O}_2, \tilde{A}_2 \rangle \in \mathbf{C}_{\mathbf{L}}^{\overline{\#}}(\mathbf{K})$, $\langle \tilde{O}_1, \tilde{A}_1 \rangle \leq_{\overline{\#}} \langle \tilde{O}_2, \tilde{A}_2 \rangle$ iff $\tilde{O}_2 \subseteq \tilde{O}_1$ (or $\tilde{A}_1 \subseteq \tilde{A}_2$), and
$$\langle \tilde{O}_1, \tilde{A}_1 \rangle \wedge_{\overline{\#}} \langle \tilde{O}_2, \tilde{A}_2 \rangle = \langle \tilde{O}_1 \cup \tilde{O}_2, (\tilde{A}_1 \cap \tilde{A}_2)^{\overline{\#\#}} \rangle,$$
$$\langle \tilde{O}_1, \tilde{A}_1 \rangle \vee_{\overline{\#}} \langle \tilde{O}_2, \tilde{A}_2 \rangle = \langle (\tilde{O}_1 \cap \tilde{O}_2)^{\overline{\#\#}}, \tilde{A}_1 \cup \tilde{A}_2 \rangle.$$

According to Propositions 1(4) and 2(4), it is clear that $(\mathbf{C}_{\mathbf{L}}^{?}(\mathbf{K}), \wedge_?, \vee_?)$ is a lattice, where $? = *$, $\bar{*}$, $\overline{\Box}\overline{\Diamond}$, $\Box\Diamond$, $\overline{\Diamond}\overline{\Box}$, $\Diamond\Box$, $\overline{\#}$, and $\#$. We present the main theorem of the $\mathbf{L}$2W concepts.

Theorem 2 *Given an* $\mathbf{L}$*-context*, $\mathbf{K} = (OB, AT, \tilde{R}, L)$, $(\mathbf{C}_{\mathbf{L}}^{?}(\mathbf{K}), \wedge_?, \vee_?)$ *is a complete lattice, where* $? = *$, $\bar{*}$, $\overline{\Box}\overline{\Diamond}$, $\Box\Diamond$, $\overline{\Diamond}\overline{\Box}$, $\Diamond\Box$, $\overline{\#}$, *and* $\#$.

Proof To prove that $\mathbf{C}_{\mathbf{L}}^{*}(\mathbf{K})$ is a complete lattice, we assume $\langle \tilde{O}_i, \tilde{A}_i \rangle \in \mathbf{C}_{\mathbf{L}}^{*}(\mathbf{K})$, $i \in \Lambda$. Items (3) and (4) of Proposition 1 show that $\langle \bigcap_{i\in\Lambda} \tilde{O}_i, (\bigcup_{i\in\Lambda} \tilde{A}_i)^{**} \rangle \in \mathbf{C}_{\mathbf{L}}^{*}(\mathbf{K})$ and $\langle \bigcap_{i\in\Lambda} \tilde{O}_i, (\bigcup \tilde{A}_i)^{**} \rangle \leq_* \langle \tilde{O}_i, \tilde{A}_i \rangle$ for each $i \in \Lambda$. Next, we prove that $\langle \bigcap_{i\in\Lambda} \tilde{O}_i, (\bigcup \tilde{A}_i)^{**} \rangle$ is the infimum. If not, suppose that $\langle \tilde{O}, \tilde{A} \rangle \leq_* \langle \tilde{O}_i, \tilde{A}_i \rangle$ for each $i \in \Lambda$ and $\langle \bigcap_{i\in\Lambda} \tilde{O}_i, (\bigcup \tilde{A}_i)^{**} \rangle \leq_* \langle \tilde{O}, \tilde{A} \rangle$. Then, it follows that $\tilde{O} \subseteq \tilde{O}_i$ for each $i \in \Lambda$ and $\bigcap_{i\in\Lambda} \tilde{O}_i \subseteq \tilde{O}$; consequently, $\bigcap_{i\in\Lambda} \tilde{O}_i = \tilde{O}$. Proposition 1(4) reveals that $\tilde{A} = \tilde{O}^* = (\bigcap_{i\in\Lambda} \tilde{O}_i)^* = (\bigcap_{i\in\Lambda} \tilde{A}_i^*)^* = (\bigcup_{i\in\Lambda} \tilde{A}_i)^{**}$. That is, $\langle \bigcap_{i\in\Lambda} \tilde{O}_i, (\bigcup_{i\in\Lambda} \tilde{A}_i)^{**} \rangle$ is the infimum of $\langle \tilde{O}_i, \tilde{A}_i \rangle \in \mathbf{C}_{\mathbf{L}}^{*}(\mathbf{K})$, $i \in \Lambda$. Similarly, we prove that $\langle (\bigcup_{i\in\Lambda} \tilde{O}_i)^{**}, \bigcap_{i\in\Lambda} \tilde{A}_i \rangle$ is the supremum of $\langle \tilde{O}_i, \tilde{A}_i \rangle \in \mathbf{C}_{\mathbf{L}}^{*}(\mathbf{K})$, $i \in \Lambda$. Therefore, $(\mathbf{C}_{\mathbf{L}}^{*}(\mathbf{K}), \wedge_*, \vee_*)$ is a complete lattice.

The others can be similarly proved.

Theorem 2 demonstrates that each $\mathbf{L}$2W concept lattice is a complete lattice. Based on the results of Theorem 1 and Corollary 1, the eight types of $\mathbf{L}$2W concept lattices can be divided into two groups.

Theorem 3 *Let* $\mathbf{K} = (OB, AT, \tilde{R}, L)$ *be an* $\mathbf{L}$*-context. Then, the following hold*:

1. $\mathbf{C}_{\mathbf{L}}^{*}(\mathbf{K}) \cong \mathbf{C}_{\mathbf{L}}^{\overline{\Diamond}\overline{\Box}}(\mathbf{K}) \cong \mathbf{C}_{\mathbf{L}}^{\overline{\Box}\overline{\Diamond}}(\mathbf{K}) \cong \mathbf{C}_{\mathbf{L}}^{\#}(\mathbf{K})$;
2. $\mathbf{C}_{\mathbf{L}}^{\bar{*}}(\mathbf{K}) \cong \mathbf{C}_{\mathbf{L}}^{\Diamond\Box}(\mathbf{K}) \cong \mathbf{C}_{\mathbf{L}}^{\Box\Diamond}(\mathbf{K}) \cong \mathbf{C}_{\mathbf{L}}^{\overline{\#}}(\mathbf{K})$.

The notation $\cong$ represents isomorphic relation.[4]

Proof (1) We assume that $f : \mathbf{C}_{\mathbf{L}}^{\overline{\Box}\overline{\Diamond}}(\mathbf{K}) \longrightarrow \mathbf{C}_{\mathbf{L}}^{\#}(\mathbf{K})$ such that $f(\langle \tilde{O}, \tilde{A} \rangle) = \langle \tilde{O}, \tilde{A}^c \rangle$ for $\langle \tilde{O}, \tilde{A} \rangle \in \mathbf{C}_{\mathbf{L}}^{\overline{\Box}\overline{\Diamond}}(\mathbf{K})$. Theorem 1 indicates that f is a bijection between $\mathbf{C}_{\mathbf{L}}^{\overline{\Box}\overline{\Diamond}}(\mathbf{K})$ and $\mathbf{C}_{\mathbf{L}}^{\#}(\mathbf{K})$. Next, we suppose that $\langle \tilde{O}_1, \tilde{A}_1 \rangle, \langle \tilde{O}_2, \tilde{A}_2 \rangle \in \mathbf{C}_{\mathbf{L}}^{\overline{\Box}\overline{\Diamond}}(\mathbf{K})$. Table 2 provides the following assertions:

$$\begin{aligned}
&f(\langle \tilde{O}_1, \tilde{A}_1 \rangle \wedge_{\overline{\Box}\overline{\Diamond}} \langle \tilde{O}_2, \tilde{A}_2 \rangle)\\
&= f(\langle \tilde{O}_1 \cup \tilde{O}_2, (\tilde{A}_1 \cup \tilde{A}_2)^{\overline{\Diamond}\overline{\Box}} \rangle)\\
&= \langle \tilde{O}_1 \cup \tilde{O}_2, (\tilde{A}_1 \cup \tilde{A}_2)^{\overline{\Diamond}\overline{\Box}c} \rangle\\
&= \langle \tilde{O}_1 \cup \tilde{O}_2, (\tilde{A}_1 \cup \tilde{A}_2)^{\overline{\Diamond}\#} \rangle,\\
&f(\langle \tilde{O}_1, \tilde{A}_1 \rangle \vee_{\overline{\Box}\overline{\Diamond}} \langle \tilde{O}_2, \tilde{A}_2 \rangle)\\
&= f(\langle (\tilde{O}_1 \cap \tilde{O}_2)^{\overline{\Box}\overline{\Diamond}}, \tilde{A}_1 \cap \tilde{A}_2 \rangle)\\
&= \langle (\tilde{O}_1 \cap \tilde{O}_2)^{\overline{\Box}\overline{\Diamond}}, (\tilde{A}_1 \cap \tilde{A}_2)^c \rangle.
\end{aligned}$$

[4] For two lattices $\mathbf{L}_1 = (L_1, \wedge_1, \vee_1)$ and $\mathbf{L}_2 = (L_2, \wedge_2, \vee_2)$, we say $\mathbf{L}_1$ is isomorphic with $\mathbf{L}_2$ (denoted as $\mathbf{L}_1 \cong \mathbf{L}_2$) if and only if there exists a bijection $f : L_1 \longrightarrow L_2$ such that $f(a \wedge_1 b) = f(a) \wedge_2 f(b)$ and $f(a \vee_1 b) = f(a) \vee_2 f(b)$, and f is called an isomorphism between $\mathbf{L}_1$ and $\mathbf{L}_2$. In the view of graphs, we say two graphs G_1 and G_2 are isomorphic if there exists a matching between their vertices so that two vertices are connected by an edge in G_1 if and only if corresponding vertices are connected by an edge in G_2. For example, the following two graphs are isomorphic.

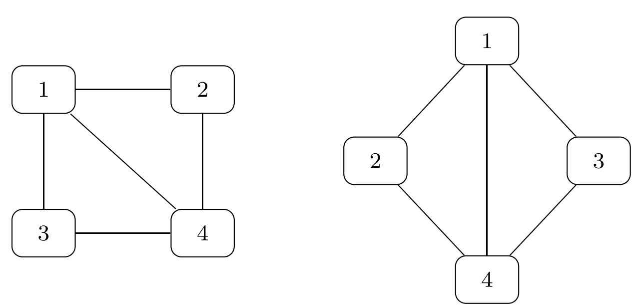


Table 2 still supports that

$$
\begin{aligned}
&f(\langle \tilde{O}_1, \tilde{A}_1 \rangle) \wedge_{\#} f(\langle \tilde{O}_2, \tilde{A}_2 \rangle) \\
&= \langle \tilde{O}_1, \tilde{A}_1^c \rangle \wedge_{\#} \langle \tilde{O}_2, \tilde{A}_2^c \rangle \\
&= \langle \tilde{O}_1 \cup \tilde{O}_2, (\tilde{A}_1^c \cap \tilde{A}_2^c)^{\#\#} \rangle \\
&= \langle \tilde{O}_1 \cup \tilde{O}_2, (\tilde{A}_1 \cup \tilde{A}_2)^{c\#\#} \rangle \\
&= \langle \tilde{O}_1 \cup \tilde{O}_2, (\tilde{A}_1 \cup \tilde{A}_2)^{\overline{\Diamond}\#} \rangle, \\
&f(\langle \tilde{O}_1, \tilde{A}_1 \rangle) \vee_{\#} f(\langle \tilde{O}_2, \tilde{A}_2 \rangle) \\
&= \langle \tilde{O}_1, \tilde{A}_1^c \rangle \vee_{\#} \langle \tilde{O}_2, \tilde{A}_2^c \rangle \\
&= \langle (\tilde{O}_1 \cap \tilde{O}_2)^{\#\#}, \tilde{A}_1^c \cup \tilde{A}_2^c \rangle.
\end{aligned}
$$

Table 3 suggests that $(\tilde{O}_1 \cap \tilde{O}_2)^{\#\#} = (\tilde{O}_1 \cup \tilde{O}_2)^{\overline{\Box}\overline{\Diamond}}$, which means f is $\wedge$-preserving and $\vee$-preserving. By letting $f(\langle \tilde{O}, \tilde{A} \rangle) = \langle \tilde{O}, \tilde{A}^c \rangle$ and $f(\langle \tilde{O}, \tilde{A} \rangle) = \langle \tilde{O}^c, \tilde{A}^c \rangle$, one proves the isomorphism between $\mathbf{C}_{\mathbf{L}}^{*}(\mathbf{K})$ and $\mathbf{C}_{\mathbf{L}}^{\overline{\Diamond}\overline{\Box}}(\mathbf{K})$ and between $\mathbf{C}_{\mathbf{L}}^{\overline{\Diamond}\overline{\Box}}(\mathbf{K})$ and $\mathbf{C}_{\mathbf{L}}^{\overline{\Box}\overline{\Diamond}}(\mathbf{K})$, respectively.

(2) The proof is similar to that of Item (1).

Theorem 3 reveals the isomorphic relationship between the **L**2W concept lattices. The eight types of **L**2W concept lattices constitute two groups; each concept lattice is isomorphic with the other three in the same group. Using the results in Theorems 1 and 3 and Corollary 1, we only need to construct a type of concept lattice in each group. The following is a brief example of the construction of **L**2W concept lattices.

Example 3 Given an **L**-context, $\mathbf{K} = (OB, AT, \tilde{R}, L)$, with four objects and three attributes, the **L**-relation is represented as

$$
\tilde{R} = \begin{array}{c} \\ o_1 \\ o_2 \\ o_3 \\ o_4 \end{array}
\begin{array}{c}
\begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\
\begin{pmatrix} 0 & 1 & 0.5 \\ 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0 \\ 0.5 & 0.5 & 1 \end{pmatrix}
\end{array}.
$$

Let $L = \{0, 0.5, 1\}$ be the truth-value set, and define $\lambda \otimes \mu = \max\{\lambda + \mu - 1, 0\}$ and $\lambda \to \mu = \min\{1 - \lambda + \mu, 1\}$ for $\lambda, \mu \in L$. The method proposed by Bězlohlávek [57] is adopted to construct the $\mathbf{L}^*$-2W concept lattice. The extent and intent of the $\mathbf{L}^*$-2W concepts are listed in Table 4. Figure 2 shows the Hasse diagram of the $\mathbf{L}^*$-2W concept lattice. The number of nodes corresponds to the number of each $\mathbf{L}^*$-2W concept in Table 4. A line connects two concepts, in which the lower concept is a sub-concept of the upper one.

Based on the results of Theorems 1 and 3, $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$-, $\mathbf{L}^{\overline{\Box}\overline{\Diamond}}$-, and $\mathbf{L}^{\#}$-2W concept lattices can be easily obtained. For example, by computing the complement of $\mathbf{L}^*$-2W concept intent, we can obtain all $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$-2W concepts (see Table 5).

Table 4 $\mathbf{L}^*$-2W concepts

No.	$\tilde{O}$	$\tilde{A}$
1	(1.0, 1.0, 1.0, 1.0)	(0.0, 0.5, 0.0)
2	(1.0, 1.0, 0.5, 1.0)	(0.0, 0.5, 0.5)
3	(1.0, 0.5, 1.0, 0.5)	(0.0, 1.0, 0.0)
4	(1.0, 0.5, 0.5, 0.5)	(0.0, 1.0, 0.5)
5	(0.5, 1.0, 1.0, 1.0)	(0.5, 0.5, 0.0)
6	(0.5, 1.0, 0.5, 1.0)	(0.5, 0.5, 0.5)
7	(0.5, 0.5, 0.0, 1.0)	(0.5, 0.5, 1.0)
8	(0.5, 0.5, 1.0, 0.5)	(0.5, 1.0, 0.0)
9	(0.5, 0.5, 0.5, 0.5)	(0.5, 1.0, 0.5)
10	(0.5, 0.5, 0.0, 0.5)	(0.5, 1.0, 1.0)
11	(0.0, 1.0, 0.5, 0.5)	(1.0, 0.5, 0.5)
12	(0.0, 0.5, 0.5, 0.5)	(1.0, 1.0, 0.5)
13	(0.0, 0.5, 0.0, 0.5)	(1.0, 1.0, 1.0)

According to Theorem 3, the $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$-2W concept lattice structure is the same as that of the $\mathbf{L}^*$-2W concept lattice.

Relationship Between L3W Concept Lattices

Early traces of **L**3W concept analysis can be found in [58], where Bartl and Konecny studied the **L**-concept with positive and negative attributes and used a pair of antitone and isotone concept-forming operators to define an **L**3W concept. Subsequently, He, Wei, and She [59] used a pair of antitone concept-forming operators to represent a type of **L**O3W concept and a type of **L**A3W concept. Singh [42] proposed a three-way fuzzy concept within the neutrosophic context, which consists of a pair of neutrosophic sets. A neutrosophic set, N, consists of three functions, T_N, I_N, and F_N, namely, the truth-membership function, indeterminacy-membership function, and falsity-membership function, respectively. The number 'three' of 'three-way fuzzy concept' means that three membership functions represent both the extent and intent of a three-way fuzzy concept. However, the **L**3W concepts in this study have a different meaning, which is the main focus of this section.

L3W Operators

Considering three-way operators in a fuzzy set view, we can have four types of **L**O3W operators and four types of **L**A3W operators and their inverses (see Table 6). The relationships between **L**O3W operators and between **L**O3W inverse operators are illustrated in Fig. 3a and b, respectively. One can convert the operators, connected by a double-arrowed line, by taking the operation with

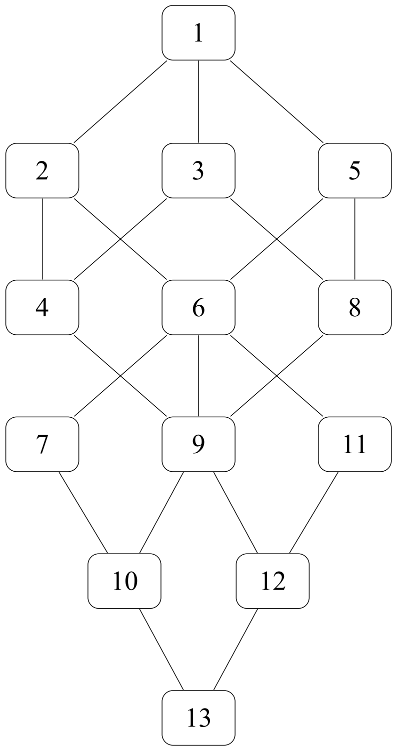


Fig. 2 $\mathbf{L}^{*}$-2W concept lattice

the line. For example, $\tilde{O}^{\lessdot}$ can be obtained by substituting $\tilde{O}^{c}$ for $\tilde{O}$ in $\tilde{O}^{\bigtriangledown}$, namely, $\tilde{O}^{\lessdot}=(\tilde{O}^{c})^{\bigtriangledown}=\tilde{O}^{c\bigtriangledown}$; $(\tilde{O}_1,\tilde{O}_2)^{\gtrdot}$ can be obtained by replacing $(\tilde{O}_1,\tilde{O}_2)$ with $(\tilde{O}_1,\tilde{O}_2)^{c}$ in $(\tilde{O}_1,\tilde{O}_2)^{\blacktriangle}$, i.e. $(\tilde{O}_1,\tilde{O}_2)^{\gtrdot}=((\tilde{O}_1,\tilde{O}_2)^{c})^{\blacktriangle}=(\tilde{O}_1,\tilde{O}_2)^{c\blacktriangle}$. For a better understanding, Table 7 lists the equivalences of **L** O3W operators and **L**O3W inverse operators. Appendix II provides proof of these equations. We omit the results for **L**A3W operators and their inverses for simplicity. Moreover, Table 8 reveals the connections between the two applications of **L**3W operators. According to the properties of $\mathbf{L}^{*}$- and $\mathbf{L}^{\bar{*}}$-2W operators, the basic properties of $\mathbf{L}^{\lessdot}$-3W operators and associated inverses are presented as follows:

Proposition 4 *For* $\tilde{O},\tilde{O}_i,\tilde{O}_j,\tilde{O}_{ik}\in L^{OB},\tilde{A},\tilde{A}_i,\tilde{A}_j,\tilde{A}_{ik}\in L^{AT}$ $(i\in\Lambda,\ j=1,2,3,4,$ *and* $k=1,2)$, *the following hold*:

1. *If* $\tilde{O}_1\subseteq\tilde{O}_2$, *then* $\tilde{O}_2^{\lessdot}\subseteq\tilde{O}_1^{\lessdot}$, *if* $\tilde{A}_1\subseteq\tilde{A}_2$, *then* $\tilde{A}_2^{\lessdot}\subseteq\tilde{A}_1^{\lessdot}$;
2. $\tilde{O}\subseteq\tilde{O}^{\lessdot\gtrdot},\tilde{A}\subseteq\tilde{A}^{\lessdot\gtrdot}$;
3. $\tilde{O}^{\lessdot}=\tilde{O}^{\lessdot\gtrdot\lessdot},\tilde{A}^{\lessdot}=\tilde{A}^{\lessdot\gtrdot\lessdot}$;
4. $(\bigcup_{i\in\Lambda}\tilde{O}_i)^{\lessdot}=\bigcap_{i\in\Lambda}\tilde{O}_i^{\lessdot},(\bigcup_{i\in\Lambda}\tilde{A}_i)^{\lessdot}=\bigcap_{i\in\Lambda}\tilde{A}_i^{\lessdot}$;
5. $(\bigcap_{i\in\Lambda}\tilde{O}_i)^{\lessdot}\supseteq\bigcup_{i\in\Lambda}\tilde{O}_i^{\lessdot},(\bigcap_{i\in\Lambda}\tilde{A}_i)^{\lessdot}\supseteq\bigcup_{i\in\Lambda}\tilde{A}_i^{\lessdot}$;
6. *If* $(\tilde{O}_1,\tilde{O}_2)\subseteq(\tilde{O}_3,\tilde{O}_4)$, *then* $(\tilde{O}_3,\tilde{O}_4)^{\gtrdot}\subseteq(\tilde{O}_1,\tilde{O}_2)^{\gtrdot}$; *if* $(\tilde{A}_1,\tilde{A}_2)\subseteq(\tilde{A}_3,\tilde{A}_4)$, *then* $(\tilde{A}_3,\tilde{A}_4)^{\gtrdot}\subseteq(\tilde{A}_1,\tilde{A}_2)^{\gtrdot}$;
7. $(\tilde{O}_1,\tilde{O}_2)\subseteq(\tilde{O}_1,\tilde{O}_2)^{\gtrdot\lessdot},(\tilde{A}_1,\tilde{A}_2)\subseteq(\tilde{A}_1,\tilde{A}_2)^{\gtrdot\lessdot}$;
8. $(\tilde{O}_1,\tilde{O}_2)^{\gtrdot}=(\tilde{O}_1,\tilde{O}_2)^{\gtrdot\lessdot\gtrdot},(\tilde{A}_1,\tilde{A}_2)^{\gtrdot}=(\tilde{A}_1,\tilde{A}_2)^{\gtrdot\lessdot\gtrdot}$;
9. $\left(\bigcup_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})\right)^{\gtrdot}=\bigcap_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})^{\gtrdot}$, $\left(\bigcup_{i\in\Lambda}(\tilde{A}_{i1},\tilde{A}_{i2})\right)^{\gtrdot}=\bigcap_{i\in\Lambda}(\tilde{A}_{i1},\tilde{A}_{i2})^{\gtrdot}$;
10. $\left(\bigcap_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})\right)^{\gtrdot}\supseteq\bigcup_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})^{\gtrdot}$, $\left(\bigcap_{i\in\Lambda}(\tilde{A}_{i1},\tilde{A}_{i2})\right)^{\gtrdot}\supseteq\bigcup_{i\in\Lambda}(\tilde{A}_{i1},\tilde{A}_{i2})^{\gtrdot}$.

Table 5 $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$-2W concepts

No.	$\tilde{O}$	$\tilde{A}$
1	(1.0, 1.0, 1.0, 1.0)	(1.0, 0.5, 1.0)
2	(1.0, 1.0, 0.5, 1.0)	(1.0, 0.5, 0.5)
3	(1.0, 0.5, 1.0, 0.5)	(1.0, 0.0, 1.0)
4	(1.0, 0.5, 0.5, 0.5)	(1.0, 0.0, 0.5)
5	(0.5, 1.0, 1.0, 1.0)	(0.5, 0.5, 1.0)
6	(0.5, 1.0, 0.5, 1.0)	(0.5, 0.5, 0.5)
7	(0.5, 0.5, 0.0, 1.0)	(0.5, 0.5, 0.0)
8	(0.5, 0.5, 1.0, 0.5)	(0.5, 0.0, 1.0)
9	(0.5, 0.5, 0.5, 0.5)	(0.5, 0.0, 0.5)
10	(0.5, 0.5, 0.0, 0.5)	(0.5, 0.0, 0.0)
11	(0.0, 1.0, 0.5, 0.5)	(0.0, 0.5, 0.5)
12	(0.0, 0.5, 0.5, 0.5)	(0.0, 0.0, 0.5)
13	(0.0, 0.5, 0.0, 0.5)	(0.0, 0.0, 0.0)

Proof (1) Propositions 1(1) and 2(1) support that $\tilde{O}_2^{\lessdot}=(\tilde{O}_2^{*},\tilde{O}_2^{\bar{*}})\subseteq(\tilde{O}_1^{*},\tilde{O}_1^{\bar{*}})=\tilde{O}_1^{\lessdot}$ for $\tilde{O}_1^{*}\subseteq\tilde{O}_2^{*}$.

(2) Given $\tilde{O}\subseteq L^{OB}$, based on Propositions 1(2) and 2(2), it follows that $\tilde{O}^{\lessdot\gtrdot}=(\tilde{O}^{*},\tilde{O}^{\bar{*}})^{\gtrdot}=\tilde{O}^{**}\cap\tilde{O}^{\bar{*}\bar{*}}\supseteq\tilde{O}\cap\tilde{O}=\tilde{O}$.

(3) Items (1) and (2) indicate that $\tilde{O}^{\lessdot\gtrdot\lessdot}\subseteq\tilde{O}^{\lessdot}$ for $\tilde{O}\in L^{OB}$. Moreover, from Propositions 1(5) and 2(5), it holds that $\tilde{O}^{\lessdot\gtrdot\lessdot}=(\tilde{O}^{*},\tilde{O}^{\bar{*}})^{\gtrdot\lessdot}=(\tilde{O}^{**}\cap\tilde{O}^{\bar{*}\bar{*}})^{\lessdot}$ $=((\tilde{O}^{**}\cap\tilde{O}^{\bar{*}\bar{*}})^{*},(\tilde{O}^{**}\cap\tilde{O}^{\bar{*}\bar{*}})^{\bar{*}})\supseteq(\tilde{O}^{***}\cup\tilde{O}^{\bar{*}\bar{*}*},\tilde{O}^{**\bar{*}}\cup\tilde{O}^{\bar{*}\bar{*}\bar{*}})\supseteq(\tilde{O}^{***},\tilde{O}^{\bar{*}\bar{*}\bar{*}})=(\tilde{O}^{*},\tilde{O}^{\bar{*}})=\tilde{O}^{\lessdot}$.

Table 6 **L**3W operators

LO3W operator	**L**O3W inverse operator	**L**A3W operator	**L**A3W inverse operator
$\tilde{O}^{\lessdot} = (\tilde{O}^{*}, \tilde{O}^{\bar{*}})$	$(\tilde{O}_1, \tilde{O}_2)^{\gtrdot} = \tilde{O}_1^{*} \cap \tilde{O}_2^{\bar{*}}$	$\tilde{A}^{\lessdot} = (\tilde{A}^{*}, \tilde{A}^{\bar{*}})$	$(\tilde{A}_1, \tilde{A}_2)^{\gtrdot} = \tilde{A}_1^{*} \cap \tilde{A}_2^{\bar{*}}$
$\tilde{O}^{\triangledown} = (\tilde{O}^{\overline{\square}}, \tilde{O}^{\square})$	$(\tilde{O}_1, \tilde{O}_2)^{\triangle} = \tilde{O}_1^{\overline{\Diamond}} \cup \tilde{O}_2^{\Diamond}$	$\tilde{A}^{\triangledown} = (\tilde{A}^{\overline{\square}}, \tilde{A}^{\square})$	$(\tilde{A}_1, \tilde{A}_2)^{\triangle} = \tilde{A}_1^{\overline{\Diamond}} \cup \tilde{A}_2^{\Diamond}$
$\tilde{O}^{\blacktriangledown} = (\tilde{O}^{\overline{\Diamond}}, \tilde{O}^{\Diamond})$	$(\tilde{O}_1, \tilde{O}_2)^{\blacktriangle} = \tilde{O}_1^{\overline{\square}} \cap \tilde{O}_2^{\square}$	$\tilde{A}^{\blacktriangledown} = (\tilde{A}^{\overline{\Diamond}}, \tilde{A}^{\Diamond})$	$(\tilde{A}_1, \tilde{A}_2)^{\blacktriangle} = \tilde{A}_1^{\overline{\square}} \cap \tilde{A}_2^{\square}$
$\tilde{O}^{\rhd} = (\tilde{O}^{\#}, \tilde{O}^{\bar{\#}})$	$(\tilde{O}_1, \tilde{O}_2)^{\lhd} = \tilde{O}_1^{\#} \cup \tilde{O}_2^{\bar{\#}}$	$\tilde{A}^{\rhd} = (\tilde{A}^{\#}, \tilde{A}^{\bar{\#}})$	$(\tilde{A}_1, \tilde{A}_2)^{\lhd} = \tilde{A}_1^{\#} \cup \tilde{A}_2^{\bar{\#}}$

(4) Propositions 1(4) and 2(4) show that $(\bigcup_{i\in\Lambda} \tilde{O}_i)^{\lessdot} = \big((\bigcup_{i\in\Lambda}\tilde{O}_i)^{*}, (\bigcup_{i\in\Lambda} \tilde{O}_i)^{\bar{*}}\big) = \big(\bigcap_{i\in\Lambda} \tilde{O}_i^{*}, \bigcap_{i\in\Lambda} \tilde{O}_i^{\bar{*}}\big) = \bigcap_{i\in\Lambda} (\tilde{O}_i^{*}, \tilde{O}_i^{\bar{*}}) = \bigcap_{i\in\Lambda} \tilde{O}_i^{\lessdot}$.

(5) It is obvious from Propositions 1(5) and 2(5).

(6) For $(\tilde{O}_1, \tilde{O}_2) \subseteq (\tilde{O}_3, \tilde{O}_4)$ which is equivalent to $\tilde{O}_1 \subseteq \tilde{O}_3$ and $\tilde{O}_2 \subseteq \tilde{O}_4$, it holds that $(\tilde{O}_3, \tilde{O}_4)^{\lessdot} = \tilde{O}_3^{*} \cap \tilde{O}_4^{\bar{*}} \subseteq \tilde{O}_1^{*} \cap \tilde{O}_2^{\bar{*}} = (\tilde{O}_1, \tilde{O}_2)^{\lessdot}$ by Propositions 1(1) and 2(1).

(7) Proposition 1(2) and (5) and Proposition 2(2) and (5) support that $(\tilde{O}_1, \tilde{O}_2)^{\gtrdot\lessdot} = (\tilde{O}_1^{*} \cap \tilde{O}_2^{\bar{*}})^{\lessdot} = \big((\tilde{O}_1^{*} \cap \tilde{O}_2^{\bar{*}})^{*}, (\tilde{O}_1^{*} \cap \tilde{O}_2^{\bar{*}})^{\bar{*}}\big) \supseteq (\tilde{O}_1^{**} \cup \tilde{O}_2^{\bar{*}*}, \tilde{O}_1^{*\bar{*}} \cup \tilde{O}_2^{\bar{*}\bar{*}}) \supseteq (\tilde{O}_1^{**}, \tilde{O}_2^{\bar{*}\bar{*}}) \supseteq (\tilde{O}_1, \tilde{O}_2)$.

(8) Items (6) and (7) show that $(\tilde{O}_1, \tilde{O}_2)^{\gtrdot\lessdot\gtrdot} \subseteq (\tilde{O}_1, \tilde{O}_2)^{\gtrdot}$. By contrast, letting $\tilde{A} = (\tilde{O}_1, \tilde{O}_2)^{\gtrdot}$ follows that $\tilde{A} \subseteq \tilde{A}^{\lessdot\gtrdot}$ by Item (2), namely, $(\tilde{O}_1, \tilde{O}_2)^{\gtrdot} \subseteq (\tilde{O}_1, \tilde{O}_2)^{\gtrdot\lessdot\gtrdot}$.

(9) Proposition 1(4) and Proposition 2(4) verifies that $(\bigcup_{i\in\Lambda}(\tilde{O}_{i1}, \tilde{O}_{i2}))^{\gtrdot} = (\bigcup_{i\in\Lambda} \tilde{O}_{i1}, \bigcup_{i\in\Lambda} \tilde{O}_{i2})^{\gtrdot} = (\bigcup_{i\in\Lambda}\tilde{O}_{i1})^{*} \cap (\bigcup_{i\in\Lambda} \tilde{O}_{i2})^{\bar{*}} = (\bigcap_{i\in\Lambda} \tilde{O}_{i1}^{*}) \cap (\bigcap_{i\in\Lambda} \tilde{O}_{i2}^{\bar{*}}) = \bigcap_{i\in\Lambda}(\tilde{O}_{i1}^{*} \cap \tilde{O}_{i2}^{\bar{*}}) = \bigcap_{i\in\Lambda}(\tilde{O}_{i1}, \tilde{O}_{i2})^{\gtrdot}$.

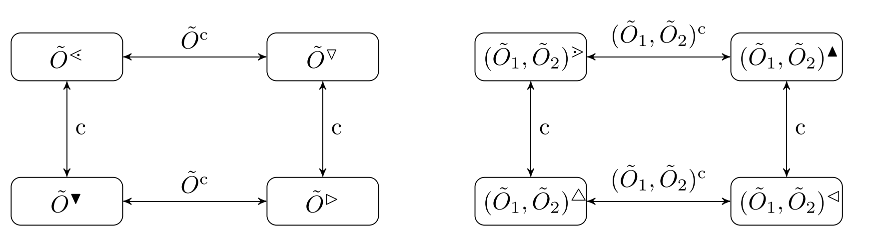
(10) From Proposition 1(5) and Proposition 2(5), it follows that $(\bigcap_{i\in\Lambda}(\tilde{O}_{i1}, \tilde{O}_{i2}))^{\gtrdot} = (\bigcap_{i\in\Lambda} \tilde{O}_{i1}, \bigcap_{i\in\Lambda} \tilde{O}_{i2})^{\gtrdot} = (\bigcap_{i\in\Lambda} \tilde{O}_{i1})^{*} \cap (\bigcap_{i\in\Lambda} \tilde{O}_{i2})^{\bar{*}} \supseteq (\bigcup_{i\in\Lambda} \tilde{O}_{i1}^{*}) \cap (\bigcup_{i\in\Lambda} \tilde{O}_{i2}^{\bar{*}}) \supseteq \bigcup_{i\in\Lambda}(\tilde{O}_{i1}^{*} \cap \tilde{O}_{i2}^{\bar{*}}) = \bigcup_{i\in\Lambda}(\tilde{O}_{i1}, \tilde{O}_{i2})^{\gtrdot}$.

Items (1) and (6) show the monotonic property of the $\mathbf{L}^{\lessdot}$-O3W operator and its inverse. The properties of Items (2) and (7) reveal that the application of $\lessdot$ and $\gtrdot$ (or $\gtrdot$ and $\lessdot$) successively will increase the **L**-set. The properties of Items (3) and (8) indicate that three applications of the **L**3W operators successively achieves the same result as the first application. The properties of Items (4), (5), (9), and (10) show the distributivity of the $\mathbf{L}^{\lessdot}$-3W operator and its inverse.

The results in Proposition 4 together with the equivalence between **L**3W operators yield the properties of other **L**3W operators.

Proposition 5 *For* $\tilde{O}, \tilde{O}_i, \tilde{O}_j, \tilde{O}_{ik} \subseteq L^{OB}$ $(i \in \Lambda,\ j = 1, 2, 3, 4,$ *and* $k = 1, 2)$, *the following hold*:

1. *If* $\tilde{O}_1 \subseteq \tilde{O}_2$, *then* $\tilde{O}_1^{\triangledown} \subseteq \tilde{O}_2^{\triangledown}, \tilde{O}_1^{\blacktriangledown} \subseteq \tilde{O}_2^{\blacktriangledown}, \tilde{O}_2^{\rhd} \subseteq \tilde{O}_1^{\rhd}$;
2. $\tilde{O} \supseteq \tilde{O}^{\triangledown\triangle}, \tilde{O} \subseteq \tilde{O}^{\blacktriangledown\blacktriangle}, \tilde{O} \supseteq \tilde{O}^{\rhd\lhd}$;
3. $\tilde{O}^{\triangledown} = \tilde{O}^{\triangledown\triangle\triangledown}, \tilde{O}^{\blacktriangledown} = \tilde{O}^{\blacktriangledown\blacktriangle\blacktriangledown}, \tilde{O}^{\rhd} = \tilde{O}^{\rhd\lhd\rhd}$;
4. $\big(\bigcap_{i\in\Lambda} \tilde{O}_i\big)^{\triangledown} = \bigcap_{i\in\Lambda} \tilde{O}_i^{\triangledown}, \big(\bigcup_{i\in\Lambda} \tilde{O}_i\big)^{\blacktriangledown} = \bigcup_{i\in\Lambda} \tilde{O}_i^{\blacktriangledown}, \big(\bigcap_{i\in\Lambda} \tilde{O}_i\big)^{\rhd} = \bigcup_{i\in\Lambda} \tilde{O}_i^{\rhd}$;
5. $\big(\bigcup_{i\in\Lambda} \tilde{O}_i\big)^{\triangledown} \supseteq \bigcup_{i\in\Lambda} \tilde{O}_i^{\triangledown}, \big(\bigcap_{i\in\Lambda} \tilde{O}_i\big)^{\blacktriangledown} \subseteq \bigcup_{i\in\Lambda} \tilde{O}_i^{\blacktriangledown}, \big(\bigcup_{i\in\Lambda} \tilde{O}_i\big)^{\rhd} \subseteq \bigcap_{i\in\Lambda} \tilde{O}_i^{\rhd}$;
6. *If* $(\tilde{O}_1, \tilde{O}_2) \subseteq (\tilde{O}_3, \tilde{O}_4)$, *then* $(\tilde{O}_1, \tilde{O}_2)^{\triangle} \subseteq (\tilde{O}_3, \tilde{O}_4)^{\triangle}, (\tilde{O}_1, \tilde{O}_2)^{\blacktriangle} \subseteq (\tilde{O}_3, \tilde{O}_4)^{\blacktriangle}, (\tilde{O}_3, \tilde{O}_4)^{\lhd} \subseteq (\tilde{O}_1, \tilde{O}_2)^{\lhd}$;
7. $(\tilde{O}_1, \tilde{O}_2) \subseteq (\tilde{O}_1, \tilde{O}_2)^{\triangle\triangledown}, \quad (\tilde{O}_1, \tilde{O}_2) \supseteq (\tilde{O}_1, \tilde{O}_2)^{\blacktriangle\blacktriangledown}, (\tilde{O}_1, \tilde{O}_2) \supseteq (\tilde{O}_1, \tilde{O}_2)^{\lhd\rhd}$;
8. $(\tilde{O}_1, \tilde{O}_2)^{\triangle} = (\tilde{O}_1, \tilde{O}_2)^{\triangle\triangledown\triangle}, (\tilde{O}_1, \tilde{O}_2)^{\blacktriangle} = (\tilde{O}_1, \tilde{O}_2)^{\blacktriangle\blacktriangledown\blacktriangle}, (\tilde{O}_1, \tilde{O}_2)^{\lhd} = (\tilde{O}_1, \tilde{O}_2)^{\lhd\rhd\lhd}$;



(a) Relationship between **L**O3W operators. **(b)** Relationship between **L**O3W inverse operators.

Fig. 3 Relationship between **L**3W operators

Table 7 Equivalences of **L**O3W operators and **L**O3W inverse operators

	$\lessdot$	$\triangledown$	$\blacktriangledown$	$\triangleright$	$\gtrdot$	$\triangle$	$\blacktriangle$	$\triangleleft$
$\lessdot$	–	$\tilde{O}^{\lessdot} = \tilde{O}^{c\triangledown}$	$\tilde{O}^{\lessdot} = \tilde{O}^{\blacktriangledown c}$	$\tilde{O}^{\lessdot} = \tilde{O}^{c\triangleright c}$				
$\triangledown$	$\tilde{O}^{\triangledown} = \tilde{O}^{c\lessdot}$	–	$\tilde{O}^{\triangledown} = \tilde{O}^{c\blacktriangledown c}$	$\tilde{O}^{\triangledown} = \tilde{O}^{\triangleright c}$				
$\blacktriangledown$	$\tilde{O}^{\blacktriangledown} = \tilde{O}^{\lessdot c}$	$\tilde{O}^{\blacktriangledown} = \tilde{O}^{c\triangledown c}$	–	$\tilde{O}^{\blacktriangledown} = \tilde{O}^{c\triangleright}$				
$\triangleright$	$\tilde{O}^{\triangleright} = \tilde{O}^{c\lessdot c}$	$\tilde{O}^{\triangleright} = \tilde{O}^{\triangledown c}$	$\tilde{O}^{\triangleright} = \tilde{O}^{c\blacktriangledown}$	–				
$\gtrdot$					–	$(\tilde{O}_1,\tilde{O}_2)^{\gtrdot} = (\tilde{O}_1,\tilde{O}_2)^{\triangle c}$	$(\tilde{O}_1,\tilde{O}_2)^{\gtrdot} = (\tilde{O}_1,\tilde{O}_2)^{c\blacktriangle}$	$(\tilde{O}_1,\tilde{O}_2)^{\gtrdot} = (\tilde{O}_1,\tilde{O}_2)^{c\triangleleft c}$
$\triangle$					$(\tilde{O}_1,\tilde{O}_2)^{\triangle} = (\tilde{O}_1,\tilde{O}_2)^{\gtrdot c}$	–	$(\tilde{O}_1,\tilde{O}_2)^{\triangle} = (\tilde{O}_1,\tilde{O}_2)^{c\blacktriangle c}$	$(\tilde{O}_1,\tilde{O}_2)^{\triangle} = (\tilde{O}_1,\tilde{O}_2)^{c\triangleleft}$
$\blacktriangle$					$(\tilde{O}_1,\tilde{O}_2)^{\blacktriangle} = (\tilde{O}_1,\tilde{O}_2)^{c\gtrdot}$	$(\tilde{O}_1,\tilde{O}_2)^{\blacktriangle} = (\tilde{O}_1,\tilde{O}_2)^{c\triangle c}$	–	$(\tilde{O}_1,\tilde{O}_2)^{\blacktriangle} = (\tilde{O}_1,\tilde{O}_2)^{\triangleleft c}$
$\triangleleft$					$(\tilde{O}_1,\tilde{O}_2)^{\triangleleft} = (\tilde{O}_1,\tilde{O}_2)^{c\gtrdot c}$	$(\tilde{O}_1,\tilde{O}_2)^{\triangleleft} = (\tilde{O}_1,\tilde{O}_2)^{c\angle}$	$(\tilde{O}_1,\tilde{O}_2)^{\triangleleft} = (\tilde{O}_1,\tilde{O}_2)^{\blacktriangle c}$	–

9. $\left(\bigcup_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})\right)^{\triangle} = \bigcup_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})^{\triangle}\left(\bigcap_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})\right)^{\blacktriangle} = \bigcap_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})^{\blacktriangle}\left(\bigcap_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})\right)^{\triangleleft} = \bigcup_{i\in\Lambda},(\tilde{O}_{i1},\tilde{O}_{i2})^{\triangleleft}$;

10. $\left(\bigcap_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})\right)^{\triangle} \subseteq \bigcap_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})^{\triangle}\left(\bigcup_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})\right)^{\blacktriangle} \supseteq \bigcup_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})^{\blacktriangle}\left(\bigcup_{i\in\Lambda}(\tilde{O}_{i1},\tilde{O}_{i2})\right)^{\triangleleft} \subseteq \bigcap_{i\in\Lambda},(\tilde{O}_{i1},\tilde{O}_{i2})^{\triangleleft}$.

The properties in Proposition 5 also hold for fuzzy attribute sets.

LO3W Concepts and LO3W Concept Lattices

A type of **L**O3W operator, together with its inverse, defines a type of **L**O3W concept.

Definition 3 Let $\mathbf{K} = (OB, AT, \tilde{R}, L)$ be an **L**-context. For $\tilde{O} \in L^{OB}$ and $\tilde{A}_1, \tilde{A}_2 \in L^{AT}$, $\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle$ is called an

1. $\mathbf{L}^{\lessdot}$-O3W concept if $\tilde{O}^{\lessdot} = (\tilde{A}_1, \tilde{A}_2)$ and $(\tilde{A}_1, \tilde{A}_2)^{\gtrdot} = \tilde{O}$;
2. $\mathbf{L}^{\triangledown}$-O3W concept if $\tilde{O}^{\triangledown} = (\tilde{A}_1, \tilde{A}_2)$ and $(\tilde{A}_1, \tilde{A}_2)^{\triangle} = \tilde{O}$;
3. $\mathbf{L}^{\blacktriangledown}$-O3W concept if $\tilde{O}^{\blacktriangledown} = (\tilde{A}_1, \tilde{A}_2)$ and $(\tilde{A}_1, \tilde{A}_2)^{\blacktriangle} = \tilde{O}$;
4. $\mathbf{L}^{\triangleright}$-O3W concept if $\tilde{O}^{\triangleright} = (\tilde{A}_1, \tilde{A}_2)$ and $(\tilde{A}_1, \tilde{A}_2)^{\triangleleft} = \tilde{O}$.

From Definition 2, two **L**-sets defined on AT comprise the intent of the **L**O3W concept. These two **L**-sets represent opposite meanings. For example, given an $\mathbf{L}^{\lessdot}$-O3W concept $\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle$, $\tilde{A}_1(a)$ characterises the degree of attribute a shared by all objects in $\tilde{O}$ and $\tilde{A}_2(a)$ characterises the degree of attribute a not shared by any objects in $\tilde{O}$. It is worth noting that the statement 'objects in $\tilde{O}$' is also a fuzzy statement.

Example 4 (Continued from Example 1) Suppose a couple intends to buy a house. The agent invites one of them to rate each factor's importance, and the other to rate the non-importance of each aspect. The results are represented by two fuzzy sets: $\tilde{A}_1 = \frac{0.2}{a_1} + \frac{0.8}{a_2} + \frac{0.4}{a_3} + \frac{0.6}{a_4}$ and $\tilde{A}_2 = \frac{0.4}{a_1} + \frac{0.2}{a_2} + \frac{0.2}{a_3} + \frac{0}{a_4}$. By computation, we have $(\tilde{A}_1, \tilde{A}_2)^{\gtrdot} = \tilde{A}^{*} \cap \tilde{A}^{\bar{*}} = \tilde{O} = \frac{0.8}{o_1} + \frac{0.8}{o_2} + \frac{1}{o_3} + \frac{0.4}{o_4} + \frac{0.6}{o_5}$. The result suggests that the agent can recommend the estate o_3 to the couple. Moreover, because $\tilde{O}^{\lessdot} = (\tilde{O}^{*}, \tilde{O}^{\bar{*}}) = (\tilde{A}_1, \tilde{A}_2)$, it holds that $\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle$ is an $\mathbf{L}^{\lessdot}$-O3W concept.

For an **L**-context $\mathbf{K} = (OB, AT, \tilde{R}, L)$, we denote

Table 8 Equivalences of two applications of **L**3W operators

	$\gtrdot$	$\triangle$	$\blacktriangle$	$\triangleleft$	$\lessdot$	$\triangledown$	$\blacktriangledown$	$\triangleright$
$\lessdot$	$\tilde{O}^{\lessdot\gtrdot} = \tilde{O}^{\blacktriangledown\blacktriangle}$	$\tilde{O}^{\lessdot\triangle} = \tilde{O}^{\blacktriangledown\triangleleft}$	$\tilde{O}^{\lessdot\blacktriangle} = \tilde{O}^{\blacktriangledown\gtrdot}$	$\tilde{O}^{\lessdot\triangleleft} = \tilde{O}^{\blacktriangledown\triangle}$				
$\triangledown$	$\tilde{O}^{\triangledown\gtrdot} = \tilde{O}^{\triangleright\blacktriangle}$	$\tilde{O}^{\triangledown\triangle} = \tilde{O}^{\triangleright\triangleleft}$	$\tilde{O}^{\triangledown\blacktriangle} = \tilde{O}^{\triangleright\gtrdot}$	$\tilde{O}^{\triangledown\triangleleft} = \tilde{O}^{\triangleright\triangle}$				
$\blacktriangledown$	$\tilde{O}^{\blacktriangledown\gtrdot} = \tilde{O}^{\lessdot\blacktriangle}$	$\tilde{O}^{\blacktriangledown\triangle} = \tilde{O}^{\lessdot\triangleleft}$	$\tilde{O}^{\blacktriangledown\blacktriangle} = \tilde{O}^{\lessdot\gtrdot}$	$\tilde{O}^{\blacktriangledown\triangleleft} = \tilde{O}^{\lessdot\triangle}$				
$\triangleright$	$\tilde{O}^{\triangleright\gtrdot} = \tilde{O}^{\triangledown\blacktriangle}$	$\tilde{O}^{\triangleright\triangle} = \tilde{O}^{\triangledown\triangleleft}$	$\tilde{O}^{\triangleright\blacktriangle} = \tilde{O}^{\triangledown\gtrdot}$	$\tilde{O}^{\triangleright\triangleleft} = \tilde{O}^{\triangledown\triangle}$				
$\gtrdot$					$(\tilde{O}_1,\tilde{O}_2)^{\gtrdot\lessdot} = (\tilde{O}_1,\tilde{O}_2)^{\triangle\triangledown}$	$(\tilde{O}_1,\tilde{O}_2)^{\gtrdot\triangledown} = (\tilde{O}_1,\tilde{O}_2)^{\triangle\lessdot}$	$(\tilde{O}_1,\tilde{O}_2)^{\gtrdot\blacktriangledown} = (\tilde{O}_1,\tilde{O}_2)^{\triangle\triangleright}$	$(\tilde{O}_1,\tilde{O}_2)^{\gtrdot\triangleright} = (\tilde{O}_1,\tilde{O}_2)^{\triangle\blacktriangledown}$
$\triangle$					$(\tilde{O}_1,\tilde{O}_2)^{\triangle\lessdot} = (\tilde{O}_1,\tilde{O}_2)^{\gtrdot\triangledown}$	$(\tilde{O}_1,\tilde{O}_2)^{\triangle\triangledown} = (\tilde{O}_1,\tilde{O}_2)^{\gtrdot\lessdot}$	$(\tilde{O}_1,\tilde{O}_2)^{\triangle\blacktriangledown} = (\tilde{O}_1,\tilde{O}_2)^{\gtrdot\triangleright}$	$(\tilde{O}_1,\tilde{O}_2)^{\triangle\triangleright} = (\tilde{O}_1,\tilde{O}_2)^{\gtrdot\blacktriangledown}$
$\blacktriangle$					$(\tilde{O}_1,\tilde{O}_2)^{\blacktriangle\lessdot} = (\tilde{O}_1,\tilde{O}_2)^{\triangleleft\triangledown}$	$(\tilde{O}_1,\tilde{O}_2)^{\blacktriangle\triangledown} = (\tilde{O}_1,\tilde{O}_2)^{\triangleleft\lessdot}$	$(\tilde{O}_1,\tilde{O}_2)^{\blacktriangle\blacktriangledown} = (\tilde{O}_1,\tilde{O}_2)^{\triangleleft\triangleright}$	$(\tilde{O}_1,\tilde{O}_2)^{\blacktriangle\triangleright} = (\tilde{O}_1,\tilde{O}_2)^{\triangleleft\blacktriangledown}$
$\triangleleft$					$(\tilde{O}_1,\tilde{O}_2)^{\triangleleft\lessdot} = (\tilde{O}_1,\tilde{O}_2)^{\blacktriangle\triangledown}$	$(\tilde{O}_1,\tilde{O}_2)^{\triangleleft\triangledown} = (\tilde{O}_1,\tilde{O}_2)^{\blacktriangle\lessdot}$	$(\tilde{O}_1,\tilde{O}_2)^{\triangleleft\blacktriangledown} = (\tilde{O}_1,\tilde{O}_2)^{\blacktriangle\triangleright}$	$(\tilde{O}_1,\tilde{O}_2)^{\triangleleft\triangleright} = (\tilde{O}_1,\tilde{O}_2)^{\blacktriangle\blacktriangledown}$

$$\mathbf{C}^{\lessdot}_{OL}(\mathbf{K}) = \{\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle \mid \tilde{O}^{\lessdot} = (\tilde{A}_1, \tilde{A}_2), \\ (\tilde{A}_1, \tilde{A}_2)^{\gtrdot} = \tilde{O}\},$$

$$\mathbf{C}^{\bigtriangledown}_{OL}(\mathbf{K}) = \{\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle \mid \tilde{O}^{\bigtriangledown} = (\tilde{A}_1, \tilde{A}_2), \\ (\tilde{A}_1, \tilde{A}_2)^{\bigtriangleup} = \tilde{O}\},$$

$$\mathbf{C}^{\blacktriangledown}_{OL}(\mathbf{K}) = \{\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle \mid \tilde{O}^{\blacktriangledown} = (\tilde{A}_1, \tilde{A}_2), \\ (\tilde{A}_1, \tilde{A}_2)^{\blacktriangle} = \tilde{O}\},$$

$$\mathbf{C}^{\rhd}_{OL}(\mathbf{K}) = \{\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle \mid \tilde{O}^{\rhd} = (\tilde{A}_1, \tilde{A}_2), \\ (\tilde{A}_1, \tilde{A}_2)^{\lhd} = \tilde{O}\}$$

as the sets of $\mathbf{L}^{\lessdot}$-, $\mathbf{L}^{\bigtriangledown}$-, $\mathbf{L}^{\blacktriangledown}$-, and $\mathbf{L}^{\rhd}$-O3W concepts, respectively.

Theorem 4 *For $\tilde{O} \in L^{OB}$ and $\tilde{A}_1, \tilde{A}_2 \in L^{AT}$, the following statements are equivalent*:

1. $\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle \in \mathbf{C}^{\lessdot}_{OL}(\mathbf{K})$;
2. $\langle \tilde{O}^c, (\tilde{A}_1, \tilde{A}_2)\rangle \in \mathbf{C}^{\bigtriangledown}_{OL}(\mathbf{K})$;
3. $\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)^c\rangle \in \mathbf{C}^{\blacktriangledown}_{OL}(\mathbf{K})$;
4. $\langle \tilde{O}^c, (\tilde{A}_1, \tilde{A}_2)^c\rangle \in \mathbf{C}^{\rhd}_{OL}(\mathbf{K})$.

Proof For $\tilde{O} \in L^{OB}$ and $\tilde{A}_1, \tilde{A}_2 \in L^{AT}$, it follows from Definition 2 and Table 7 that

$$\begin{aligned}
&\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)^c\rangle \in \mathbf{C}^{\blacktriangledown}_{OL}(\mathbf{K}) \\
&\Leftrightarrow \tilde{O}^{\blacktriangledown} = (\tilde{A}_1, \tilde{A}_2)^c,\ ((\tilde{A}_1, \tilde{A}_2)^c)^{\blacktriangle} = \tilde{O} \\
&\Leftrightarrow \tilde{O}^{c\rhd} = (\tilde{A}_1, \tilde{A}_2)^c,\ ((\tilde{A}_1, \tilde{A}_2)^c)^{\lhd c} = \tilde{O} \\
&\Leftrightarrow (\tilde{O}^c)^{\rhd} = (\tilde{A}_1, \tilde{A}_2)^c,\ ((\tilde{A}_1, \tilde{A}_2)^c)^{\lhd} = \tilde{O}^c \\
&\Leftrightarrow \langle \tilde{O}^c, (\tilde{A}_1, \tilde{A}_2)^c\rangle \in \mathbf{C}^{\rhd}_{OL}(\mathbf{K}).
\end{aligned}$$

The other equivalences are similarly proved.

Theorem 4 establishes the connections between **L**O3W concepts. Utlising these results, one starts with any **L**O3W concept to obtain the other three types. The following are the definitions of the order, infimum, and supremum of each type of **L**O3W concept.

Definition 4 Let $\mathbf{K} = (OB, AT, \tilde{R}, L)$ be an **L**-context, $\tilde{O}_i \in L^{OB}$, and $\tilde{A}_{ij} \in L^{AT}$ $(i, j = 1, 2)$.

1. Suppose $\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle, \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle \in \mathbf{C}^{\lessdot}_{OL}(\mathbf{K})$, then $\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \leq_{\lessdot} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle$ iff $\tilde{O}_1 \subseteq \tilde{O}_2$ (or $(\tilde{A}_{21}, \tilde{A}_{22}) \subseteq (\tilde{A}_{11}, \tilde{A}_{12})$), and
$$\begin{aligned}
&\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \wedge_{\lessdot} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle \\
&= \left\langle \tilde{O}_1 \cap \tilde{O}_2, ((\tilde{A}_{11}, \tilde{A}_{12}) \cup (\tilde{A}_{21}, \tilde{A}_{22}))^{\gtrdot\lessdot} \right\rangle, \\
&\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \vee_{\lessdot} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle \\
&= \left\langle (\tilde{O}_1 \cup \tilde{O}_2)^{\lessdot\gtrdot}, (\tilde{A}_{11}, \tilde{A}_{12}) \cap (\tilde{A}_{21}, \tilde{A}_{22}) \right\rangle.
\end{aligned}$$
2. Suppose $\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle, \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle \in \mathbf{C}^{\bigtriangledown}_{OL}(\mathbf{K})$, then $\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \leq_{\bigtriangledown} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle$ iff $\tilde{O}_2 \subseteq \tilde{O}_1$ (or $(\tilde{A}_{21}, \tilde{A}_{22}) \subseteq (\tilde{A}_{11}, \tilde{A}_{12})$), and
$$\begin{aligned}
&\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \wedge_{\bigtriangledown} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle \\
&= \left\langle \tilde{O}_1 \cup \tilde{O}_2, ((\tilde{A}_{11}, \tilde{A}_{12}) \cup (\tilde{A}_{21}, \tilde{A}_{22}))^{\bigtriangleup\bigtriangledown} \right\rangle, \\
&\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \vee_{\bigtriangledown} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle \\
&= \left\langle (\tilde{O}_1 \cap \tilde{O}_2)^{\bigtriangledown\bigtriangleup}, (\tilde{A}_{11}, \tilde{A}_{12}) \cap (\tilde{A}_{21}, \tilde{A}_{22}) \right\rangle.
\end{aligned}$$
3. Suppose $\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle, \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle \in \mathbf{C}^{\blacktriangledown}_{OL}(\mathbf{K})$, then $\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \leq_{\blacktriangledown} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle$ iff $\tilde{O}_1 \subseteq \tilde{O}_2$ (or $(\tilde{A}_{11}, \tilde{A}_{12}) \subseteq (\tilde{A}_{21}, \tilde{A}_{22})$), and
$$\begin{aligned}
&\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \wedge_{\blacktriangledown} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle \\
&= \left\langle \tilde{O}_1 \cap \tilde{O}_2, ((\tilde{A}_{11}, \tilde{A}_{12}) \cap (\tilde{A}_{21}, \tilde{A}_{22}))^{\blacktriangle\blacktriangledown} \right\rangle, \\
&\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \vee_{\blacktriangledown} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle \\
&= \left\langle (\tilde{O}_1 \cup \tilde{O}_2)^{\blacktriangledown\blacktriangle}, (\tilde{A}_{11}, \tilde{A}_{12}) \cup (\tilde{A}_{21}, \tilde{A}_{22}) \right\rangle.
\end{aligned}$$
4. Suppose $\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle, \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle \in \mathbf{C}^{\rhd}_{OL}(\mathbf{K})$, then $\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \leq_{\rhd} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle$ iff $\tilde{O}_2 \subseteq \tilde{O}_1$ (or $(\tilde{A}_{11}, \tilde{A}_{12}) \subseteq (\tilde{A}_{21}, \tilde{A}_{22})$), and
$$\begin{aligned}
&\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \wedge_{\rhd} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle \\
&= \left\langle \tilde{O}_1 \cup \tilde{O}_2, ((\tilde{A}_{11}, \tilde{A}_{12}) \cap (\tilde{A}_{21}, \tilde{A}_{22}))^{\lhd\rhd} \right\rangle, \\
&\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \vee_{\rhd} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle \\
&= \left\langle (\tilde{O}_1 \cap \tilde{O}_2)^{\rhd\lhd}, (\tilde{A}_{11}, \tilde{A}_{12}) \cup (\tilde{A}_{21}, \tilde{A}_{22}) \right\rangle.
\end{aligned}$$

For two **L**O3W concepts of the same type, such as $\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle$ and $\langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle$, if $\tilde{O}_1 = \tilde{O}_2$, then $(\tilde{A}_{11}, \tilde{A}_{12}) = (\tilde{A}_{21}, \tilde{A}_{22})$; the converse also holds. Obviously, $(\mathbf{C}^{?}_{OL}(\mathbf{K}), \wedge_?, \vee_?)$ is a lattice by Propositions 4 and 5. The following is the main theorem of **L**O3W concepts.

Theorem 5 *Given an* **L**-*context* $\mathbf{K} = (OB, AT, \tilde{R}, L)$, $(\mathbf{C}^{?}_{OL}(\mathbf{K}), \wedge_?, \vee_?)$ *is a complete lattice, where* $? = \lessdot, \bigtriangledown, \blacktriangledown$, *and* $\rhd$.

Proof To prove that $(\mathbf{C}^{\lessdot}_{OL}(\mathbf{K}), \wedge_{\lessdot}, \vee_{\lessdot})$ is a complete lattice, we assume that $\langle \tilde{O}_i, (\tilde{A}_{i1}, \tilde{A}_{i2})\rangle \in \mathbf{C}^{\lessdot}_{OL}(\mathbf{K})$, $i \in \Lambda$. As a consequence of Proposition 4 it holds

$$\begin{aligned}
&\langle \textstyle\bigcap_{i\in\Lambda} \tilde{O}_i, (\bigcup_{i\in\Lambda} (\tilde{A}_{i1}, \tilde{A}_{i2}))^{\gtrdot\lessdot}\rangle \\
&= \langle \textstyle\bigcap_{i\in\Lambda} \tilde{O}_i, (\bigcap_{i\in\Lambda} \tilde{O}_i)^{\lessdot}\rangle \in \mathbf{C}^{\lessdot}_{OL}(\mathbf{K}) \text{ and} \\
&\langle (\textstyle\bigcup_{i\in\Lambda} \tilde{O}_i)^{\lessdot\gtrdot}, \bigcap_{i\in\Lambda} (\tilde{A}_{i1}, \tilde{A}_{i2})\rangle \\
&= \langle (\textstyle\bigcap_{i\in\Lambda} (\tilde{A}_{i1}, \tilde{A}_{i2}))^{\gtrdot}, \bigcap_{i\in\Lambda} (\tilde{A}_{i1}, \tilde{A}_{i2})\rangle \in \mathbf{C}^{\lessdot}_{OL}(\mathbf{K}).
\end{aligned}$$

Next, we prove that $\langle \bigcap_{i\in\Lambda} \tilde{O}_i, (\bigcup_{i\in\Lambda} (\tilde{A}_{i1}, \tilde{A}_{i2}))^{\gtrdot\lessdot}\rangle$ is the infimum of $\langle \tilde{O}_i, (\tilde{A}_{i1}, \tilde{A}_{i2})\rangle$, $i \in \Lambda$. If not, assume that there exists an $\mathbf{L}^{\lessdot}$-O3W concept $\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle$ such that

$\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle \leq_{\lessdot} \langle \tilde{O}_i, (\tilde{A}_{i1}, \tilde{A}_{i2})\rangle$ for $i \in \Lambda$ and

$\langle \bigcap_{i\in\Lambda} \tilde{O}_i, (\bigcup_{i\in\Lambda} (\tilde{A}_{i1}, \tilde{A}_{i2}))^{\gtrdot\lessdot}\rangle \leq_{\lessdot} \langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle$.

Then, by Definition 4(1), it follows that $\tilde{O} \subseteq \tilde{O}_i$ for $i \in \Lambda$ and $\bigcap_{i\in\Lambda} \tilde{O}_i \subseteq \tilde{O}$, thus, $\tilde{O} = \bigcap_{i\in\Lambda} \tilde{O}_i$ or, equivalently, $\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle = \langle \bigcap_{i\in\Lambda} \tilde{O}_i, (\bigcup_{i\in\Lambda}(\tilde{A}_{i1}, \tilde{A}_{i2}))^{\gtrdot\lessdot}\rangle$. This completes the proof of the infimum. Similarly, we prove that $\langle (\bigcup_{i\in\Lambda} \tilde{O}_i)^{\lessdot\gtrdot}, \bigcap_{i\in\Lambda}(\tilde{A}_{i1}, \tilde{A}_{i2})\rangle$ is the supremum of $\langle \tilde{O}_i, (\tilde{A}_{i1}, \tilde{A}_{i2})\rangle, i \in \Lambda$. That is to say, $(\mathbf{C}^{\lessdot}_{O\mathbf{L}}(\mathbf{K}), \wedge_{\lessdot}, \vee_{\lessdot})$ is a complete lattice.

The others are similarly proved.

Theorem 5 shows that each type of **L**O3W concept lattice is a complete lattice. Moreover, these concept lattices are isomorphic to each other.

Theorem 6 *Let* $\mathbf{K} = (OB, AT, \tilde{R}, L)$ *be an* **L***-context. Then,* $\mathbf{C}^{\lessdot}_{O\mathbf{L}}(\mathbf{K}) \cong \mathbf{C}^{\triangledown}_{O\mathbf{L}}(\mathbf{K}) \cong \mathbf{C}^{\blacktriangledown}_{O\mathbf{L}}(\mathbf{K}) \cong \mathbf{C}^{\triangleright}_{O\mathbf{L}}(\mathbf{K})$.

Proof To prove $\mathbf{C}^{\triangledown}_{O\mathbf{L}}(\mathbf{K}) \cong \mathbf{C}^{\blacktriangledown}_{O\mathbf{L}}(\mathbf{K})$, let $f : \mathbf{C}^{\triangledown}_{O\mathbf{L}}(\mathbf{K}) \longrightarrow \mathbf{C}^{\blacktriangledown}_{O\mathbf{L}}(\mathbf{K})$ such that $f(\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle) = \langle \tilde{O}^c, (\tilde{A}_1, \tilde{A}_2)^c\rangle$ for $\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle \in \mathbf{C}^{\triangledown}_{O\mathbf{L}}(\mathbf{K})$. Theorem 4 verifies that f is a bijection between $\mathbf{C}^{\triangledown}_{O\mathbf{L}}(\mathbf{K})$ and $\mathbf{C}^{\blacktriangledown}_{O\mathbf{L}}(\mathbf{K})$. Given $\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle, \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle \in \mathbf{C}^{\triangledown}_{O\mathbf{L}}(\mathbf{K})$, Table 7 confirms that

$$
\begin{aligned}
&f(\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \wedge_{\triangledown} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle)\\
&= f(\langle \tilde{O}_1 \cup \tilde{O}_2, ((\tilde{A}_{11}, \tilde{A}_{12}) \cup (\tilde{A}_{21}, \tilde{A}_{22}))^{\vartriangle\triangledown}\rangle)\\
&= \langle (\tilde{O}_1 \cup \tilde{O}_2)^c, ((\tilde{A}_{11}, \tilde{A}_{12}) \cup (\tilde{A}_{21}, \tilde{A}_{22}))^{\vartriangle\triangledown c}\rangle\\
&= \langle (\tilde{O}_1 \cup \tilde{O}_2)^c, ((\tilde{A}_{11}, \tilde{A}_{12}) \cup (\tilde{A}_{21}, \tilde{A}_{22}))^{\vartriangle\triangleright}\rangle,\\
&f(\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle) \wedge_{\blacktriangledown} f(\langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle)\\
&= \langle \tilde{O}_1^c, (\tilde{A}_{11}, \tilde{A}_{12})^c\rangle \wedge_{\blacktriangledown} \langle \tilde{O}_2^c, (\tilde{A}_{21}, \tilde{A}_{22})^c\rangle\\
&= \langle \tilde{O}_1^c \cap \tilde{O}_2^c, ((\tilde{A}_{11}, \tilde{A}_{12})^c \cap (\tilde{A}_{21}, \tilde{A}_{22})^c)^{\blacktriangle\blacktriangledown}\rangle\\
&= \langle (\tilde{O}_1 \cup \tilde{O}_2)^c, ((\tilde{A}_{11}, \tilde{A}_{12}) \cup (\tilde{A}_{21}, \tilde{A}_{22}))^{c\blacktriangle\blacktriangledown}\rangle\\
&= \langle (\tilde{O}_1 \cup \tilde{O}_2)^c, ((\tilde{A}_{11}, \tilde{A}_{12}) \cup (\tilde{A}_{21}, \tilde{A}_{22}))^{\gtrdot\blacktriangledown}\rangle.
\end{aligned}
$$

Furthermore, it follows from Table 8 that $((\tilde{A}_{11}, \tilde{A}_{12}) \cup (\tilde{A}_{21}, \tilde{A}_{22}))^{\vartriangle\triangleright} = ((\tilde{A}_{11}, \tilde{A}_{12}) \cup (\tilde{A}_{21}, \tilde{A}_{22}))^{\gtrdot\blacktriangledown}$ which indicates that f is $\wedge$-preserving. Table 7 also certifies that

$$
\begin{aligned}
&f(\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle \vee_{\triangledown} \langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle)\\
&= f(\langle (\tilde{O}_1 \cap \tilde{O}_2)^{\triangledown\vartriangle}, (\tilde{A}_{11}, \tilde{A}_{12}) \cap (\tilde{A}_{21}, \tilde{A}_{22})\rangle)\\
&= \langle (\tilde{O}_1 \cap \tilde{O}_2)^{\triangledown\vartriangle c}, ((\tilde{A}_{11}, \tilde{A}_{12}) \cap (\tilde{A}_{21}, \tilde{A}_{22}))^c\rangle\\
&= \langle (\tilde{O}_1 \cap \tilde{O}_2)^{\triangledown\gtrdot}, (\tilde{A}_{11}, \tilde{A}_{12})^c \cup (\tilde{A}_{21}, \tilde{A}_{22})^c\rangle,\\
&f(\langle \tilde{O}_1, (\tilde{A}_{11}, \tilde{A}_{12})\rangle) \vee_{\blacktriangledown} f(\langle \tilde{O}_2, (\tilde{A}_{21}, \tilde{A}_{22})\rangle)\\
&= \langle \tilde{O}_1^c, (\tilde{A}_{11}, \tilde{A}_{12})^c\rangle \vee_{\blacktriangledown} \langle \tilde{O}_2^c, (\tilde{A}_{21}, \tilde{A}_{22})^c\rangle\\
&= \langle (\tilde{O}_1^c \cup \tilde{O}_2^c)^{\blacktriangledown\blacktriangle}, (\tilde{A}_{11}, \tilde{A}_{12})^c \cup (\tilde{A}_{21}, \tilde{A}_{22})^c\rangle\\
&= \langle (\tilde{O}_1 \cap \tilde{O}_2)^{c\blacktriangledown\blacktriangle}, (\tilde{A}_{11}, \tilde{A}_{12})^c \cup (\tilde{A}_{21}, \tilde{A}_{22})^c\rangle\\
&= \langle (\tilde{O}_1 \cap \tilde{O}_2)^{\triangleright\blacktriangle}, (\tilde{A}_{11}, \tilde{A}_{12})^c \cup (\tilde{A}_{21}, \tilde{A}_{22})^c\rangle.
\end{aligned}
$$

Moreover, Table 8 indicates that $(\tilde{O}_1 \cap \tilde{O}_2)^{\triangledown\gtrdot} = \tilde{O}_1 \cap \tilde{O}_2)^{\triangleright\blacktriangle}$. Therefore, f is $\vee$-preserving. This completes the proof that $\mathbf{C}^{\triangledown}_{O\mathbf{L}}(\mathbf{K}) \cong \mathbf{C}^{\blacktriangledown}_{O\mathbf{L}}(\mathbf{K})$.

By setting $f(\langle \tilde{O}, (\tilde{A}_1, \tilde{A}_2)\rangle) = \langle \tilde{O}^c, (\tilde{A}_1, \tilde{A}_2)\rangle$, one can prove that $\mathbf{C}^{\lessdot}_{O\mathbf{L}}(\mathbf{K}) \cong \mathbf{C}^{\triangledown}_{O\mathbf{L}}(\mathbf{K})$ and $\mathbf{C}^{\blacktriangledown}_{O\mathbf{L}}(\mathbf{K}) \cong \mathbf{C}^{\triangleright}_{O\mathbf{L}}(\mathbf{K})$.

Theorem 6 reveals the isomorphic relationship between the four types of **L**O3W concept lattices. Using the results in Theorems 4 and 6, we can construct different **L**O3W concept lattices through existing ones.

LA3W Concepts and LA3W Concept Lattices

Similarly, we adopt a type of **L**A3W operator and its inverse to determine the type of **L**A3W concept.

Definition 5 Let $\mathbf{K} = (OB, AT, \tilde{R}, L)$ be an **L**-context. For $\tilde{O}_1, \tilde{O}_2 \in L^{OB}$ and $\tilde{A} \in L^{AT}$, $\langle (\tilde{O}_1, \tilde{O}_2), \tilde{A}\rangle$ is called an

1. $\mathbf{L}^{\lessdot}$-A3W concept if $\tilde{A}^{\lessdot} = (\tilde{O}_1, \tilde{O}_2)$ and $(\tilde{O}_1, \tilde{O}_2)^{\gtrdot} = \tilde{A}$;
2. $\mathbf{L}^{\triangledown}$-A3W concept if $\tilde{A}^{\triangledown} = (\tilde{O}_1, \tilde{O}_2)$ and $(\tilde{O}_1, \tilde{O}_2)^{\vartriangle} = \tilde{A}$;
3. $\mathbf{L}^{\blacktriangledown}$-A3W concept if $\tilde{A}^{\blacktriangledown} = (\tilde{O}_1, \tilde{O}_2)$ and $(\tilde{O}_1, \tilde{O}_2)^{\blacktriangle} = \tilde{A}$;
4. $\mathbf{L}^{\triangleright}$-A3W concept if $\tilde{A}^{\triangleright} = (\tilde{O}_1, \tilde{O}_2)$ and $(\tilde{O}_1, \tilde{O}_2)^{\triangleleft} = \tilde{A}$.

Definition 4 shows that the extent of the **L**A3W concept consists of a pair of **L**-sets, defined on OB, which represent the opposite meaning, just as the intent of the **L**O3W concept.

Remark 1 Note that the $\mathbf{L}^{\lessdot}$-O3W concept and $\mathbf{L}^{\lessdot}$-A3W concept in Definitions 2 and 4 are the same as those proposed in [59]. The main aim of this paper is to discuss the relationship between various **L**3W concept lattices, so we introduce other **L**O3W concepts and **L**A3W concepts.

For an **L**-context $\mathbf{K} = (OB, AT, \tilde{R}, L)$, we denote

$$
\begin{aligned}
\mathbf{C}^{\lessdot}_{A\mathbf{L}}(\mathbf{K}) &= \{\langle (\tilde{O}_1, \tilde{O}_2), \tilde{A}\rangle \mid \tilde{A}^{\lessdot} = (\tilde{O}_1, \tilde{O}_2),\\
&\qquad (\tilde{O}_1, \tilde{O}_2)^{\lessdot} = \tilde{A}\},\\
\mathbf{C}^{\triangledown}_{A\mathbf{L}}(\mathbf{K}) &= \{\langle (\tilde{O}_1, \tilde{O}_2), \tilde{A}\rangle \mid \tilde{A}^{\triangledown} = (\tilde{O}_1, \tilde{O}_2),\\
&\qquad (\tilde{O}_1, \tilde{O}_2)^{\vartriangle} = \tilde{A}\},\\
\mathbf{C}^{\blacktriangledown}_{A\mathbf{L}}(\mathbf{K}) &= \{\langle (\tilde{O}_1, \tilde{O}_2), \tilde{A}\rangle \mid \tilde{A}^{\blacktriangledown} = (\tilde{O}_1, \tilde{O}_2),\\
&\qquad (\tilde{O}_1, \tilde{O}_2)^{\blacktriangle} = \tilde{A}\},\\
\mathbf{C}^{\triangleright}_{A\mathbf{L}}(\mathbf{K}) &= \{\langle (\tilde{O}_1, \tilde{O}_2), \tilde{A}\rangle \mid \tilde{A}^{\triangleright} = (\tilde{O}_1, \tilde{O}_2),\\
&\qquad (\tilde{O}_1, \tilde{O}_2)^{\triangleleft} = \tilde{A}\}
\end{aligned}
$$

as the sets of $\mathbf{L}^{\lessdot}$-, $\mathbf{L}^{\triangledown}$-, $\mathbf{L}^{\blacktriangledown}$-, and $\mathbf{L}^{\triangleright}$-A3W concepts, respectively. Below shows the relationship between **L**A3W concepts.

Theorem 7 *For $\tilde{O}_1, \tilde{O}_2 \in L^{OB}$ and $\tilde{A} \in L^{AT}$, the following statements are equivalent*:

1. $\langle(\tilde{O}_1, \tilde{O}_2), A\rangle \in \mathbf{C}^{\lessdot}_{AL}(\mathbf{K})$;
2. $\langle(\tilde{O}_1, \tilde{O}_2), A^c\rangle \in \mathbf{C}^{\triangledown}_{AL}(\mathbf{K})$;
3. $\langle(\tilde{O}_1, \tilde{O}_2)^c, A\rangle \in \mathbf{C}^{\blacktriangledown}_{AL}(\mathbf{K})$;
4. $\langle(\tilde{O}_1, \tilde{O}_2)^c, A^c\rangle \in \mathbf{C}^{\rhd}_{AL}(\mathbf{K})$.

Proof The proof is similar to that of Theorem 4.

The results in Theorem 7 provide a convenient way to construct the **L**A3W concept from other concepts. For example, given an $\mathbf{L}^{\lessdot}$-A3W concept, replacing the intent of the $\mathbf{L}^{\lessdot}$-A3W concept with its complement generates an $\mathbf{L}^{\triangledown}$-A3W concept.

The order, infimum, and supremum of each type of **L** A3W concept are defined as follows:

Definition 6 Let $\mathbf{K} = (OB, AT, \tilde{R}, L)$ be an **L**-context, $\tilde{O}_{ij} \in L^{OB}$, and $\tilde{A}_i \in L^{AT}$ $(i, j = 1, 2)$.

1. Suppose $\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle, \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle \in \mathbf{C}^{\lessdot}_{AL}(\mathbf{K})$, then $\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle \leq_{\lessdot} \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle$ iff $(\tilde{O}_{11}, \tilde{O}_{12}) \subseteq (\tilde{O}_{21}, \tilde{O}_{22})$ (or $\tilde{A}_2 \subseteq \tilde{A}_1$), and
$$\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle \wedge_{\lessdot} \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle = \left\langle(\tilde{O}_{11}, \tilde{O}_{12}) \cap (\tilde{O}_{21}, \tilde{O}_{22}), (\tilde{A}_1 \cup \tilde{A}_2)^{\lessdot\gtrdot}\right\rangle,$$
$$\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle \vee_{\lessdot} \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle = \left\langle((\tilde{O}_{11}, \tilde{O}_{12}) \cup (\tilde{O}_{21}, \tilde{O}_{22}))^{\gtrdot\lessdot}, \tilde{A}_1 \cap \tilde{A}_2\right\rangle.$$
2. Suppose $\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle, \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle \in \mathbf{C}^{\triangledown}_{AL}(\mathbf{K})$, then $\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle \leq_{\triangledown} \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle$ iff $(\tilde{O}_{11}, \tilde{O}_{12}) \subseteq (\tilde{O}_{21}, \tilde{O}_{22})$ (or $\tilde{A}_1 \subseteq \tilde{A}_2$), and
$$\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle \wedge_{\triangledown} \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle = \left\langle(\tilde{O}_{11}, \tilde{O}_{12}) \cap (\tilde{O}_{21}, \tilde{O}_{22}), (\tilde{A}_1 \cap \tilde{A}_2)^{\triangledown\vartriangle}\right\rangle,$$
$$\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle \vee_{\triangledown} \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle = \left\langle((\tilde{O}_{11}, \tilde{O}_{12}) \cup (\tilde{O}_{21}, \tilde{O}_{22}))^{\vartriangle\triangledown}, \tilde{A}_1 \cup \tilde{A}_2\right\rangle.$$
3. Suppose $\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle, \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle \in \mathbf{C}^{\blacktriangledown}_{AL}(\mathbf{K})$, then $\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle \leq_{\blacktriangledown} \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle$ iff $(\tilde{O}_{21}, \tilde{O}_{22}) \subseteq (\tilde{O}_{11}, \tilde{O}_{12})$ (or $\tilde{A}_2 \subseteq \tilde{A}_1$), and
$$\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle \wedge_{\blacktriangledown} \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle = \left\langle(\tilde{O}_{11}, \tilde{O}_{12}) \cup (\tilde{O}_{21}, \tilde{O}_{22}), (\tilde{A}_1 \cup \tilde{A}_2)^{\blacktriangledown\blacktriangle}\right\rangle,$$
$$\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle \vee_{\blacktriangledown} \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle = \left\langle((\tilde{O}_{11}, \tilde{O}_{12}) \cap (\tilde{O}_{21}, \tilde{O}_{22}))^{\blacktriangle\blacktriangledown}, \tilde{A}_1 \cap \tilde{A}_2\right\rangle.$$
4. Suppose $\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle, \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle \in \mathbf{C}^{\rhd}_{AL}(\mathbf{K})$, then $\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle \leq_{\rhd} \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle$ iff $(\tilde{O}_{21}, \tilde{O}_{22}) \subseteq (\tilde{O}_{11}, \tilde{O}_{12})$ (or $\tilde{A}_1 \subseteq \tilde{A}_2$), and
$$\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle \wedge_{\rhd} \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle = \left\langle(\tilde{O}_{11}, \tilde{O}_{12}) \cup (\tilde{O}_{21}, \tilde{O}_{22}), (\tilde{A}_1 \cap \tilde{A}_2)^{\rhd\lhd}\right\rangle,$$
$$\langle(\tilde{O}_{11}, \tilde{O}_{12}), \tilde{A}_1\rangle \vee_{\rhd} \langle(\tilde{O}_{21}, \tilde{O}_{22}), \tilde{A}_2\rangle = \left\langle((\tilde{O}_{11}, \tilde{O}_{12}) \cap (\tilde{O}_{21}, \tilde{O}_{22}))^{\lhd\rhd}, \tilde{A}_1 \cup \tilde{A}_2\right\rangle.$$

Each type of **L**A3W concept forms a complete lattice, based on the infimum and supremum defined in Definition 5.

Theorem 8 *Given an* **L**-*context* $\mathbf{K} = (OB, AT, \tilde{R}, L)$, $(\mathbf{C}^{?}_{AL}(\mathbf{K}), \wedge_?, \vee_?)$ *is a complete lattice, where* $? = \lessdot, \triangledown, \blacktriangledown$, *and* $\rhd$.

Proof The proof is similar to that of Theorem 5.

The four types of **L**A3W concept lattices are isomorphic with each other.

Theorem 9 *Let* $\mathbf{K} = (OB, AT, \tilde{R}, L)$ *be an* **L**-*context. Then,* $\mathbf{C}^{\lessdot}_{AL}(\mathbf{K}) \cong \mathbf{C}^{\triangledown}_{AL}(\mathbf{K}) \cong \mathbf{C}^{\blacktriangledown}_{AL}(\mathbf{K}) \cong \mathbf{C}^{\rhd}_{AL}(\mathbf{K})$.

Proof The proof is similar to that of Theorem 6.

The results in Theorems 7 and 8 provide a convenient way to construct an **L**A3W concept lattice from other **L** A3W concept lattices.

Relationship Between L2W Concepts and L3W Concepts

A pair of **L**2W operators determine an **L**3W operator, which suggests that the **L**3W concept must connect with **L**2W concepts. The main focus in this section is to address this problem.

Relationship Between L2W Concepts and LO3W Concepts

The following theorem reveals how to obtain **L**O3W concepts from **L**2W concepts.

Theorem 10 *Given* $\tilde{O} \in L^{OB}$ *and* $\tilde{A} \in L^{AT}$, *the following hold*:

1. *If* $\langle\tilde{O}, \tilde{A}\rangle$ *is an* $\mathbf{L}^{*}$-*2W concept, then* $\langle\tilde{O}, (\tilde{A}, \tilde{O}^{\bar{*}})\rangle$ *is an* $\mathbf{L}^{\lessdot}$-*O3W concept*;
2. *If* $\langle\tilde{O}, \tilde{A}\rangle$ *is an* $\mathbf{L}^{\bar{*}}$-*2W concept, then* $\langle\tilde{O}, (\tilde{O}^{*}, \tilde{A})\rangle$ *is an* $\mathbf{L}^{\lessdot}$-*O3W concept*;

3. *If $\langle \tilde O, \tilde A\rangle$ is an $\mathbf{L}^{\overline{\Box}\overline{\Diamond}}$-2W concept, then $\langle \tilde O, (\tilde A, \tilde O^{\Box})\rangle$ is an $\mathbf{L}^{\bigtriangledown}$-O3W concept*;
4. *If $\langle \tilde O, \tilde A\rangle$ is an $\mathbf{L}^{\Box\Diamond}$-2W concept, then $\langle \tilde O, (\tilde O^{\overline{\Box}}, \tilde A)\rangle$ is an $\mathbf{L}^{\bigtriangledown}$-O3W concept*;
5. *If $\langle \tilde O, \tilde A\rangle$ is an $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$-2W concept, then $\langle \tilde O, (\tilde A, \tilde O^{\Diamond})\rangle$ is an $\mathbf{L}^{\blacktriangledown}$-O3W concept*;
6. *If $\langle \tilde O, \tilde A\rangle$ is an $\mathbf{L}^{\Diamond\Box}$-2W concept, then $\langle \tilde O, (\tilde O^{\overline{\Diamond}}, \tilde A)\rangle$ is an $\mathbf{L}^{\blacktriangledown}$-O3W concept*;
7. *If $\langle \tilde O, \tilde A\rangle$ is an $\mathbf{L}^{\#}$-2W concept, then $\langle \tilde O, (\tilde A, \tilde O^{\overline{\#}})\rangle$ is an $\mathbf{L}^{\triangleright}$-O3W concept*;
8. *If $\langle \tilde O, \tilde A\rangle$ is an $\mathbf{L}^{\overline{\#}}$-2W concept, then $\langle \tilde O, (\tilde O^{\#}, \tilde A)\rangle$ is an $\mathbf{L}^{\triangleright}$-O3W concept.*

Proof Suppose that $\langle \tilde O, \tilde A\rangle$ is an $\mathbf{L}^{*}$-2W concept. Then, $\tilde O^{*} = \tilde A$ and $\tilde A^{*} = \tilde O$, and $\tilde O^{\lessdot} = (\tilde O^{*}, \tilde O^{\bar *}) = (\tilde A, \tilde O^{\bar *})$. Meanwhile, Proposition 2(2) explains that $(\tilde A, \tilde O^{\bar *})^{\gtrdot} = \tilde A^{*} \cap \tilde O^{\bar *\bar *} = \tilde O \cap \tilde O^{\bar *\bar *} = \tilde O$, from which we can conclude that $\langle \tilde O, (\tilde A, \tilde O^{\bar *})\rangle$ is an $\mathbf{L}^{\lessdot}$-O3W concept.

The others are similarly proved.

Thinking contrarily, we obtain $\mathbf{L}$2W concepts from $\mathbf{L}$ O3W concepts.

Theorem 11 *Given $\tilde O \in L^{OB}$ and $\tilde A_1, \tilde A_2 \in L^{AT}$, the following hold*:

1. *If $\langle \tilde O, (\tilde A_1, \tilde A_2)\rangle$ is an $\mathbf{L}^{\lessdot}$-O3W concept, then $\langle \tilde A_1^{*}, \tilde A_1\rangle$ is an $\mathbf{L}^{*}$-2W concept and $\langle \tilde A_2^{\bar *}, \tilde A_2\rangle$ is an $\mathbf{L}^{\bar *}$-2W concept*;
2. *If $\langle \tilde O, (\tilde A_1, \tilde A_2)\rangle$ is an $\mathbf{L}^{\bigtriangledown}$-O3W concept, then $\langle \tilde A_1^{\Diamond}, \tilde A_1\rangle$ is an $\mathbf{L}^{\overline{\Box}\overline{\Diamond}}$-2W concept and $\langle \tilde A_2^{\Diamond}, \tilde A_2\rangle$ is an $\mathbf{L}^{\Box\Diamond}$-2W concept*;
3. *If $\langle \tilde O, (\tilde A_1, \tilde A_2)\rangle$ is an $\mathbf{L}^{\blacktriangledown}$-O3W concept, then $\langle \tilde A_1^{\overline{\Box}}, \tilde A_1\rangle$ is an $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$-2W concept and $\langle \tilde A_2^{\Box}, \tilde A_2\rangle$ is an $\mathbf{L}^{\Diamond\Box}$-2W concept*;
4. *If $\langle \tilde O, (\tilde A_1, \tilde A_2)\rangle$ is an $\mathbf{L}^{\triangleright}$-O3W concept, then $\langle \tilde A_1^{\#}, \tilde A_1\rangle$ is an $\mathbf{L}^{\#}$-2W concept, and $\langle \tilde A_2^{\overline{\#}}, \tilde A_2\rangle$ is an $\mathbf{L}^{\overline{\#}}$-2W concept.*

Proof Suppose that $\langle \tilde O, (\tilde A_1, \tilde A_2)\rangle$ is an $\mathbf{L}^{\lessdot}$-O3W concept, then $\tilde O^{\lessdot} = (\tilde O^{*}, \tilde O^{\bar *}) = (\tilde A_1, \tilde A_2)$ which means that $\tilde A_1 = \tilde O^{*}$ and $\tilde A_2 = \tilde O^{\bar *}$. Moreover, it follows from Propositions 1(3) and 2(3) that $\tilde A_1^{**} = \tilde O^{***} = \tilde O^{*} = \tilde A_1$ and $\tilde A_2^{\bar *\bar *} = \tilde O^{\bar *\bar *\bar *} = \tilde O^{\bar *} = \tilde A_2$ which means that $\langle \tilde A_1^{*}, \tilde A_1\rangle$ is an $\mathbf{L}^{*}$-2W concept and $\langle \tilde A_2^{\bar *}, \tilde A_2\rangle$ is an $\mathbf{L}^{\bar *}$-2W concept.

The others are similarly proved.

Combining the results of Theorems 10 and 11, we now introduce the equivalent relationship between the $\mathbf{L}$O3W concepts and $\mathbf{L}$2W concepts.

Theorem 12 *Given $\tilde O \in L^{OB}$ and $\tilde A_1, \tilde A_2 \in L^{AT}$, the following hold*:

1. *$\langle \tilde O, (\tilde A_1, \tilde A_2)\rangle$ is an $\mathbf{L}^{\lessdot}$-O3W concept iff there exist an $\mathbf{L}^{*}$-2W concept $\langle \tilde O_1, \tilde A'\rangle$ and an $\mathbf{L}^{\bar *}$-2W concept $\langle \tilde O_2, \tilde A''\rangle$ such that $\tilde O = \tilde O_1 \cap \tilde O_2$, $\tilde A_1 = (\tilde O_1 \cap \tilde O_2)^{*}$, and $\tilde A_2 = (\tilde O_1 \cap \tilde O_2)^{\bar *}$*;
2. *$\langle \tilde O, (\tilde A_1, \tilde A_2)\rangle$ is an $\mathbf{L}^{\bigtriangledown}$-O3W concept iff there exist an $\mathbf{L}^{\overline{\Box}\overline{\Diamond}}$-2W concept $\langle \tilde O_1, \tilde A'\rangle$ and an $\mathbf{L}^{\Box\Diamond}$-2W concept $\langle \tilde O_2, \tilde A''\rangle$ such that $\tilde O = \tilde O_1 \cup \tilde O_2$, $\tilde A_1 = (\tilde O_1 \cup \tilde O_2)^{\overline{\Box}}$, and $\tilde A_2 = (\tilde O_1 \cup \tilde O_2)^{\Box}$*;
3. *$\langle \tilde O, (\tilde A_1, \tilde A_2)\rangle$ is an $\mathbf{L}^{\blacktriangledown}$-O3W concept iff there exist an $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$-2W concept $\langle \tilde O_1, \tilde A'\rangle$ and an $\mathbf{L}^{\Diamond\Box}$-2W concept $\langle \tilde O_2, \tilde A''\rangle$ such that $\tilde O = \tilde O_1 \cap \tilde O_2$, $\tilde A_1 = (\tilde O_1 \cap \tilde O_2)^{\overline{\Diamond}}$, and $\tilde A_2 = (\tilde O_1 \cap \tilde O_2)^{\Diamond}$*;
4. *$\langle \tilde O, (\tilde A_1, \tilde A_2)\rangle$ is an $\mathbf{L}^{\triangleright}$-O3W concept iff there exists an $\mathbf{L}^{\#}$-2W concept $\langle \tilde O_1, \tilde A'\rangle$ and an $\mathbf{L}^{\overline{\#}}$-2W concept $\langle \tilde O_2, \tilde A''\rangle$ such that $\tilde O = \tilde O_1 \cup \tilde O_2$, $\tilde A_1 = (\tilde O_1 \cup \tilde O_2)^{\#}$, and $\tilde A_2 = (\tilde O_1 \cup \tilde O_2)^{\overline{\#}}$.*

Proof To prove this necessity, we assume that $\langle \tilde O, (\tilde A_1, \tilde A_2)\rangle$ is an $\mathbf{L}^{\lessdot}$-O3W concept, and let $\tilde O_1 = \tilde A_1^{*}$, $\tilde A' = \tilde A_1$ and $\tilde O_2 = \tilde A_2^{\bar *}$, $\tilde A'' = \tilde A_2$. Then, according to Theorem 11(1), $\langle \tilde O_1, \tilde A'\rangle$ is an $\mathbf{L}^{*}$-2W concept, and $\langle \tilde O_2, \tilde A''\rangle$ is an $\mathbf{L}^{\bar *}$-2W concept. It is clear that $\tilde O_1 \cap \tilde O_2 = \tilde A_1^{*} \cap \tilde A_2^{\bar *} = (\tilde A_1, \tilde A_2)^{\gtrdot} = \tilde O$. Furthermore, $(\tilde O_1 \cap \tilde O_2)^{*} = \tilde O^{*} = \tilde A_1$ and $(\tilde O_1 \cap \tilde O_2)^{\bar *} = \tilde O^{\bar *} = \tilde A_2$.

To prove the sufficiency, we suppose that $\langle \tilde O_1, \tilde A'\rangle$ is an $\mathbf{L}^{*}$-2W concept and $\langle \tilde O_2, \tilde A''\rangle$ is an $\mathbf{L}^{\bar *}$-2W concept, and let $\tilde O = \tilde O_1 \cap \tilde O_2, \tilde A_1 = (\tilde O_1 \cap \tilde O_2)^{*}$, and $\tilde A_2 = (\tilde O_1 \cap \tilde O_2)^{\bar *}$. Clearly, $\tilde O^{\lessdot} = (\tilde O_1 \cap \tilde O_2)^{\lessdot} = ((\tilde O_1 \cap \tilde O_2)^{*}, (\tilde O_1 \cap \tilde O_2)^{\bar *}) = (\tilde A_1, \tilde A_2)$ and $(\tilde A_1, \tilde A_2)^{\gtrdot} = \tilde A_1^{*} \cap \tilde A_2^{\bar *} = (\tilde O_1 \cap \tilde O_2)^{**} \cap (\tilde O_1 \cap \tilde O_2)^{\bar *\bar *}$. Propositions 1(2) and 2(2) confirm that $(\tilde A_1, \tilde A_2)^{\gtrdot} \supseteq \tilde O_1 \cap \tilde O_2$. Moreover, $(\tilde O_1 \cap \tilde O_2)^{**} \subseteq \tilde O_1$ because of $\tilde O_1 \cap \tilde O_2 \subseteq \tilde O_1$ and $\tilde O_1^{**} = \tilde O_1$. Similarly, $(\tilde O_1 \cap \tilde O_2)^{\bar *\bar *} \subseteq \tilde O_2$. Therefore, $(\tilde A_1, \tilde A_2)^{\gtrdot} = (\tilde O_1 \cap \tilde O_2)^{**} \cap (\tilde O_1 \cap \tilde O_2)^{\bar *\bar *} \subseteq \tilde O_1 \cap \tilde O_2$. Finally, it is true that $\tilde O_1 \cap \tilde O_2 = (\tilde A_1, \tilde A_2)^{\gtrdot} = \tilde O$, which indicates that $\langle \tilde O, (\tilde A_1, \tilde A_2)\rangle$ is an $\mathbf{L}^{\lessdot}$-O3W concept.

The other equivalences are proved in a similar way.

Theorem 12 provides a method for generating $\mathbf{L}$O3W concept lattices from $\mathbf{L}$2W concept lattices. For example, an $\mathbf{L}^{\lessdot}$-O3W concept lattice is obtained as follows: take an $\mathbf{L}^{*}$-2W concept, $\langle O_1, A_1\rangle$, from $\mathbf{C}_{\mathbf{L}}^{*}(\mathbf{K})$ and an $\mathbf{L}^{\bar *}$-2W concept, $\langle O_2, A_2\rangle$, from $\mathbf{C}_{\mathbf{L}}^{\bar *}(\mathbf{K})$; compute $O_1 \cap O_2$, $(O_1 \cap O_2)^{*}$, and $(O_1 \cap O_2)^{\bar *}$; then $\langle O_1 \cap O_2, ((O_1 \cap O_2)^{*}, (O_1 \cap O_2)^{\bar *})\rangle$ is an $\mathbf{L}^{\lessdot}$-O3W concept. The $\mathbf{L}^{\lessdot}$-O3W concept lattice is achieved by determining all non-repeated $\mathbf{L}^{\lessdot}$-O3W concepts.

Algorithm 1: Generate $\mathbf{L}^{\lessdot}$-O3W concept lattice.

input : $\mathbf{L}^{*}$-2W concept lattice: $\mathbf{C}_{\mathbf{L}}^{*}(\mathbf{K}) = \{\langle O_{i*}, A_{i*}\rangle\}$, $\mathbf{L}^{\bar{*}}$-2W concept lattice: $\mathbf{C}_{\mathbf{L}}^{\bar{*}}(\mathbf{K}) = \{\langle O_{i\bar{*}}, A_{i\bar{*}}\rangle\}$.

output: $\mathbf{L}^{\lessdot}$-O3W concept lattice: $\mathbf{C}_{O\mathbf{L}}^{\lessdot}(K = \{\langle O_i, (A_{1i}, A_{2i})\rangle\}$.

1 $n = 0$,
2 **for** $i = 1$ *to* $|\mathbf{C}_{\mathbf{L}}^{*}(\mathbf{K})|$ **do**
3   **for** $j = 1$ *to* $|\mathbf{C}_{\mathbf{L}}^{\bar{*}}(\mathbf{K})|$ **do**
4         $n = n + 1$,
5         $O_n = O_{i*} \cap O_{j\bar{*}}$,
6   **end**
7 **end**
8 Delete repeated elements in $\{O_1, O_2, \cdots\}$,
9 **for** *each* O_i **do**
10     $A_{1i} = O_i^{*}$,
11     $A_{2i} = O_i^{\bar{*}}$.
12 **end**

Algorithm 1 is applied to generate an $\mathbf{L}^{\lessdot}$-O3W concept lattice from $\mathbf{L}^{*}$- and $\mathbf{L}^{\bar{*}}$-2W concept lattices. Suppose that the number of $\mathbf{L}^{*}$-2W concepts is k and the number of $\mathbf{L}^{\bar{*}}$-2W concepts is l. Because $O_{i*}, O_{j\bar{*}} \subseteq OB$, the complexity of the double **for** loop (from line 2 to 7) is $O(kl \cdot |OB|^2)$, where $|OB|$ is the number of objects. To delete the same elements in $\{O_1, O_2, \cdots\}$, any two elements must be compared in this set. The complexity of line 8 is $O(\frac{kl(kl+1)}{2} \cdot |OB|^2) = O(k^2l^2 \cdot |OB|^2)$. The last **for** loop computes the intent of each O_i (see Eqs. (13) and (15)). The time complexity of this **for** loop is $O(kl \cdot |OB| \cdot |AT|)$, where $|AT|$ represents the number of attributes. As $k^2l^2 \cdot |OB|^2 > kl \cdot |OB| \cdot |AT|$, the time complexity of Algorithm 1 is $O(k^2l^2 \cdot |OB|^2)$.

To obtain other **L**O3W concept lattices, we only need to replace the inputs in Algorithm 1 with the corresponding **L**2W concept lattices and change the intersection in line 5 with union or maintain the same. Lines 10 and 11 are adjusted to the corresponding intent-computing formulas shown in Theorem 12.

Example 5 (Continued from Example 3) Using the method proposed by Bězlohlávek [57], we obtain 13 $\mathbf{L}^{*}$-2W concepts (see Table 4) and 17 $\mathbf{L}^{\bar{*}}$-2W concepts (see Table 9). Algorithm 1 is adopted to find all $\mathbf{L}^{\lessdot}$-O3W concepts (see Table 10). For a better understanding, the Hasen diagram of the $\mathbf{L}^{\lessdot}$-O3W concept lattice is illustrated in Fig. 4. Each number in the figure corresponds to the $\mathbf{L}^{\lessdot}$-O3W concept in Table 10. A line connects two concepts, and the lower concept is a sub-concept of the upper one.

Table 9 $\mathbf{L}^{\bar{*}}$-2W concepts

No.	$\tilde{O}$	$\tilde{A}$
1	(1.0, 1.0, 1.0, 1.0)	(0.0, 0.0, 0.0)
2	(1.0, 1.0, 1.0, 0.5)	(0.0, 0.0, 0.5)
3	(0.5, 1.0, 0.5, 1.0)	(0.0, 0.5, 0.0)
4	(0.5, 1.0, 0.5, 0.5)	(0.0, 0.5, 0.5)
5	(1.0, 0.5, 1.0, 1.0)	(0.5, 0.0, 0.0)
6	(1.0, 0.5, 1.0, 0.5)	(0.5, 0.0, 0.5)
7	(0.5, 0.5, 1.0, 0.0)	(0.5, 0.0, 1.0)
8	(0.5, 0.5, 0.5, 1.0)	(0.5, 0.5, 0.0)
9	(0.5, 0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)
10	(0.5, 0.5, 0.5, 0.0)	(0.5, 0.5, 1.0)
11	(0.0, 0.5, 0.0, 0.5)	(0.5, 1.0, 0.5)
12	(0.0, 0.5, 0.0, 0.0)	(0.5, 1.0, 1.0)
13	(1.0, 0.0, 0.5, 0.5)	(1.0, 0.0, 0.5)
14	(0.5, 0.0, 0.5, 0.5)	(1.0, 0.5, 0.5)
15	(0.5, 0.0, 0.5, 0.0)	(1.0, 0.5, 1.0)
16	(0.0, 0.0, 0.0, 0.5)	(1.0, 1.0, 0.5)
17	(0.0, 0.0, 0.0, 0.0)	(1.0, 1.0, 1.0)

Relationship Between L2W Concepts and LA3W Concepts

The **L**2W concepts also produce **L**A3W concepts in a manner similar to that of the **L**O3W concepts. The following is a detailed presentation.

Theorem 13 *Given* $\tilde{O} \in L^{OB}$ *and* $\tilde{A} \in L^{AT}$, *the following hold*:

1. *If* $\langle \tilde{O}, \tilde{A}\rangle$ *is an* $\mathbf{L}^{*}$*-2W concept, then* $\langle(\tilde{O}, \tilde{A}^{\bar{*}}), \tilde{A}\rangle$ *is an* $\mathbf{L}^{\lessdot}$*-A3W concept*;
2. *If* $\langle \tilde{O}, \tilde{A}\rangle$ *is an* $\mathbf{L}^{\bar{*}}$*-2W concept, then* $\langle(\tilde{A}^{*}, \tilde{O}), \tilde{A}\rangle$ *is an* $\mathbf{L}^{\lessdot}$*-A3W concept*;
3. *If* $\langle \tilde{O}, \tilde{A}\rangle$ *is an* $\mathbf{L}^{\overline{\Box}\overline{\Diamond}}$*-2W concept, then* $\langle(\tilde{O}, \tilde{A}^{\Box}), \tilde{A}\rangle$ *is an* $\mathbf{L}^{\triangledown}$*-A3W concept*;
4. *If* $\langle \tilde{O}, \tilde{A}\rangle$ *is an* $\mathbf{L}^{\Box\Diamond}$*-2W concept, then* $\langle(\tilde{A}^{\overline{\Box}}, \tilde{O}), \tilde{A}\rangle$ *is an* $\mathbf{L}^{\triangledown}$*-A3W concept*;
5. *If* $\langle \tilde{O}, \tilde{A}\rangle$ *is an* $\mathbf{L}^{\overline{\Diamond}\overline{\Box}}$*-2W concept, then* $\langle(\tilde{O}, \tilde{A}^{\Diamond}), \tilde{A}\rangle$ *is an* $\mathbf{L}^{\blacktriangledown}$*-A3W concept*;
6. *If* $\langle \tilde{O}, \tilde{A}\rangle$ *is an* $\mathbf{L}^{\Diamond\Box}$*-2W concept, then* $\langle(\tilde{A}^{\overline{\Diamond}}, \tilde{O}), \tilde{A}\rangle$ *is an* $\mathbf{L}^{\blacktriangledown}$*-A3W concept*;
7. *If* $\langle \tilde{O}, \tilde{A}\rangle$ *is an* $\mathbf{L}^{\#}$*-2W concept, then* $\langle(\tilde{O}, \tilde{A}^{\bar{\#}}), \tilde{A}\rangle$ *is an* $\mathbf{L}^{\triangleright}$*-A3W concept*;
8. *If* $\langle \tilde{O}, \tilde{A}\rangle$ *is an* $\mathbf{L}^{\bar{\#}}$*-2W concept, then* $\langle(\tilde{A}^{\#}, \tilde{O}), \tilde{A}\rangle$ *is an* $\mathbf{L}^{\triangleright}$*-A3W concept*.

Proof The proof is similar to that of Theorem 10.

Table 10 $\mathbf{L}^{\lessdot}$-O3W concepts

No.	$\tilde{O}$	$\tilde{A}_1$	$\tilde{A}_2$
1	(1.0, 1.0, 1.0, 1.0)	(0.0, 0.5, 0.0)	(0.0, 0.0, 0.0)
2	(1.0, 1.0, 1.0, 0.5)	(0.0, 0.5, 0.0)	(0.0, 0.0, 0.5)
3	(1.0, 1.0, 0.5, 1.0)	(0.0, 0.5, 0.5)	(0.0, 0.0, 0.0)
4	(1.0, 1.0, 0.5, 0.5)	(0.0, 0.5, 0.5)	(0.0, 0.0, 0.5)
5	(1.0, 0.5, 1.0, 1.0)	(0.0, 0.5, 0.0)	(0.5, 0.0, 0.0)
6	(1.0, 0.5, 1.0, 0.5)	(0.0, 1.0, 0.0)	(0.5, 0.0, 0.5)
7	(1.0, 0.5, 0.5, 1.0)	(0.0, 0.5, 0.5)	(0.5, 0.0, 0.0)
8	(1.0, 0.5, 0.5, 0.5)	(0.0, 1.0, 0.5)	(0.5, 0.0, 0.5)
9	(1.0, 0.0, 0.5, 0.5)	(0.0, 1.0, 0.5)	(1.0, 0.0, 0.5)
10	(0.5, 1.0, 1.0, 1.0)	(0.5, 0.5, 0.0)	(0.0, 0.0, 0.0)
11	(0.5, 1.0, 1.0, 0.5)	(0.5, 0.5, 0.0)	(0.0, 0.0, 0.5)
12	(0.5, 1.0, 0.5, 1.0)	(0.5, 0.5, 0.5)	(0.0, 0.5, 0.0)
13	(0.5, 1.0, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.0, 0.5, 0.5)
14	(0.5, 0.5, 1.0, 1.0)	(0.5, 0.5, 0.0)	(0.5, 0.0, 0.0)
15	(0.5, 0.5, 1.0, 0.5)	(0.5, 1.0, 0.0)	(0.5, 0.0, 0.5)
16	(0.5, 0.5, 1.0, 0.0)	(0.5, 1.0, 0.0)	(0.5, 0.0, 1.0)
17	(0.5, 0.5, 0.5, 1.0)	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.0)
18	(0.5, 0.5, 0.5, 0.5)	(0.5, 1.0, 0.5)	(0.5, 0.5, 0.5)
19	(0.5, 0.5, 0.5, 0.0)	(0.5, 1.0, 0.5)	(0.5, 0.5, 1.0)
20	(0.5, 0.5, 0.0, 1.0)	(0.5, 0.5, 1.0)	(0.5, 0.5, 0.0)
21	(0.5, 0.5, 0.0, 0.5)	(0.5, 1.0, 1.0)	(0.5, 0.5, 0.5)
22	(0.5, 0.5, 0.0, 0.0)	(0.5, 1.0, 1.0)	(0.5, 0.5, 1.0)
23	(0.5, 0.0, 0.5, 0.5)	(0.5, 1.0, 0.5)	(1.0, 0.5, 0.5)
24	(0.5, 0.0, 0.5, 0.0)	(0.5, 1.0, 0.5)	(1.0, 0.5, 1.0)
25	(0.5, 0.0, 0.0, 0.5)	(0.5, 1.0, 1.0)	(1.0, 0.5, 0.5)
26	(0.5, 0.0, 0.0, 0.0)	(0.5, 1.0, 1.0)	(1.0, 0.5, 1.0)
27	(0.0, 1.0, 0.5, 0.5)	(1.0, 0.5, 0.5)	(0.0, 0.5, 0.5)
28	(0.0, 0.5, 0.5, 0.5)	(1.0, 1.0, 0.5)	(0.5, 0.5, 0.5)
29	(0.0, 0.5, 0.5, 0.0)	(1.0, 1.0, 0.5)	(0.5, 0.5, 1.0)
30	(0.0, 0.5, 0.0, 0.5)	(1.0, 1.0, 1.0)	(0.5, 1.0, 0.5)
31	(0.0, 0.5, 0.0, 0.0)	(1.0, 1.0, 1.0)	(0.5, 1.0, 1.0)
32	(0.0, 0.0, 0.5, 0.5)	(1.0, 1.0, 0.5)	(1.0, 0.5, 0.5)
33	(0.0, 0.0, 0.5, 0.0)	(1.0, 1.0, 0.5)	(1.0, 0.5, 1.0)
34	(0.0, 0.0, 0.0, 0.5)	(1.0, 1.0, 1.0)	(1.0, 1.0, 0.5)
35	(0.0, 0.0, 0.0, 0.0)	(1.0, 1.0, 1.0)	(1.0, 1.0, 1.0)

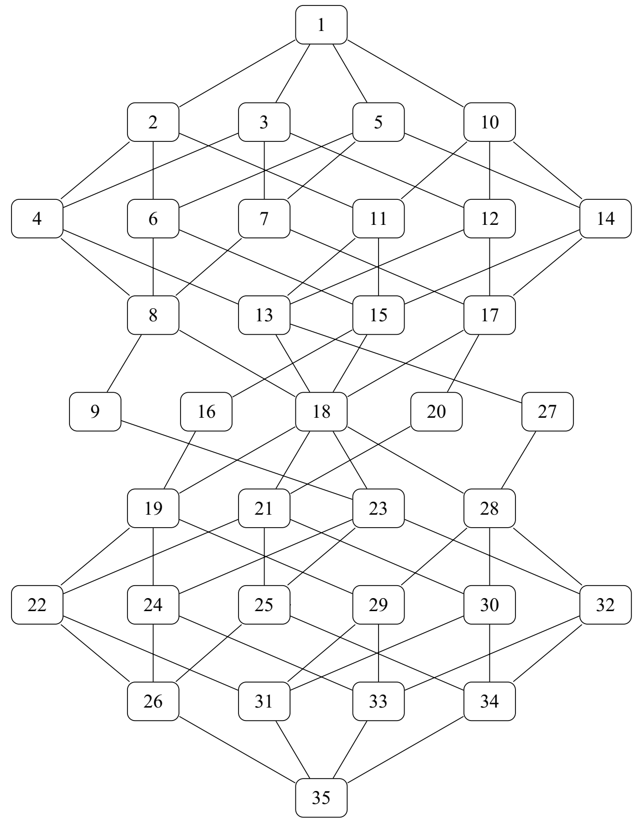


Fig. 4 $\mathbf{L}^{\lessdot}$-O3W concept lattice

Theorem 13 explains how to obtain the $\mathbf{L}$A3W concepts from $\mathbf{L}$2W concepts. For example, the $\mathbf{L}^{*}$-2W concept, $\langle \tilde{O}, \tilde{A} \rangle$, generates an $\mathbf{L}^{\lessdot}$-A3W concept by letting the extent as $(\tilde{O}, \tilde{A}^{\bar{*}})$ and the intent as $\tilde{A}$. The following holds by investigating conversely.

Theorem 14 *Given $\tilde{O}_1, \tilde{O}_2 \in L^{OB}$ and $\tilde{A} \in L^{AT}$, the following hold*:

1. *If $\langle (\tilde{O}_1, \tilde{O}_2), \tilde{A} \rangle$ is an $\mathbf{L}^{\lessdot}$-A3W concept, then $\langle \tilde{O}_1, \tilde{O}_1^{*} \rangle$ is an $\mathbf{L}^{*}$-2W concept and $\langle \tilde{O}_2, \tilde{O}_2^{\bar{*}} \rangle$ is an $\mathbf{L}^{\bar{*}}$-2W concept*;
2. *If $\langle (\tilde{O}_1, \tilde{O}_2), \tilde{A} \rangle$ is an $\mathbf{L}^{\bigtriangledown}$-A3W concept, then $\langle \tilde{O}_1, \tilde{O}_1^{\overline{\Diamond}} \rangle$ is an $\mathbf{L}^{\overline{\Diamond\Box}}$-2W concept and $\langle \tilde{O}_2, \tilde{O}_2^{\Diamond} \rangle$ is an $\mathbf{L}^{\Diamond\Box}$-2W concept*;
3. *If $\langle (\tilde{O}_1, \tilde{O}_2), \tilde{A} \rangle$ is an $\mathbf{L}^{\blacktriangledown}$-A3W concept, then $\langle \tilde{O}_1, \tilde{O}_1^{\overline{\Box}} \rangle$ is an $\mathbf{L}^{\overline{\Box\Diamond}}$-2W concept and $\langle \tilde{O}_2, \tilde{O}_2^{\Box} \rangle$ is an $\mathbf{L}^{\Box\Diamond}$-2W concept*;
4. *If $\langle (\tilde{O}_1, \tilde{O}_2), \tilde{A} \rangle$ is an $\mathbf{L}^{\triangleright}$-A3W concept, then $\langle \tilde{O}_1, \tilde{O}_1^{\#} \rangle$ is an $\mathbf{L}^{\#}$-2W concept, and $\langle \tilde{O}_2, \tilde{O}_2^{\overline{\#}} \rangle$ is an $\mathbf{L}^{\overline{\#}}$-2W concept.*

Proof The proof is similar to that of Theorem 11.

Theorem 14 shows how to obtain $\mathbf{L}$2W concepts through $\mathbf{L}$A3W concepts. The $\mathbf{L}$A3W concept can generate two related $\mathbf{L}$2W concepts. For example, the $\mathbf{L}^{\bigtriangledown}$-A3W concept forms both the $\mathbf{L}^{\overline{\Diamond\Box}}$-2W concept and the $\mathbf{L}^{\Diamond\Box}$-2W concept. Based on the combination of the results of Theorems 13 and 14, the following equivalent relationship between $\mathbf{L}$2W concepts and $\mathbf{L}$A3W concepts is valid.

Theorem 15 *Given $\tilde{O}_1, \tilde{O}_2 \in L^{OB}$ and $\tilde{A} \in L^{AT}$, the following hold*:

1. $\langle(\tilde{O}_1, \tilde{O}_2), \tilde{A}\rangle$ *is an* $\mathbf{L}^{\lessdot}$*-A3W concept iff there exist an* $\mathbf{L}^{*}$*-2W concept* $\langle\tilde{O}', \tilde{A}_1\rangle$ *and an* $\mathbf{L}^{\bar{*}}$*-2W concept* $\langle\tilde{O}'', \tilde{A}_2\rangle$ *such that* $\tilde{A} = \tilde{A}_1 \cap \tilde{A}_2, \tilde{O}_1 = (\tilde{A}_1 \cap \tilde{A}_2)^{*}$*, and* $\tilde{O}_2 = (\tilde{A}_1 \cap \tilde{A}_2)^{\bar{*}}$;
2. $\langle(\tilde{O}_1, \tilde{O}_2), \tilde{A}\rangle$ *is an* $\mathbf{L}^{\triangledown}$*-A3W concept iff there exist an* $\mathbf{L}^{\Diamond\overline{\Box}}$*-2W concept* $\langle\tilde{O}', \tilde{A}_1\rangle$ *and an* $\mathbf{L}^{\Diamond\Box}$*-2W concept* $\langle\tilde{O}'', \tilde{A}_2\rangle$ *such that* $\tilde{A} = \tilde{A}_1 \cup \tilde{A}_2$, $\tilde{O}_1 = (\tilde{A}_1 \cup \tilde{A}_2)^{\overline{\Box}}$*, and* $\tilde{O}_2 = (\tilde{A}_1 \cup \tilde{A}_2)^{\Box}$;
3. $\langle(\tilde{O}_1, \tilde{O}_2), \tilde{A}\rangle$ *is an* $\mathbf{L}^{\blacktriangledown}$*-A3W concept iff there exist an* $\mathbf{L}^{\overline{\Box}\Diamond}$*-2W concept* $\langle\tilde{O}', \tilde{A}_1\rangle$ *and an* $\mathbf{L}^{\Box\Diamond}$*-2W concept* $\langle\tilde{O}'', \tilde{A}_2\rangle$ *such that* $\tilde{A} = \tilde{A}_1 \cap \tilde{A}_2$, $\tilde{O}_1 = (\tilde{A}_1 \cap \tilde{A}_2)^{\Diamond}$*, and* $\tilde{O}_2 = (\tilde{A}_1 \cap \tilde{A}_2)^{\Diamond}$;
4. $\langle(\tilde{O}_1, \tilde{O}_2), \tilde{A}\rangle$ *is an* $\mathbf{L}^{\triangleright}$*-A3W concept iff there exist an* $\mathbf{L}^{\#}$*-2W concept* $\langle\tilde{O}', \tilde{A}_1\rangle$ *and an* $\mathbf{L}^{\overline{\#}}$*-2W concept* $\langle\tilde{O}'', \tilde{A}_2\rangle$ *such that* $\tilde{A} = \tilde{A}_1 \cup \tilde{A}_2, \tilde{O}_1 = (\tilde{A}_1 \cup \tilde{A}_2)^{\#}$*, and* $\tilde{O}_2 = (\tilde{A}_1 \cup \tilde{A}_2)^{\overline{\#}}$.

Proof The proof is similar to that of Theorem 12.

The results in Theorem 15 provide a way to construct the **L**A3W concept lattice from **L**2W concept lattices. For example, an $\mathbf{L}^{\triangleright}$-A3W concept lattice can be obtained as follows: take an $\mathbf{L}^{\#}$-2W concept, $\langle O_1, A_1\rangle$, from $\mathbf{C}^{\#}_{\mathbf{L}}(\mathbf{K})$ and an $\mathbf{L}^{\overline{\#}}$-2W concept, $\langle O_2, A_2\rangle$, from $\mathbf{C}^{\overline{\#}}_{\mathbf{L}}(\mathbf{K})$; compute $A_1 \cup A_2$, $(A_1 \cup A_2)^{\#}$, and $(A_1 \cup A_2)^{\overline{\#}}$; then, $\langle((A_1 \cup A_2)^{\#}, (A_1 \cup A_2)^{\overline{\#}}), A_1 \cup A_2\rangle$ is an $\mathbf{L}^{\triangleright}$-A3W concept. The $\mathbf{L}^{\triangleright}$-A3W concept lattice is created by determining all non-repeated $\mathbf{L}^{\triangleright}$-A3W concepts. Algorithm 2 is used to compute the $\mathbf{L}^{\lessdot}$-A3W concept lattice from the $\mathbf{L}^{*}$-2W concept lattice and $\mathbf{L}^{\bar{*}}$-2W concept lattice. Similar to Algorithm 1, we prove that the time complexity of Algorithm 2 is $O(k^2l^2 \cdot |AT|^2)$.

Algorithm 2: Generate $\mathbf{L}^{\lessdot}$-A3W concept lattice.

input : $\mathbf{L}^{*}$-2W concept lattice: $\mathbf{C}^{*}_{\mathbf{L}}(\mathbf{K}) = \{\langle O_{i*}, A_{i*}\rangle\}$, $\mathbf{L}^{\bar{*}}$-2W concept lattice: $\mathbf{C}^{\bar{*}}_{\mathbf{L}}(\mathbf{K}) = \{\langle O_{i\bar{*}}, A_{i\bar{*}}\rangle\}$.
output: $\mathbf{L}^{\lessdot}$-A3W concept lattice: $\mathbf{C}^{\lessdot}_{A\mathbf{L}}(K = \{\langle(O_{1i}, O_{2i}), A_i\rangle\}$.

$n = 0$,
for $i = 1$ *to* $|\mathbf{C}^{*}_{\mathbf{L}}(\mathbf{K})|$ **do**
  for $j = 1$ *to* $|\mathbf{C}^{\bar{*}}_{\mathbf{L}}(\mathbf{K})|$ **do**
    $n = n + 1$,
    $A_n = A_{i*} \cap A_{j\bar{*}}$,
  end
end
Delete repeated elements in $\{A_1, A_2, \cdots\}$,
for *each* A_i **do**
  $O_{1i} = A_i^{*}$,
  $O_{2i} = A_i^{\bar{*}}$.
end

Example 6 (Continued from Example 5) Using $\mathbf{L}^{*}$-2W concepts and $\mathbf{L}^{\bar{*}}$-2W concepts as inputs, we obtain 20 $\mathbf{L}^{\lessdot}$-A3W concepts (see Table 11) based on Algorithm 2. The Hasse diagram of the $\mathbf{L}^{\lessdot}$-A3W concept lattice is shown in Fig. 5. Each number in the figure corresponds to the $\mathbf{L}^{\lessdot}$-A3W concept in Table 11, and a line connects two concepts in which the lower concept is a sub-concept of the upper one.

Table 11 $\mathbf{L}^{\lessdot}$-A3W concepts

No.	$\tilde{O}_1$	$\tilde{O}_2$	$\tilde{A}$
1	(1.0, 1.0, 1.0, 1.0)	(1.0, 1.0, 1.0, 1.0)	(0.0, 0.0, 0.0)
2	(1.0, 1.0, 0.5, 1.0)	(1.0, 1.0, 1.0, 0.5)	(0.0, 0.0, 0.5)
3	(1.0, 1.0, 1.0, 1.0)	(0.5, 1.0, 0.5, 1.0)	(0.0, 0.5, 0.0)
4	(1.0, 1.0, 0.5, 1.0)	(0.5, 1.0, 0.5, 0.5)	(0.0, 0.5, 0.5)
5	(1.0, 0.5, 1.0, 0.5)	(0.0, 0.5, 0.0, 0.5)	(0.0, 1.0, 0.0)
6	(1.0, 0.5, 0.5, 0.5)	(0.0, 0.5, 0.0, 0.5)	(0.0, 1.0, 0.5)
7	(0.5, 1.0, 1.0, 1.0)	(1.0, 0.5, 1.0, 1.0)	(0.5, 0.0, 0.0)
8	(0.5, 1.0, 0.5, 1.0)	(1.0, 0.5, 1.0, 0.5)	(0.5, 0.0, 0.5)
9	(0.5, 0.5, 0.0, 1.0)	(0.5, 0.5, 1.0, 0.0)	(0.5, 0.0, 1.0)
10	(0.5, 1.0, 1.0, 1.0)	(0.5, 0.5, 0.5, 1.0)	(0.5, 0.5, 0.0)
11	(0.5, 1.0, 0.5, 1.0)	(0.5, 0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)
12	(0.5, 0.5, 0.0, 1.0)	(0.5, 0.5, 0.5, 0.0)	(0.5, 0.5, 1.0)
13	(0.5, 0.5, 1.0, 0.5)	(0.0, 0.5, 0.0, 0.5)	(0.5, 1.0, 0.0)
14	(0.5, 0.5, 0.5, 0.5)	(0.0, 0.5, 0.0, 0.5)	(0.5, 1.0, 0.5)
15	(0.5, 0.5, 0.0, 0.5)	(0.0, 0.5, 0.0, 0.0)	(0.5, 1.0, 1.0)
16	(0.0, 1.0, 0.5, 0.5)	(1.0, 0.0, 0.5, 0.5)	(1.0, 0.0, 0.5)
17	(0.0, 1.0, 0.5, 0.5)	(0.5, 0.0, 0.5, 0.5)	(1.0, 0.5, 0.5)
18	(0.0, 0.5, 0.0, 0.5)	(0.5, 0.0, 0.5, 0.0)	(1.0, 0.5, 1.0)
19	(0.0, 0.5, 0.5, 0.5)	(0.0, 0.0, 0.0, 0.5)	(1.0, 1.0, 0.5)
20	(0.0, 0.5, 0.0, 0.5)	(0.0, 0.0, 0.0, 0.0)	(1.0, 1.0, 1.0)

Conclusion

Concept learning plays a key role in human cognition. It may be easy for humans to distinguish among a set of actual objects. However, it will be a little complex for humans to figure out different concepts just from a batch of data. With the help of FCA, we can easily mine hidden patterns from the data.

In this study, several new types of **L**2W concept and **L**3W concept are proposed, and the isomorphic relationship between **L**-concept lattices is discussed. The results demonstrate that the eight types of **L**2W concept lattices, as well as the eight types of **L**3W concept lattices, can be divided into two isomorphic groups respectively. The isomorphic relationships help us to investigate and construct new concept lattices through existing ones. The equivalent relationship between **L**2W concepts and **L**3W concepts helps to generate **L**3W concept lattices through **L**2W concept lattices.

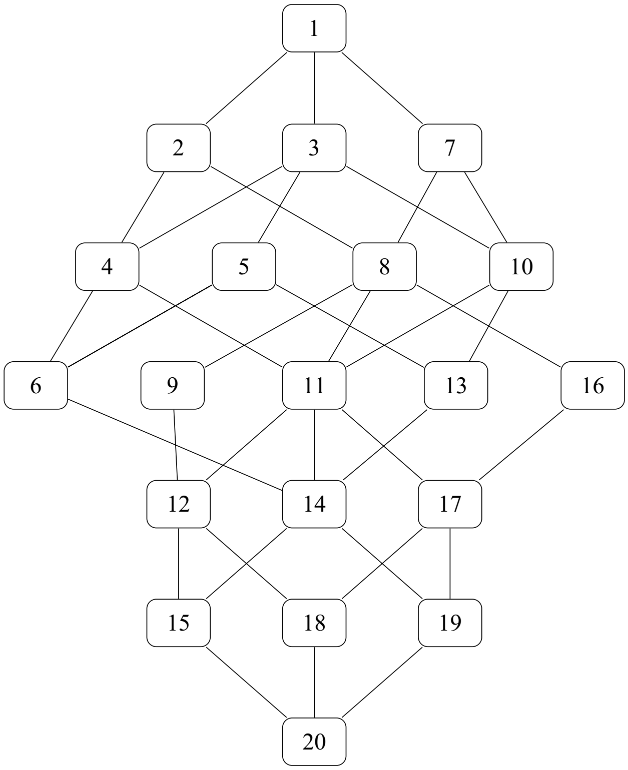


Fig. 5 $\mathbf{L}^{\lessdot}$-A3W concept lattice

In future work, we will focus on two areas. First, as there are different ways to extend classical operations to fuzzy cases, there are various ways to investigate **L**-concept analysis. Thus, we will explore the isomorphic relationship between **L**-concept lattices obtained through other methods. Second, we will apply this work to multi-attribute decision-making and classification problems.

Appendix I

The following are the verification of the equations in Table 2:

Clearly, for $\tilde{O} \in L^{OB}$, it holds that $\tilde{O}^{\bar{*}} = \tilde{O}^{*}_{\tilde{R}^c}$, $\tilde{O}^{\Box} = \tilde{O}^{\overline{\Box}}_{\tilde{R}^c}$, $\tilde{O}^{\overline{\Diamond}} = \tilde{O}^{\Diamond}_{\tilde{R}^c}$, $\tilde{O}^{\#} = \tilde{O}^{\bar{\#}}_{\tilde{R}^c}$, $\tilde{O}^{\overline{\Box}} = \tilde{O}^{c*}$, $\tilde{O}^{\Box} = \tilde{O}^{c\bar{*}}$, $\tilde{O}^{\bar{\#}} = \tilde{O}^{c\Diamond}$, and $\tilde{O}^{\#} = \tilde{O}^{c\overline{\Diamond}}$.

Moreover, the regularity of **L** ensures that $\tilde{O}^{*} = \tilde{O}^{\bar{*}}_{\tilde{R}^c}$, $\tilde{O}^{\overline{\Box}} = \tilde{O}^{\Box}_{\tilde{R}^c}$, $\tilde{O}^{\Diamond} = \tilde{O}^{\overline{\Diamond}}_{\tilde{R}^c}$, $\tilde{O}^{\bar{\#}} = \tilde{O}^{\#}_{\tilde{R}^c}$, $\tilde{O}^{*} = \tilde{O}^{c\overline{\Box}}$, $\tilde{O}^{\bar{*}} = \tilde{O}^{c\Box}$, $\tilde{O}^{\Diamond} = \tilde{O}^{c\bar{\#}}$, and $\tilde{O}^{\overline{\Diamond}} = \tilde{O}^{c\#}$.

For $\tilde{O} \in L^{OB}$ and $a \in AT$, items (2) and (3) of Lemma 1 verify that

$$
\begin{aligned}
&\tilde{O}^{\overline{\Diamond}c}(a) = \neg\left(\tilde{O}^{\overline{\Diamond}}(a)\right)\\
&= \neg\big(\bigvee\nolimits_{o\in OB} \big(\tilde{O}(o) \otimes \tilde{R}^c(o,a)\big)\big)\\
&= \neg\big(\bigvee\nolimits_{o\in OB} \big(\tilde{O}(o) \otimes \neg\tilde{R}(o,a)\big)\big)\\
&= \neg\big(\bigvee\nolimits_{o\in OB} \neg\big(\tilde{O}(o) \to \tilde{R}(o,a)\big)\big)\\
&= \neg\neg\big(\bigwedge\nolimits_{o\in OB} \big(\tilde{O}(o) \to \tilde{R}(o,a)\big)\big)\\
&= \bigwedge\nolimits_{o\in OB} \big(\tilde{O}(o) \to \tilde{R}(o,a)\big) = \tilde{O}^{*}(a),\\
&\tilde{O}^{*c}(a) = \neg\big(\tilde{O}^{*}(a)\big)\\
&= \neg\big(\bigwedge\nolimits_{o\in OB} \big(\tilde{O}(o) \to \tilde{R}(o,a)\big)\big)\\
&= \neg\big(\bigwedge\nolimits_{o\in OB} \big(\tilde{O}(o) \to \neg\neg\tilde{R}(o,a)\big)\big)\\
&= \neg\big(\bigwedge\nolimits_{o\in OB} \neg\big(\tilde{O}(o) \otimes \neg\tilde{R}(o,a)\big)\big)\\
&= \bigvee\nolimits_{o\in OB} \neg\neg\big(\tilde{O}(o) \otimes \neg\tilde{R}(o,a)\big)\\
&= \bigvee\nolimits_{o\in OB} \big(\tilde{O}(o) \otimes \tilde{R}^c(o,a)\big) = \tilde{O}^{\overline{\Diamond}}(a),\\
&\tilde{O}^{\Diamond c}(a) = \neg\big(\tilde{O}^{\Diamond}(a)\big)\\
&= \neg\big(\bigvee\nolimits_{o\in OB} \big(\tilde{O}(o) \otimes \tilde{R}(o,a)\big)\big)\\
&= \neg\big(\bigvee\nolimits_{o\in OB} \big(\tilde{O}(o) \otimes \neg\neg\tilde{R}(o,a)\big)\big)\\
&= \neg\big(\bigvee\nolimits_{o\in OB} \neg\big(\tilde{O}(o) \to \neg\tilde{R}(o,a)\big)\big)\\
&= \neg\neg\big(\bigwedge\nolimits_{o\in OB} \big(\tilde{O}(o) \to \neg\tilde{R}(o,a)\big)\big)\\
&= \bigwedge\nolimits_{o\in OB} \big(\tilde{O}(o) \to \tilde{R}^c(o,a)\big) = \tilde{O}^{\bar{*}}(a),\\
&\tilde{O}^{\bar{*}c}(a) = \neg\big(\tilde{O}^{\bar{*}}(a)\big)\\
&= \neg\big(\bigwedge\nolimits_{o\in OB} \big(\tilde{O}(o) \to \tilde{R}^c(o,a)\big)\big)\\
&= \neg\big(\bigwedge\nolimits_{o\in OB} \big(\tilde{O}(o) \to \neg\tilde{R}(o,a)\big)\big)\\
&= \neg\big(\bigwedge\nolimits_{o\in OB} \neg\big(\tilde{O}(o) \otimes \tilde{R}(o,a)\big)\big)\\
&= \bigvee\nolimits_{o\in OB} \neg\neg\big(\tilde{O}(o) \otimes \tilde{R}(o,a)\big)\\
&= \bigvee\nolimits_{o\in OB} \big(\tilde{O}(o) \otimes \tilde{R}(o,a)\big) = \tilde{O}^{\Diamond}(a),\\
&\tilde{O}^{\overline{\Box}c}(a) = \neg\left(\tilde{O}^{\overline{\Box}}(a)\right)\\
&= \neg\big(\bigwedge\nolimits_{o\in OB} \big(\tilde{O}^c(o) \to \tilde{R}(o,a)\big)\big)\\
&= \neg\big(\bigwedge\nolimits_{o\in OB} \big(\neg\tilde{O}(o) \to \tilde{R}(o,a)\big)\big)\\
&= \neg\big(\bigwedge\nolimits_{o\in OB} \neg\big(\neg\tilde{O}(o) \otimes \neg\tilde{R}(o,a)\big)\big)\\
&= \bigvee\nolimits_{o\in OB} \neg\neg\big(\neg\tilde{O}(o) \otimes \neg\tilde{R}(o,a)\big)\\
&= \bigvee\nolimits_{o\in OB} \big(\tilde{O}^c(o) \otimes \tilde{R}^c(o,a)\big) = \tilde{O}^{\#}(a),\\
&\tilde{O}^{\#c}(a) = \neg\big(\tilde{O}^{\#}(a)\big)\\
&= \neg\big(\bigvee\nolimits_{o\in OB} \big(\tilde{O}^c(o) \otimes \tilde{R}^c(o,a)\big)\big)\\
&= \neg\big(\bigvee\nolimits_{o\in OB} \big(\neg\tilde{O}(o) \otimes \neg\tilde{R}(o,a)\big)\big)\\
&= \neg\big(\bigvee\nolimits_{o\in OB} \neg\big(\neg\tilde{O}(o) \to \tilde{R}(o,a)\big)\big)\\
&= \neg\neg\big(\bigwedge\nolimits_{o\in OB} \big(\neg\tilde{O}(o) \to \tilde{R}(o,a)\big)\big)\\
&= \bigwedge\nolimits_{o\in OB} \big(\tilde{O}^c(o) \to \tilde{R}(o,a)\big) = \tilde{O}^{\overline{\Box}}(a),\\
&\tilde{O}^{\Box c}(a) = \neg\big(\tilde{O}^{\Box}(a)\big)\\
&= \neg\big(\bigwedge\nolimits_{o\in OB} \big(\tilde{O}^c(o) \to \tilde{R}^c(o,a)\big)\big)\\
&= \neg\big(\bigwedge\nolimits_{o\in OB} \big(\neg\tilde{O}(o) \to \neg\tilde{R}(o,a)\big)\big)\\
&= \neg\big(\bigwedge\nolimits_{o\in OB} \neg\big(\neg\tilde{O}(o) \otimes \tilde{R}(o,a)\big)\big)\\
&= \bigvee\nolimits_{o\in OB} \neg\neg\big(\neg\tilde{O}(o) \otimes \tilde{R}(o,a)\big)\\
&= \bigvee\nolimits_{o\in OB} \big(\tilde{O}^c(o) \otimes \tilde{R}(o,a)\big) = \tilde{O}^{\bar{\#}}(a),\\
&\tilde{O}^{\bar{\#}c}(a) = \neg\left(\tilde{O}^{\bar{\#}}(a)\right)\\
&= \neg\big(\bigvee\nolimits_{o\in OB} \big(\tilde{O}^c(o) \otimes \tilde{R}(o,a)\big)\big)\\
&= \neg\big(\bigvee\nolimits_{o\in OB} \big(\neg\tilde{O}(o) \otimes \tilde{R}(o,a)\big)\big)\\
&= \neg\big(\bigvee\nolimits_{o\in OB} \neg\big(\neg\tilde{O}(o) \to \neg\tilde{R}(o,a)\big)\big)\\
&= \neg\neg\big(\bigwedge\nolimits_{o\in OB} \big(\neg\tilde{O}(o) \to \neg\tilde{R}(o,a)\big)\big)\\
&= \bigwedge\nolimits_{o\in OB} \big(\tilde{O}^c(o) \to \tilde{R}^c(o,a)\big) = \tilde{O}^{\Box}(a).
\end{aligned}
$$

Appendix II

Briefly, we provide the proof of the equivalence of **L**3W operators and **L**3W inverse operators, as shown in Table 7.

Based on the relationship between **L**2W operators (see Fig. 1) and by the definition of **L**3W operators, the following assertions hold. For $\tilde{O}, \tilde{O}_1, \tilde{O}_2 \in L^{OB}$,

$$
\begin{aligned}
(\tilde{O}^c)^{\lessdot} &= (\tilde{O}^{c*}, \tilde{O}^{c\bar{*}}) = (\tilde{O}^{\overline{\Box}}, \tilde{O}^{\Box}) = \tilde{O}^{\bigtriangledown},\\
(\tilde{O}^{\bigtriangledown})^c &= (\tilde{O}^{\overline{\Box}}, \tilde{O}^{\Box})^c = (\tilde{O}^{\overline{\Box}c}, \tilde{O}^{\Box c}) = (\tilde{O}^{\#}, \tilde{O}^{\bar{\#}}) = \tilde{O}^{\triangleright},\\
(\tilde{O}^c)^{\triangleright} &= (\tilde{O}^{c\#}, \tilde{O}^{c\bar{\#}}) = (\tilde{O}^{\overline{\Diamond}}, \tilde{O}^{\Diamond}) = \tilde{O}^{\blacktriangledown},\\
(\tilde{O}^{\blacktriangledown})^c &= (\tilde{O}^{\overline{\Diamond}}, \tilde{O}^{\Diamond})^c = (\tilde{O}^{\overline{\Diamond}c}, \tilde{O}^{\Diamond c}) = (\tilde{O}^{*}, \tilde{O}^{\bar{*}}) = \tilde{O}^{\lessdot},
\end{aligned}
$$

and

$$
\begin{aligned}
((\tilde{O}_1, \tilde{O}_2)^c)^{\gtrdot} &= (\tilde{O}_1^c, \tilde{O}_2^c)^{\gtrdot} = \tilde{O}_1^{c*} \cap \tilde{O}_2^{c\bar{*}}\\
&= \tilde{O}_1^{\overline{\Box}} \cap \tilde{O}_2^{\Box} = (\tilde{O}_1, \tilde{O}_2)^{\blacktriangle},\\
((\tilde{O}_1, \tilde{O}_2)^{\blacktriangle})^c &= (\tilde{O}_1^{\overline{\Box}} \cap \tilde{O}_2^{\Box})^c = \tilde{O}_1^{\overline{\Box}c} \cup \tilde{O}_2^{\Box c}\\
&= \tilde{O}_1^{\#} \cup \tilde{O}_2^{\bar{\#}} = (\tilde{O}_1, \tilde{O}_2)^{\triangleleft},\\
((\tilde{O}_1, \tilde{O}_2)^c)^{\triangleleft} &= (\tilde{O}_1^c, \tilde{O}_2^c)^{\triangleleft} = \tilde{O}_1^{c\#} \cup \tilde{O}_2^{c\bar{\#}}\\
&= \tilde{O}_1^{\overline{\Diamond}} \cup \tilde{O}_2^{\Diamond} = (\tilde{O}_1, \tilde{O}_2)^{\bigtriangleup},\\
((\tilde{O}_1, \tilde{O}_2)^{\bigtriangleup})^c &= (\tilde{O}_1^{\overline{\Diamond}} \cup \tilde{O}_2^{\Diamond})^c = \tilde{O}_1^{\overline{\Diamond}c} \cap \tilde{O}_2^{\Diamond c}\\
&= \tilde{O}_1^{*} \cap \tilde{O}_2^{\bar{*}} = (\tilde{O}_1, \tilde{O}_2)^{\gtrdot}.
\end{aligned}
$$

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Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

Ethical Approval This article does not contain any studies with human participants or animals performed by any of the authors

Informed Consent Informed consent was obtained from all individual participants included in the study.

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