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ORIGINAL RESEARCH

Rule acquisition of three-way semi-concept lattices in formal decision context

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Abstract

Three-way concept analysis is an important tool for information processing, and rule acquisition is one of the research hotspots of three-way concept analysis. However, compared with three-way concept lattices, three-way semi-concept lattices have three-way operators with weaker constraints, which can generate more concepts. In this article, the problem of rule acquisition for three-way semi-concept lattices is discussed in general. The authors construct the finer relation of three-way semi-concept lattices, and propose a method of rule acquisition for three-way semi-concept lattices. The authors also discuss the set of decision rules and the relationships of decision rules among object-induced three-way semi-concept lattices, object-induced three-way concept lattices, classical concept lattices and semi-concept lattices. Finally, examples are provided to illustrate the validity of our conclusions.

KEYWORDS

finer relation, rule acquisition, three-way concept analysis, three-way semi-concept lattices

1 | INTRODUCTION

In general, a formal concept consists of extension and intension, and is the formal and mathematical description of concepts in philosophy. Since Wille [1] proposed formal concept analysis (FCA) for constructing lattice theory in 1982, FCA has been widely used in data analysis [2–4], information retrieval [5], data mining [6], medical diagnosis [7], machine learning [8] and many other fields [9, 10]. The semi-concept proposed by Luksch and Wille [11] in 1991 is a branch of FCA, and the binary relation expressed by it is unidirectional, which makes any set of objects and attributes are semi-concepts. Vormbrock [12] explained the relationship between semi-concept and classical concept, and proved two important theorems of semi-concept.

Three-way decisions [13], proposed by Yao, is a decision-making theory, which describes the decision-making behaviour of decision maker on uncertain things. The three-way decision theory considers decision problems from the perspective of acceptance, rejection and non-commitment, and can well explain many decision problems in practice. Qi et al. [14] combined FCA with three-way decisions and proposed the concept of three-way concept analysis, and divided the object set (or attribute set) into three parts by the information of 'common possession' and 'common not possession'. Compared with the traditional FCA, three-way concept analysis can better mine the hidden information in the formal context, and is more in line with people's decision-making behaviour and cognitive process, which makes it more conducive to the data expression. Zhai et al. [15] developed a structure theorem for three-way concept lattice which mathematically describes the relationships between concept lattices. Mao et al. [16] combined three-way decision with rough semi-concepts, defined the three-way rough semi-concepts, and gave two approximation operators to describe the three-way rough semi-concepts. On this basis, they also defined two forms of the three-way semi-concept (*OE*-semi-concept and *AE*-semi-concept) [17].

In order to obtain the decision rules of concept lattices, Zhang et al. [18] and Wei et al. [19] explored the Formal Decision Context (FDC), and then Liu et al. [20] discussed the problem of rule acquisition of concept lattice based on their research and rough set theory. Rule acquisition is also an important content of three-way concept analysis. For the consistent decision formal context, Wei et al. [21] acquired the decision rules of three-way concept lattices, and gave their semantic interpretation. Liu et al. [22] designed a rule acquisition method for three-way concept lattice under the inconsistent formal context, and presented some important properties of three-way rules. Li et al. [23] proposed a theoretical framework of object compressing for FDC without losing decision rules. Xu and Huang [24] introduced a methodological analysis of rule extraction of three-way concept lattices, and their purpose was to build shared conceptual models of the real world that support knowledge-intensive domain applications. Zhai et al. [25] introduced variable decision implications as a fundamental form of knowledge reasoning in FCA for decision-making problems. They extended the concept of decision implications to uncertain decision implications and provided semantic explanations. Zhai et al. [26] proposed three inference rules for deduction on decision implications and studied their properties. Li et al. [27] developed the minimal closed label concept lattice to extract rules from concept lattices. They demonstrated through numerical experiments, that limitary decision implications can acquire rules more easily than decision implications. Wu et al. [28] investigated the rule extraction of multi-scale formal context, and presented a method of local optimal scale selections to obtain more concise decision rules for different objects.

This work aims to establish the mathematical basis for information storage of three-way semi-concepts using the three-way semi-concept lattice. Compared with the three-way concept, the binary relationship of the three-way semi-concept is unidirectional. Our work complements that of Wei et al. by considering a complementary context that was not accounted for in their study. Additionally, we provide a detailed comparison of decision rule sets for semi-concept lattices, concept lattices, three-way concept lattices, and three-way semi-concept lattices from a generalised perspective.

The rest of this paper is organised as follows: Section 2 reviews some basic concepts related to formal concept, semi-concept, three-way concept, three-way semi-concept and rule acquisition of three-way concept lattice. Section 3 provides a method of rule acquisition of object-induced three-way semi-concept lattice and discusses the relations of semi-consistency, consistency, object-induced three-way semi-consistency and object-induced three-way consistency of FDC. Finally, conclusions are drawn in Section 4.

2 | RELATED CONCEPTS

This section mainly reviews some necessary concepts about formal concept, semi-concept, three-way concept, three-way semi-concept, and rules acquisition of three-way concept lattices.

2.1 | Formal concept analysis

Definition 1 [1] Assume that $K = (G, V, I)$ is a formal context, where G is a non-empty finite set of objects, V is a non-empty finite set of attributes, and I is the relationship of G and V, where $I \subseteq G \times V$. If an object $u \in G$ possesses an attribute $v \in V$, we denote the relation of u and v as uIv (or $(u, v) \in I$).

For convenience of expression, we denote formal context by *FC*.

Definition 2 [1] Suppose $K = (G, V, I)$ is a *FC*. For any $Y \subseteq G$ and $B \subseteq V$, a pair of positive operators are defined in the following:

$$^*: P(G) \rightarrow P(V), Y^* = \{v \in V | \, \forall y \in Y(yIv)\} = \{v \in V | Y \subseteq Iv\} \tag{1}$$

$$^*: P(V) \rightarrow P(G), B^* = \{u \in G | \, \forall b \in B(uIb)\} = \{u \in G | B \subseteq uI\} \tag{2}$$

where $P(G)$ and $P(V)$ are the power sets (the set of all subsets) of G and V respectively. Y^* is the set of all attributes shared by all objects in Y and B^* is the set of all objects having all attributes in B.

If $Y^* = B$ and $B^* = Y$, then (Y, B) is called as a formal concept, and then Y and B are called the extension and the intension of the formal concept respectively. The two operators * defined above are called positive operators and concepts induced by positive operators are also called positive concepts (for short, P-concepts). The complete lattice consisting of all P-concepts is denoted as $L(G, V, I)$, $L_E(G, V, I)$ and $L_I(G, V, I)$ represent the set of intensions, the set of the components of extensions in $L(G, V, I)$ respectively.

2.2 | Three-way concept analysis

We sometimes have to pay attention not only to the number of supporting, but also to the number of opposing. When the number of opposing exceeds the number of supporting, we will not be able to achieve our goals. To characterise this phenomenon, the scholars have defined a pair of negative operators.

Definition 3 [14] Assume that $K = (G, V, I)$ is a *FC*. (G, V, I^C) is the complement context of K (here $I^C = G \times V - I$), namely, for any $x \in U$ and $a \in V$, $(x, a) \in R^C$ (also noted as xR^Ca) means that the object x does not have the attribute a.

Definition 4 [14] Suppose $K = (G, V, I)$ is a *FC*. For any $Y \subseteq G$ and $B \subseteq V$, a pair of negative operators, $^{\bar{*}}: P(G) \rightarrow P(V)$ and $^{\bar{*}}: P(V) \rightarrow P(G)$, are defined as:

$$
\begin{aligned}
Y^{\bar{*}} &= \{v \in V |\ \forall y \in Y(\neg(yIv))\} \\
&= \{v \in V |\ \forall y \in Y(yI^c v)\} \\
&= \{v \in V | Y \subseteq I^c v\}
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
B^{\bar{*}} &= \{u \in G |\ \forall b \in B(\neg(uIb))\} \\
&= \{u \in G |\ \forall b \in B(uI^c b)\} \\
&= \{u \in G | B \subseteq uI^c\}
\end{aligned} \tag{4}
$$

Correspondingly, negative concept can be obtained by Definition 4.

Definition 5 [14] Assume that $K = (G, V, I)$ is a *FC*. For any $Y \subseteq G$ and $B \subseteq V$, if $Y^{\bar{*}} = B$ and $B^{\bar{*}} = Y$, then (Y, B) is called a negative concept (noted as N-concept). Similarly, Y and B are called the extension and the intension of (Y, B) respectively.

It is easy to see that $\left(Y^{\bar{*}\bar{*}}, Y^{\bar{*}}\right)$ and $\left(B^{\bar{*}}, B^{\bar{*}\bar{*}}\right)$ are N-concepts, and all N-concepts form a complete lattice (denote by $NL(G, V, I)$).

With the operators * and $^{\bar{*}}$, we can then construct a pair of three-way operators to describe the information of 'common possession' and 'common not possession'.

Definition 6 [14] Suppose $K = (G, V, I)$ is a *FC*. For any $Y \subseteq G$ and B, $C \subseteq V$, a pair of object-induced three-way operators (for short, *OE*-operators), $(OE1)^{\lhd}: P(G) \to DP(V)$ and $(OE2)^{\rhd}: DP(V) \to P(G)$, are defined as:

$$Y^{\lhd} = \left(Y^*, Y^{\bar{*}}\right) \tag{5}$$

$$(B, C)^{\rhd} = \left\{u \in G \middle| u \in B^*, u \in C^{\bar{*}}\right\} = B^* \cap C^{\bar{*}} \tag{6}$$

where $DP(G)$ is the set of all pairs of subsets of U, that is, $P(G) \times P(G)$, and $DP(V)$ is the set of all pairs of subsets of V, that is, $P(V) \times P(V)$. For any $A \subseteq V$ and X, $Y \subseteq G$, a pair of attribute-induced three-way operators (for short, *AE*-operators), $(AE1)^{\lhd}: P(V) \to DP(G)$ and $(AE2)^{\rhd}: DP(G) \to P(V)$, are defined as:

$$A^{\lhd} = \left(A^*, A^{\bar{*}}\right) \tag{7}$$

$$(X, Y)^{\rhd} = \left\{v \in V \middle| v \in A^*, v \in B^{\bar{*}}\right\} = X^* \cap Y^{\bar{*}} \tag{8}$$

Definition 7 [14] Assume that $K = (G, V, I)$ is a *FC*. For any $Y \subseteq G$, B, $C \subseteq V$, if $Y^{\lhd} = (B, C)$ and $(B, C)^{\rhd} = Y$, then $(Y, (B, C))$ is called an object-induced three-way concept (for short, OE-concept), and then Y is called the extension and B, C are called positive intension and negative intension of $(Y, (B, C))$ respectively. The complete lattice consisting of all OE-concepts is denoted as $OEL(G, V, I)$.

Theorem 1 *[29] Let $K = (G, V, I)$ be a FC, the following relations hold.*

(1) $L_E(G, V, I) \subseteq OEL_E(G, V, I)$,
$NL_E(G, V, I) \subseteq OEL_E(G, V, I)$
(2) $L_I(G, V, I) = OEL_I^+(G, V, I)$,
$NL_I(G, V, I) = OEL_I^-(G, V, I)$

Theorem 1 shows that there are some concrete connections between three-way concept lattices and classical concept lattices.

Definition 8 [14] Suppose $K = (G, V, I)$ is a *FC*. For any Y, $Z \subseteq G$, $B \subseteq V$, if $(Y, Z)^{\rhd} = B$ and $B^{\lhd} = (Y, Z)$, then $((Y, Z), B)$ is called an attribute-induced three-way concept (for short, AE-concept), and then B is called the intension and Y, Z are called the positive extension and negative extension of $((Y, Z), B)$ respectively. The complete lattice consisting of all AE-concepts is denoted as $AEL(G, V, I)$.

We will illustrate the above-mentioned concepts by Example 1.

Example 1 A *FC* is shown in Table 1, and its complement context is shown in Table 2.

Figures 1–4 show the concept lattice in *FC*, the concept lattice in FC's complement context, $OEL(G, V, I)$ and $AEL(G, V, I)$ respectively.

2.3 | Semi-concept

The unidirectional relation is an important binary relation. In this subsection, we will review the formal expression of unidirectional relation by semi-concept.

Definition 9 [11, 12] Assume $K = (G, V, I)$ is a *FC*. For any Y, $Z \subseteq G$, B, $C \subseteq V$, if $Y^* = B$, then (Y, B) is called $\cap$-semi-concept, and then Y and B are called the extension and

TABLE 1 A formal context (G, V, I).

	a_1	a_2	a_3	a_4
1	1	1	0	0
2	0	0	0	1
3	1	1	1	0

TABLE 2 Complement context (G, V, I^C).

	a_1	a_2	a_3	a_4
1	0	0	1	1
2	1	1	1	0
3	0	0	0	1

intension of (Y, B) respectively. Alternatively, (Z, C) is called ∪-semi-concept if $Z* = C$. The complete lattice consisting of all semi-concepts is denoted as *SL(G, V, I)*.

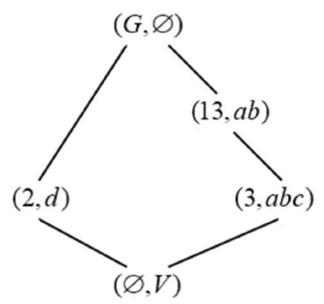

FIGURE 1 $L(G, V, I)$ produced from Table 1.

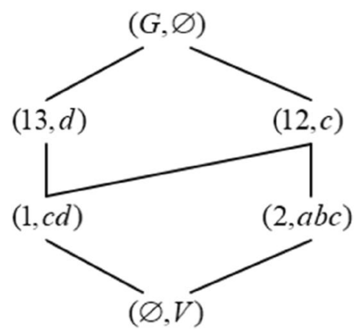

FIGURE 2 $NL(G, V, I)$ produced from Table 2.

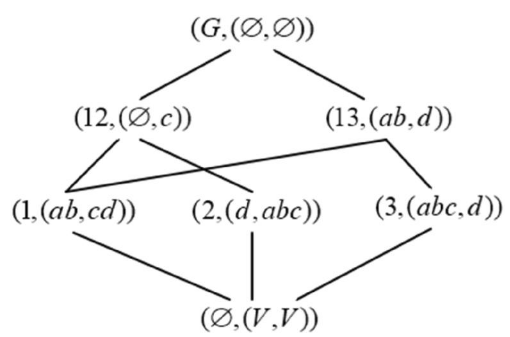

FIGURE 3 $OEL(G, V, I)$ produced from Tables 1 and 2.

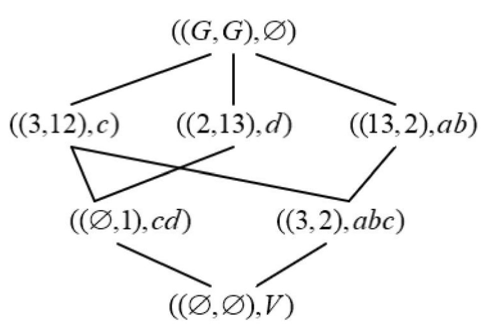

FIGURE 4 $AEL(G, V, I)$ produced from Tables 1 and 2.

To clearly depict the semi-concepts, we illustrate it through Example 2.

Example 2 Consider Table 1 as a formal context. By the Definition 9, $(\varnothing, V)$, $(1, ab)$, $(2, d)$, $(3, abc)$, $(12, \varnothing)$, $(13, ab)$, $(23,\varnothing)$ and $(G,\varnothing)$ are ∩-semi-concepts, $(G, \varnothing)$, $(13, a)$, $(13, b)$, $(3, c)$, $(2, d)$, $(13, ab)$, $(3, ac)$, $(\varnothing, ad)$, $(3, bc)$, $(\varnothing, bd)$, $(\varnothing, cd)$, $(3, abc)$, $(\varnothing, abd)$, $(\varnothing, acd)$, $(\varnothing, bcd)$ and $(\varnothing, V)$ are ∪-semi-concepts.

2.4 | Three-way semi-concept

Due to the restriction of three-way operator is weakened, three-way semi-concept can produce more abundant information than three-way concept.

Definition 10 [17] Suppose $K = (G, V, I)$ is a *FC*. For any $Y \subseteq G$, B, $C \subseteq V$, if $Y^{\lessdot} = (B, C)$, then $(Y, (B, C))$ is called object-induced three-way semi-concept (for short, *OE*-semi-concept), and then Y and (B, C) are called the extension and intension of $(Y, (B, C))$ respectively.

For OE-semi-concept, the following conclusions hold.

Proposition 1 *[17] For any Z_1, $Z_2 \subseteq G$, these properties hold as follows.*

(1) $Z_1 \subseteq Z_2 \Rightarrow Z_2^{\lessdot} \subseteq Z_1^{\lessdot}$
(2) $(Z_1 \cup Z_2)^{\lessdot} = Z_1^{\lessdot} \cap Z_2^{\lessdot}$
(3) $(Z_1 \cap Z_2)^{\lessdot} \supseteq Z_1^{\lessdot} \cap Z_2^{\lessdot}$

Definition 11 [17] Assume $K = (G, V, I)$ is a *FC*. For any Y, $Z \subseteq G$, $B \subseteq V$, $((Y, Z), B)$ is called attribute-induced three-way semi-concept (for short, *AE*-semi-concept) if $B^{\gtrdot} = (Y, Z)$, and then (Y, Z) and B are called the extension and intension of $((Y, Z), B)$ respectively.

Analogously, the following conclusions of AE - semi-concept hold.

Proposition 2 *[17] For any B_1, $B_2 \subseteq V$, these following properties hold.*

(1) $B_1 \subseteq B_2 \Rightarrow B_2^{\gtrdot} \subseteq B_1^{\gtrdot}$
(2) $(B_1 \cup B_2)^{\gtrdot} = B_1^{\gtrdot} \cap B_2^{\gtrdot}$
(3) $(B_1 \cap B_2)^{\gtrdot} \supseteq B_1^{\gtrdot} \cap B_2^{\gtrdot}$

Example 3 The *FC* is shown in Table 1, $(\varnothing, (V, V))$, $(1, (ab, cd))$, $(2, (d, abc))$, $(3, (abc, d))$, $(12, (\varnothing, c))$, $(13, (ab, d))$, $(23, (\varnothing, \varnothing))$ and $(G, (\varnothing,\varnothing))$ are *OE*-semi-concepts, and $((G, G),\varnothing)$, $((13, 2), a)$, $((13, 2), b)$, $((3, 12), c)$, $((2, 13), d)$, $((13, 2), ab)$, $((3, 2), ac)$, $((\varnothing,\varnothing), ad)$, $((3, 2), bc)$, $((\varnothing,\varnothing), bd)$, $((\varnothing, 1), cd)$, $((3, 2), abc)$, $((\varnothing,\varnothing), abd)$, $((\varnothing,\varnothing), acd)$, $((\varnothing,\varnothing), bcd)$ and $((\varnothing,\varnothing), V)$ are *AE*-semi-concepts.

2.5 | Rule acquisition of three-way concept lattices in consistent formal decision context

Formal Decision Context (for short, *FDC*) includes consistent formal decision context (for short, *CFDC*) and inconsistent formal decision context (for short, *NCFDC*). In this work, we will focus on rules acquisition of three-way concept lattices in *CFDC*.

Definition 12 [18] Assume $L(G, V, I)$ and $L(G, N, J)$ are two concept lattices. If for any $(Y, B) \in L(G, N, J)$, there is a formal concept $(Z, C) \in L(G, V, I)$ such that $Y = Z$, then we say that $L(G, V, I)$ is finer than $L(G, N, J)$, and express as $L(G, V, I) \leq L(G, N, J)$.

Definition 13 [18] A Formal Decision Context (for short, *FDC*) is composed of (G, V, I, N, J), (G, V, I) and (G, N, J) are *FC*. V and N are called condition attribute set and decision attribute set respectively. Moreover, if $L(G, V, I) \leq L(G, N, J)$, then the (G, V, I, N, J) is a *CFDC*.

In CFDC, there is object-induced three-way CFDC.

Definition 14 [21] Assume (G, V, I, N, J) is a *FDC*, $OEL(G, V, I)$ and $OEL(G, N, J)$ are two *OE*-concept lattices. For any $(Y, (C, D)) \in OEL(G, N, J)$, if there exists $(Z, (A, B)) \in OEL(G, V, I)$ such that $Y = Z$, we say that $OEL(G, V, I)$ is finer than $OEL(G, N, J)$, and express as $OEL(G, V, I) \leq OEL(G, N, J)$, and then (G, V, I, N, J) is called object-induced three-way consistent formal decision context (for short, $OE - CFDC$).

Definition 14 states that the finer a concept lattice is, the more information it expresses.

Definition 15 [21] Suppose (G, V, I, N, J) is a *FDC* with $OEL(G, V, I) \leq OEL(G, N, J)$. If $(Y, (B, C)) \in OEL(G, V, I)$, $(Z, (D, E)) \in OEL(G, N, J)(Z \neq \varnothing, G)$ satisfying $Y \subseteq Z$, then we have a positive object-induced three-way decision rule (for short, $OE - P$ decision rule): $B \rightarrow D(D \neq \varnothing)$, read as *if B then D* and a negative object-induced three-way decision rule (for short, $OE - N$ decision rule): *not* $C \rightarrow$ *not* $E(E \neq \varnothing)$, read as: *if not* $C \rightarrow$ *then not* E. The set of all $OE - P$ decision rules is denoted by $OE - PR$ and the set of all $OE - N$ decision rules is denoted by $OE - NR$. $OE - P$ and $OE - N$ are collectively called object-induced three-way rules (for short, *OE*-decision rules), and all the *OE*-decision rules are expressed by $OE - R$.

Definition 16 [21] Assume (G, V, I, N, J) is a *FDC* with $OEL(G, V, I) \leq OEL(G, N, J)$. For $OE - P$ decision rules, if $B \rightarrow D$ and $B' \rightarrow D'$ such that $B \subseteq B'$ and $D' \subseteq D$, then we say $B \rightarrow D$ can derive $B' \rightarrow D'$, and the rule $B' \rightarrow D'$ is redundant. For $OE - N$ decision rules, if *not* $C \rightarrow$ *not* E and *not* $C' \rightarrow$ *not* E' such that $C \subseteq C'$ and $E' \subseteq E$, then we say *not* $C \rightarrow$ *not* E can deduce *not* $C' \rightarrow$ *not* E', and the rule *not* $C' \rightarrow$ *not* E' is redundant.

To clearly depict the above-mentioned the decision rules and the irredundant decision rules in $OE - CFDC$, we illustrate them through Example 4.

Example 4 A *FDC* is shown in Table 3. Figures 5 and 6 are *OE*-concept lattices respectively.

Intuitively, Figures 5 and 6 show that $OEL(G, V, I) \leq OEL(G, N, J)$, then the FDC is *OE*-consistent. By the Definitions 15 and 16, the object-induced three-way rules $OE - R$ is shown in Table 4 and the irredundant object-induced three-way rules is shown in Table 5.

Dually, one can define *AE*-consistent FDC (for short, $AE - CFDC$) and acquire decision rules of attribute-induced three-way concept lattice in $AE - CFDC$. For simplicity, we only discuss rules obtainment of object-induced three-way concept lattice in $OE - CFDC$.

3 | RULE ACQUISITION OF THREE-WAY SEMI-CONCEPT LATTICES IN *FDC*

Concept lattice is an important method of information storage, but also a mathematical means of data analysis. Obtaining decision rules of concept lattice is of great significance for data mining. In reality, a lot of data have unidirectional relation, so it

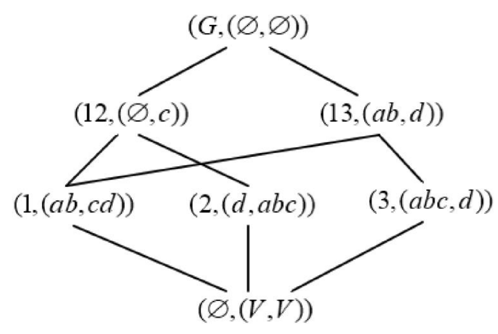

FIGURE 5 $OEL(G, V, I)$ produced from Table 3.

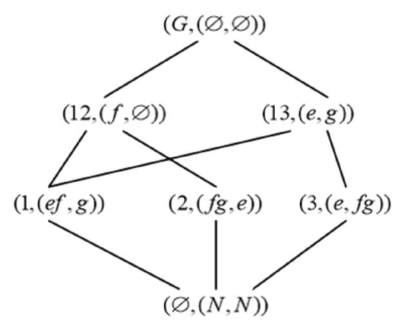

FIGURE 6 $OEL(G, N, J)$ produced from Table 3.

TABLE 3 A formal decision context (G, V, I, N, J).

	a	b	c	d	e	f	g
1	1	1	0	0	1	1	0
2	0	0	0	1	0	1	1
3	1	1	1	0	1	0	0

TABLE 4 The object-induced three-way rules $OE - R$.

$ab \to e$	$not\ cd \to not\ g$
$abc \to e$	$not\ d \to not\ g$
$d \to f$	$not\ abc \to not\ e$
$ab \to f$	$not\ d \to not\ fg$
$d \to fg$	
$ab \to ef$	

TABLE 5 The irredundant object-induced three-way rules.

$ab \to e$	$not\ d \to not\ g$
$d \to f$	$not\ abc \to not\ e$
$ab \to f$	$not\ d \to not\ fg$
$d \to fg$	
$ab \to ef$	

is of great practical significance to obtain decision rules of three-way semi-concept lattice. This section explores the problem of obtaining decision rules of three-way semi-concept lattices in *FDC*.

3.1 | Rule acquisition of semi-concept lattices in *FDC*

Definition 17 Assume $SL(G, V, I)$ and $SL(G, N, J)$ are two semi-concept lattices. If for any $(Y, B) \in SL(G, N, J)$, there is a formal concept $(Z, C) \in SL(G, V, I)$ such that $Y = Z$, then we say $SL(G, V, I)$ is finer than $SL(G, N, J)$, and express as $SL(G, V, I) \leq SL(G, N, J)$.

In fact, the finer relation between semi-concept lattices essentially exhibits a kind of thickness about knowledge. In other words, the finer the semi-concept lattice is, the more abundant information it contains.

Definition 18 [18] Suppose $L(G, V_1, I_1)$ and $L(G, V_2, I_2)$ are two concept lattices. If for any $(Y, B) \in L(G, V_2, I_2)$, there exists $(Y', B') \in L(G, V_1, I_1)$ such that $Y' = Y$, then we say $L(G, V_1, I_1)$ is finer than $L(G, V_2, I_2)$, and express as:

$$L(G, V_1, I_1) \leq L(G, V_2, I_2) \tag{9}$$

If $L(G, V_1, I_1) \leq L(G, V_2, I_2)$ and $L(G, V_2, I_2) \leq L(G, V_1, I_1)$, then those two concept lattices are deemed to be isomorphic to each other, and denoted by

$$L(G, V_1, I_1) \cong L(G, V_2, I_2) \tag{10}$$

Definition 18 shows that the finer the concept lattice is, the more abundant information it contains. Definition 18 also describes an effective method to prove the isomorphic relationship between two concept lattices without computing the infimum and supremum of the concept lattice.

Definition 19 Assume (G, V, I, N, J) is a *FDC*. If $SL(G, V, I) \leq SL(G, N, J)$, then we say (G, V, I, N, J) is a semi-consistent formal decision context (for short, *SFDC*).

Definition 19 provides a formal description of semi-consistent formal decision context (SFDC). In the following, we will give some important theorems about semi-concept lattices before discussing the relationships between formal decision contexts of consistency and semi-consistency.

Theorem 2 *Suppose* (G, V, I) *is a FC, then* $L(G, V, I) \subseteq SL(G, V, I)$ *holds.*

Proof For any $(Y, B) \in L(G, V, I)$, there exist $Y^* = B$ and $B^* = Y$. From the Definition 9, then $(Y, B) \in SL(G, V, I)$. Therefore, $L(G, V, I) \subseteq SL(G, V, I)$.

Theorem 2 shows that in FC, the set of formal concepts is a subset of the set of semi-concepts.

Theorem 3 *Let* (G, V, I, N, J) *be a FDC, then* $SL(G, V, I) \cong SL(G, N, J)$.

Proof Intuitively, according to the Definition 9, $SL_E(G, V, I) = SL_E(G, N, J)$ holds, where $SL_E(G, V, I)$ and $SL_E(G, N, J)$ represent the set of extensions of $SL(G, V, I)$ and $SL(G, N, J)$ respectively. According to the Definition 18, $SL(G, V, I) \cong SL(G, N, J)$ holds.

Accordingly, by Theorem 2, the following lemma holds.

Lemma 1 *If a FDC* is consistent, then it is semi-consistent.

Proof Assume that a *FDC* is consistent, there exists $L(G, V, I) \leq L(G, N, J)$. By the Theorem 2 and Definition 9, we have $SL(G, V, I) \leq SL(G, N, J)$. Therefore, the *FDC* is semi-consistent.

It is worth noting that the Lemma 1 reveals that a *FDC* is consistent, then it is semi-consistent, but the converse is not all true. We illustrate the above-mentioned semi-concept lattice and *SFDC* by Example 5.

Example 5 A *FDC* is shown in Table 3. Lattices plotted in Figures 7 and 8 are semi-concept lattices, and lattices plotted in Figures 9 and 10 are concept lattices.

Obviously, by the Definitions 12, 13, 17, 19 and Figures 7–10, one can conclude that the *FDC* is semi-consistent and is not consistent.

Next, we will introduce a rule acquisition method of semi-concept lattices.

Definition 20 Assume (G, V, I, N, J) is a *FDC* with $SL(G, V, I) \leq SL(G, N, J)$. If $(Y, B) \in SL(G, V, I)$, $(Z, C) \in SL(G, N, J)$ $(Z \neq \varnothing, G)$ and $Y \subseteq Z$, then we say $B \to C$ is a semi-decision rule (for short, S-decision rule), read as: *if B then C.*

Henceforth, the set of decision rules consisting of all semi-decision rules determined by Definition 20 is denoted as SR, and the set of decision rules consisting of all semi-decision rules in their complementary context is denoted by SR^C.

Definition 21 Suppose (G, V, I, N, J) is a *FDC* with $SL(G, V, I) \leq SL(G, N, J)$. If S-decision rules $B \rightarrow C$ and $B' \rightarrow C'$ such that $B \subseteq B'$ and $C' \subseteq C$, then we say $B \rightarrow C$ implies $B' \rightarrow C'$, and the decision rule $B' \rightarrow C'$ is redundant.

Definition 21 gives a basis for eliminating redundant decision rules, and combined with Definitions 14 and 19, the following conclusions can be drawn.

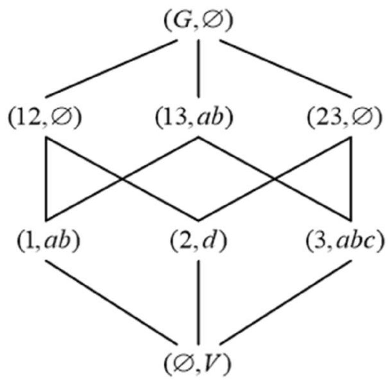

FIGURE 7 $SL(G, V, I)$ produced from Table 3.

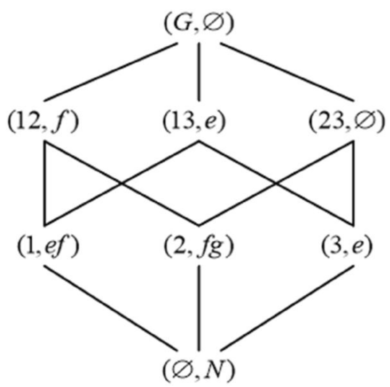

FIGURE 8 $SL(G, N, J)$ produced from Table 3.

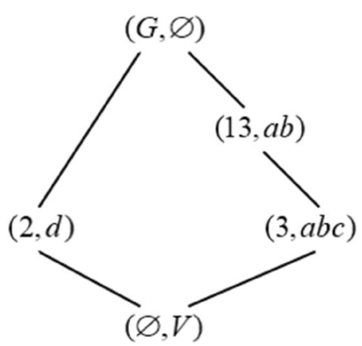

FIGURE 9 $L(G, V, I)$ produced from Table 3.

Lemma 2 *If a FDC is OE-consistent, then it is semi-consistent.*

Proof Intuitively, according to Lemma 3, the conclusion holds.

Lemma 2 reveals that a *FDC* is *OE*-consistent, then it be semi-consistent, but the converse is not all true. Example 6 below effectively demonstrates this situation.

Example 6 A *FDC* is shown in Table 6. Lattices plotted in Figures 11 and 12 are semi-concept lattices, and lattices plotted in Figures 9 and 10 are *OE*-concept lattices.

Intuitively, by the Definitions 14, 19 and Figures 11–14, one can conclude that the *FDC* is semi-consistent and it is not *OE*-consistent.

Subsequently, we will discuss the relationships between the set of decision rules of *SFDC*(*CFDC*) and the set of decision rules of its complement context.

Theorem 4 *Assume* (G, V, I, N, J) *is a FDC with* $SL(G, V, I) \leq SL(G, N, J)$, $L(G, V, I) \leq L(G, N, J)$. *The following conclusions hold.*

(1) $R \subseteq SR$, $R \subseteq SR^C$
(2) $R^C \subseteq SR$, $R^C \subseteq SR^C$

Proof (1a) For $R \subseteq SR$, we will disprove it. Suppose $R \supset SR$, then for $\forall B \rightarrow C \in SR$, there exist $(Y, B) \in SL(G, V, I)$ and $(Z, C) \in SL(G, N, J)$, thus, $Y \subseteq Z$. If $(Y, B) \in L(G, V, I)$, $(Z, C) \in L(G, N, J)$ and $Y \subseteq Z$, then $B \rightarrow C \in R$. However, for $\cap$-semi-concept lattice, there no exist $B^* = Y$ and $C^* = Z$. This contradicts the assumption. Therefore, (1a) holds. Similar to (1a), one can easily prove (1b) and (2).

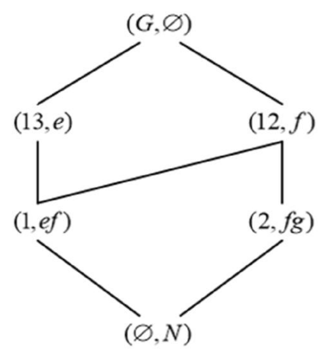

FIGURE 10 $L(G, N, J)$ produced from Table 3.

TABLE 6 A formal decision context (G, V, I, N, J).

	a	b	c	d	e	f	g
1	1	1	0	0	1	1	0
2	0	0	0	1	0	1	1
3	1	1	1	0	0	0	1

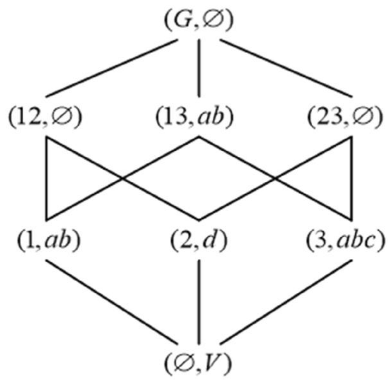

FIGURE 11 $SL(G, V, I)$ produced from Table 6.

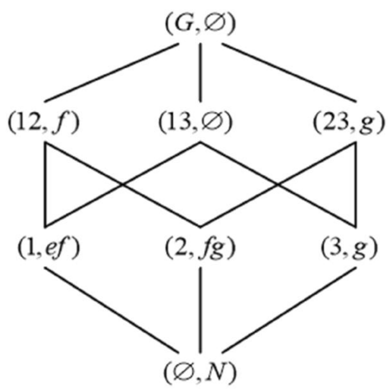

FIGURE 12 $SL(G, N, I)$ produced from Table 6.

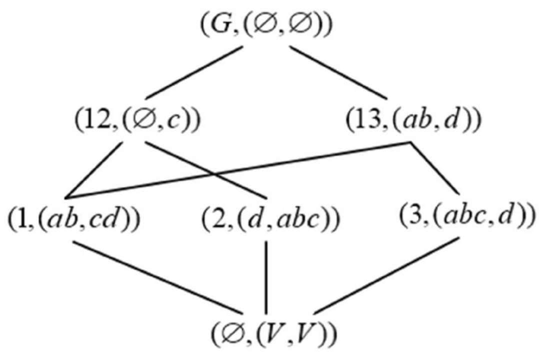

FIGURE 13 $OEL(G, V, I)$ produced from Table 6.

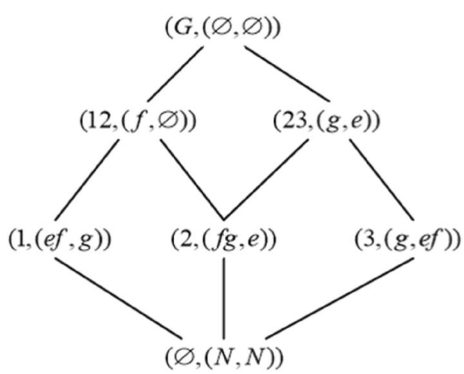

FIGURE 14 $OEL(G, N, J)$ produced from Table 6.

Just as the rules of semi-concept lattice and concept lattice are obtained in *FDC*, the decision rules of OE-semi-concept lattice (OESL) and AE-semi-concept lattice (AESL) can also be obtained easily.

3.2 | Acquiring decision rules of *OE*-semi-concept lattice in *FDC*

Extracting decision rules for OESL is an important research topic in FCA. In this work, we will discuss the decision rules acquisition of OESL in FDC.

Definition 22 Assume (G, V, I, N, J) is a *FDC*. If $OESL(G, V, I) \le OESL(G, N, J)$, then we say (G, V, I, N, J) is a *OE*-semi-consistent formal decision context (for short, $OE - SFDC$).

Definition 22 gives a formal description of the OE-semi-consistent FDC. From the definitions of semi-concept, formal concept, OE-semi-concept and OE-concept, it is also not difficult to find that there are some connections among them.

Proposition 3 *Suppose* (G, V, I) is a *FC*. The following properties hold.

(1) $|SL(G, V, I)| \ge |L(G, V, I)|$, $|SL(G, V, I)| \ge |NL(G, V, I)|$
(2) $|OESL(G, V, I)| \ge |OEL(G, V, I)|$
(3) $OEL_I^+(G,V,I) = L_I(G,V,I)$, $OEL_I^-(G,V,I) = NL_I(G,V,I)$
(4) $OESL_I^+(G,V,I) = SL_I(G,V,I)$, $OESL_I^-(G,V,I) = SNL_I(G,V,I)$

where $|\cdot|$ represents the cardinality of a set, $L_I(G, V, I)$ is the set of all intensions of $L(G, V, I)$, $OEL_I^+(G,V,I)$ is the set of all positive intensions of $OEL(G, V, I)$.

Proof Intuitively, from the Definitions 2, 5, 7, 9, and 10, one may easily verify these properties.

Proposition 3 reveals the relationship among concepts in the same formal context.

Theorem 5 *Let* (G, V, I, N, J) *be a FDC, then* $OESL(G, V, I) \cong OESL(G, N, J)$.

Proof By Definition 10, there exists $OESL_E(G, V, I) = OESL_E(G, N, J)$, where $OESL_E(G, V, I)$, $OESL_E(G, N, J)$ are extension sets of $OESL(G, V, I)$ and $OESL(G, N, J)$ respectively. According to Definition 18, $SL(G, V, I) \cong SL(G, N, J)$ holds intuitively.

From Theorem 5, it is straightforward to yield the following lemma.

Lemma 3 *Semi-consistency and OE*-semi-consistency of *FDC* are coherent, that is, a *FDC* is semi-consistent, then it is also *OE*-semi-consistent, and a *FDC* is *OE*-semi-consistent, then it is also semi-consistent.

Proof Intuitively, by the Theorems 3 and 5, one may easily prove the conclusions.

According to Lemma 3, a FDC is OE-semi-consistent if and only if it is semi-consistent. Furthermore, the relationships among consistency, OE-semi-consistency, and OE-consistency of the FDC are as follows.

Lemma 4 *A FDC* being *OE*-semi-consistent is not necessarily consistent.

Proof According to the Lemmas 1 and 3, one may easily verify the lemma.

Lemma 5 *A FDC being OE-semi-consistent is not necessarily OE-consistent.*

Lemma 5 implies that if a FDC is consistent, then it is OE-semi-consistent, but the converse is not necessarily all true.

Definition 23 Assume (G, V, I, N, J) is a *FDC* with $OESL(G, V, I) \leq OESL(G, N, J)$. If $(Y, (B, C)) \in OESL(G, V, I)$ and $(Z, (D, E)) \in OESL(G, N, J)(Z \neq \varnothing, G)$ such that $Y \subseteq Z$, then we have a positive object-induced three-way semi-decision rule (for short, $OES - P$ decision rule): $B \to D(D \neq \varnothing)$, read as *if* B *then* D and a negative object-induced three-way decision rule (for short, $OES - N$ decision rule): *not* $C \to$ *not* $E(E \neq \varnothing)$, read as: *if not* $C \to$ *then not* E.

The set of all $OES - P$ decision rules is expressed by $OES - PR$. The other is *not* $C \to$ *not* $E(E \neq \varnothing)$, the set of all $OES - N$ decision rules is denoted by $OES - NR$. $OES - P$ and $OES - N$ are collectively called object-induced three-way semi-decision rules (for short, *OES*-decision rules), and all the *OES*-decision rules are denoted by $OES - R$.

Definition 24 Suppose (G, V, I, N, J) is a *FDC* with $OESL(G, V, I) \leq OESL(G, N, J)$. For $OES - P$ decision rules, if $B \to D$ and $B' \to D'$ such that $B \subseteq B'$ and $D' \subseteq D$, then we say $B \to D$ can derive $B' \to D'$, and the rule $B' \to D'$ is redundant. For $OES - N$ decision rules, if *not* $C \to$ *not* E and *not* $C' \to$ *not* E' such that $C \subseteq C'$ and $E' \subseteq E$, then we say *not* $C \to$ *not* E can derive *not* $C' \to$ *not* E', and the rule *not* $C' \to$ *not* E' is redundant.

In general, Definition 24 allows for the elimination of redundant OES-decision rules, resulting in a set of non-redundant OES-decision rules and an OES-decision rule set. Theorem 6 is presented prior to the discussion of the relationships among R, SR, and OES-PR.

Theorem 6 *Let* (G, V, I, N, J) *be a FDC with* $SL(G, V, I) \leq SL(G, N, J)$, $OESL(G, V, I) \leq OESL(G, N, J)$.

The following conclusions hold.

(1) For any $(Y, B) \in SL(G, V, I)$, we have $\left(Y, \left(B, Y^{\bar{*}I}\right)\right) \in OESL(G, V, I)$.
(2) For any $\left(Y, \left(B, Y^{\bar{*}I}\right)\right) \in OESL(G, V, I)$, we have $(Y, B) \in SL(G, V, I)$.

Proof One can easily verify the conclusions based on Definitions 9 and 10 by intuition.

Theorem 6 provides a detailed explanation of the relationships between $SL(G, V, I)$ and $OESL(G, V, I)$. Furthermore, the relationships among decision rule sets R, R^C, SR, SR^C, $OES - PR$ and $OES - NR$ can be described as follows.

Theorem 7 *Assume* (G, V, I, N, J) *is a FDC with* $SL(G, V, I) \leq SL(G, N, J)$, $L(G, V, I) \leq L(G, N, J)$, $OESL(G, V, I) \leq OESL(G, N, J)$. *The following relations hold.*

(1) $R \subseteq SR = OES - PR$
(2) $R^C \subseteq SR^C = OES - NR$

Proof (1) For $R \subseteq SR$, Theorem 4 has been proved. For $SR = OES - PR$, we firstly prove $SR \subseteq OES - PR$. For the semi-concepts $(Y, B) \in SL(G, V, I)$, $(Z, C) \in SL(G, N, J)$ and $\forall B \to C \in SR$, we have $Y \subseteq Z$. By the Theorem 6, $\left(Y, \left(B, Y^{\bar{*}I}\right)\right) \in OESL(G, V, I)$, $\left(Z, \left(C, Z^{\bar{*}J}\right)\right) \in OESL(G, N, J)$ hold.

Thus, $B \to C \in OES - PR$. Hence, $SR \subseteq OES - PR$. We try to prove $SR \supseteq OES - PR$. For the *OE*-semi-concepts $\left(Y, \left(B, Y^{\bar{*}I}\right)\right) \in OESL(G, V, I)$, $\left(Z, \left(C, Z^{\bar{*}J}\right)\right) \in OESL(G, N, J)$ and $\forall B \to C \in OES - PR$, then $Y \subseteq Z$. According to the Theorem 6, we have $(Y, B) \in SL(G, V, I)$, $(Z, C) \in SL(G, N, J)$. Thus, $B \to C \in SR$. Hence, $SR \supseteq OES - PR$. Therefore, the conclusion is proved. Similar to (1), one can prove (2).

Theorem 7 reveals the relationships among the decision rule sets mentioned above. Specifically, it states that the decision rule set is a subset of the semi-decision rule set SR, SR is equal to the $OES - P$ decision rule set $OES - PR$, and this conclusion still holds in their corresponding complement contexts.

Theorem 8 *Let* (G, V, I, N, J) *be a FDC with* $L(G, V, I) \leq L(G, N, J)$, $OEL(G, V, I) \leq OEL(G, N, J)$, *then*

(1) For any $(Y, B) \in L(G, V, I)$, we have $\left(Y, \left(B, Y^{\bar{*}I}\right)\right) \in OEL(G, V, I)$.
(2) For any $\left(Y, \left(B, Y^{\bar{*}I}\right)\right) \in OEL(G, V, I)$, we have $(Y, B) \in L(G, V, I)$.

Proof One can easily prove these properties based on the Definitions 2 and 7 by intuition.

Theorem 8 provides a detailed explanation of the relationships between $L(G, V, I)$ and $OEL(G, V, I)$.

Theorem 9 *Assume* (G, V, I, N, J) *is a FDC with* $L(G, V, I) \leq L(G, N, J)$, $OEL(G, V, I) \leq OEL(G, N, J)$. *The following conclusions hold.*

(1) $R = OE - PR$, $R^C = OE - NR$
(2) $OE - R = R \cup R^C$

Proof (1a) For $R = OE - PR$, we firstly prove $R \subseteq OE - PR$. For $\forall B \to C \in R$, $(Y, B) \in L(G, V, I)$, $(Z, C) \in L(G, N, J)$, then $Y \subseteq Z$. From Theorem 8, $\left(Y, \left(B, Y^{\overline{*}I}\right)\right) \in OEL(G, V, I)$, $\left(Z, \left(C, Z^{\overline{*}J}\right)\right) \in OEL(G, N, J)$ hold. Thus, $B \to C \in OE - PR$. Hence, $R \subseteq OE - PR$. Then, we try to prove $R \supseteq OE - PR$. For $\forall B \to C \in OE - PR$, $\left(Y, \left(B, Y^{\overline{*}I}\right)\right) \in OEL(G, V, I)$, $\left(Z, \left(C, Z^{\overline{*}J}\right)\right) \in OEL(G, N, J)$, we have $Y \subseteq Z$. By the Theorem 8, $(Y, B) \in L(G, V, I)$ and $(Z, C) \in L(G, N, J)$ hold. Thus, $B \to C \in R$. Hence, $R \supseteq OE - PR$. Therefore, the conclusion is proven. Similarly, one can prove (1b) and (2) as well.

Theorem 9 states that the decision rule set R is equivalent to the $OE - P$ decision rule set, and this conclusion remains valid in their corresponding complement contexts. It is clear that the object-induced three-way decision rule set is the union of R and R^C.

Theorem 10 *Assume (G, V, I, N, J) is a FDC with $SL(G, V, I) \leq SL(G, N, J)$, $L(G, V, I) \leq L(G, N, J)$, $OESL(G, V, I) \leq OESL(G, N, J)$, $OEL(G, V, I) \leq OEL(G, N, J)$. The following conclusions hold.*

(1) $OE - PR \subseteq OES - PR$
(2) $OES - R = SR \cup SR^C$

Proof One can easily prove (1) using a similar method as demonstrated in Theorem 4. Similarly, (2) can be easily verified by the approach used in Theorem 9.

Theorems 9 and 10 reveal the relationships of decision rule sets among R, R^C, SR, SR^C, $OE - PR$, $OE - NR$, $OES - PR$, $OES - NR$, $OE - PR$ and $OE - NR$.

3.3 | The relationships of decision rules among semi-concept lattice, concept lattice, *OE*-semi-concept lattice and *OE*-concept lattice in *FDC*

In this work, we obtain decision rules of the OESL in *FDC* and compare decision rule sets among R, R^C, SR, SR^C, $OE - PR$, $OE - NR$, $OES - PR$, $OES - NR$, $OE - PR$ and $OE - NR$. Moreover, we aim to explore the relationships among the decision rules of these aforementioned decision rule sets, which can serve as a significant theoretical basis for data analysis and decision-making. Therefore, our main focus is to compare the relationships of decision rules among the semi-concept lattice, concept lattice, OESL, and OE-concept lattice in *FDC*. This analysis can help us make sound decisions that are acquired from the concept lattices.

Theorem 11 *Suppose (G, V, I, N, J) is a FDC with $OEL(G, V, I) \leq OEL(G, N, J)$. For $\forall B \to C \in R$, there exists a rule $D \to C \in OE - PR$ such that $D \subseteq B$.*

Proof If the concepts $(Y, B) \in L(G, V, I)$ and $(Z, C) \in L(G, N, J)$, for any rule $B \to C \in R$, then $Y \subseteq Z$. Due to $(Z, C) \in L(G, N, J)$, $\left(Z, \left(C, C^{\overline{*}J}\right)\right) \in OEL(G, N, J)$ hold. For $OEL(G, V, I) \leq OEL(G, N, J)$ and $\left(Z, \left(C, C^{\overline{*}J}\right)\right) \in OEL(G, N, J)$, there exists $\left(Z, \left(D, D^{\overline{*}I}\right)\right) \in OEL(G, V, I)$, hence, $D \to C \in OE - PR$. Since $(Y, B) \in L(G, V, I)$, thus, $\left(Y, \left(B, B^{\overline{*}I}\right)\right) \in OEL(G, V, I)$. And then $Y \subseteq Z$, we have $\left(D, D^{\overline{*}I}\right) \subseteq \left(B, B^{\overline{*}I}\right)$. Therefore, $D \subseteq B$ holds.

Theorem 11 clearly illustrates the containment relation between decision conditions and $OE - P$ decision conditions. This means that for the same conclusion, an $OE - P$ decision rule has fewer decision conditions than a decision rule.

Theorem 12 *Assume (G, V, I, N, J) is a FDC with $OEL(G, V, I) \leq OEL(G, N, J)$. For any rule $B \to C \in R$, there exists a rule $B \to D \in OE - PR$ such that $C \subseteq D$.*

Proof If the concepts $(Y, B) \in L(G, V, I)$ and $(Z, C) \in L(G, N, J)$, for any rule $B \to C \in R$, we have $Y \subseteq Z$. Due to $(Y, B) \in L(G, V, I)$, therefore, $\left(Y, \left(B, B^{\overline{*}I}\right)\right) \in OEL(G, V, I)$. For $OEL(G, V, I) \leq OEL(G, N, J)$ and $\left(Y, \left(D, D^{\overline{*}J}\right)\right) \in OEL(G, N, J)$, there exists $\left(Y, \left(B, B^{\overline{*}I}\right)\right) \in OEL(G, V, I)$. Hence, $B \to D \in OE - PR$. Since $(Z, C) \in L(G, N, J)$, thus, $\left(Z, \left(C, C^{\overline{*}J}\right)\right) \in OEL(G, N, J)$. And then $Y \subseteq Z$, $\left(C, C^{\overline{*}J}\right) \subseteq \left(D, D^{\overline{*}J}\right)$ holds. Hence, $C \subseteq D$.

Theorem 12 establishes the containment relation between decision conclusions and $OE - P$ decision conclusions under the condition of '$OEL(G, V, I) \leq OEL(G, N, J)$'. For the same condition, an $OE - P$ decision rule has more decision conclusions than a decision rule. Similarly, by following a similar approach, one can obtain some crucial relationships between semi-decision rules and $OES - P$ decision rules, which are presented below.

Theorem 13 *Suppose (G, V, I, N, J) is a FDC with $OESL(G, V, I) \leq OESL(G, N, J)$. For any decision rule $B \to C \in SR$, there exists a decision rule $D \to C \in OES - PR$ such that $D \subseteq B$.*

Proof Similar to Theorem 11, one can easily prove the conclusion by intuition.

Theorem 13 clearly illustrates the containment relation between semi-decision conditions and $OES - P$ decision conditions. This means that for the same conclusion, an $OES - P$ decision rule has fewer decision conditions than a semi-decision rule.

Theorem 14 *Assume (G, V, I, N, J) is a FDC with $OESL(G, V, I) \leq OESL(G, N, J)$. For any decision rule $B \to C \in SR$,* *there exists a decision rule $B \to D \in OES - PR$ such that $C \subseteq D$.*

Proof Similar to Theorem 12, one can easily verify the conclusion.

Theorem 14 establishes the containment relation between decision conclusions and $OE - P$ decision conclusions under the condition of '$OEL(G, V, I) \leq OEL(G, N, J)$'. For the same condition, an $OES - P$ decision rule has more decision conclusions than a semi-decision rule.

To clearly depict the decision rules and the irredundant decision rules of the aforementioned concept lattices in *CFDC*, we will illustrate them through the Example 7.

Example 7 A formal decision context (G, M, I, N, J) is shown in Table 7. Lattices plotted in Figures 15 and 16 are semi-

TABLE 7 A formal decision context (G, M, I, N, J) [30].

	a	*b*	*c*	*d*	*e*	*f*	*g*
1	1	0	0	0	1	0	0
2	1	0	0	1	1	1	0
3	1	1	0	0	1	1	0
4	0	1	1	1	0	1	1

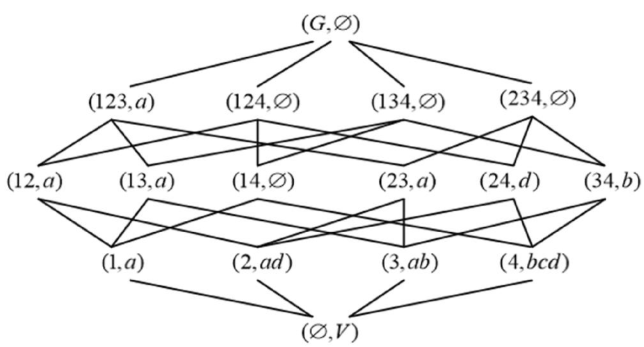

FIGURE 15 $SL(G, V, I)$ produced from Table 7.

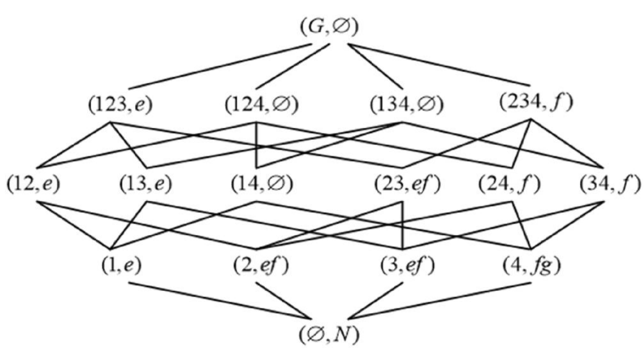

FIGURE 16 $SL(G, N, J)$ produced from Table 7.

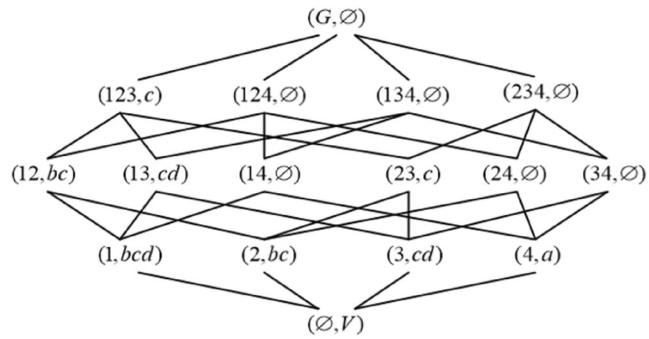

FIGURE 17 $SNL(G, V, I)$ produced from Table 7.

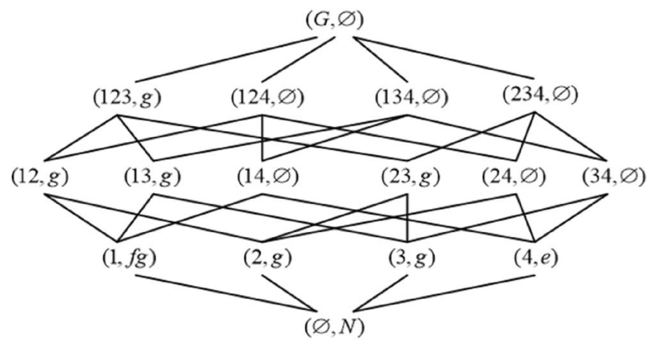

FIGURE 18 $SNL(G, N, J)$ produced from Table 7.

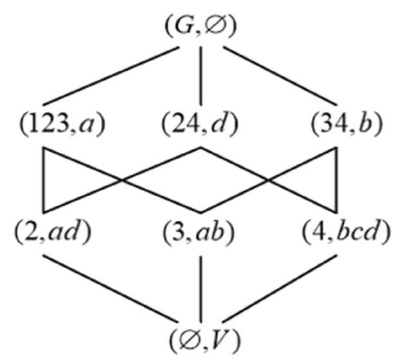

FIGURE 19 $L(G, V, I)$ [30] produced from Table 7.

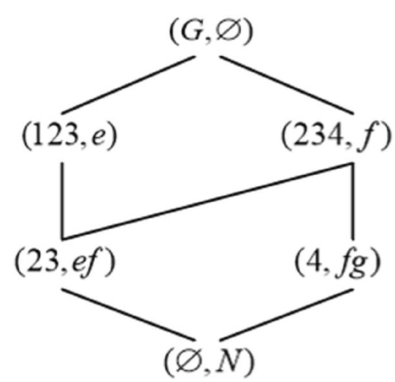

FIGURE 20 $L(G, N, J)$ [30] produced from Table 7.

concept lattices, and lattices plotted in Figures 17 and 18 are semi-concept lattices in its complement context. Additionally, concept lattices are shown in Figures 19 and 20. Concept lattices of its complement context are shown in Figures 21 and 22. Lattices plotted in Figures 23 and 24 are *OE*-semi-concept lattices, and lattices plotted in Figures 25 and 26 are *OE*-concept lattices. Tables 8–19 show the respective decision rules in different concept lattices.

4 | CONCLUSION

Obtaining decision rules of concept lattice is of great significance for data mining. The problem of decision rule acquisition for three-way concept lattices was originally proposed by Wei et al. [20]. In this work, we extended the research of decision rules acquisition from three-way concept lattices to three-way semi-concept lattices, and investigated decision rule acquisition for three-way concept lattices in the complement context, which complemented Wei et al.'s research. Additionally, we analysed the decision rule acquisition problem from a generalised perspective and compared the sets of decision rules for semi-concept lattices, concept lattices, three-way concept lattices, and three-way semi-concept lattices.

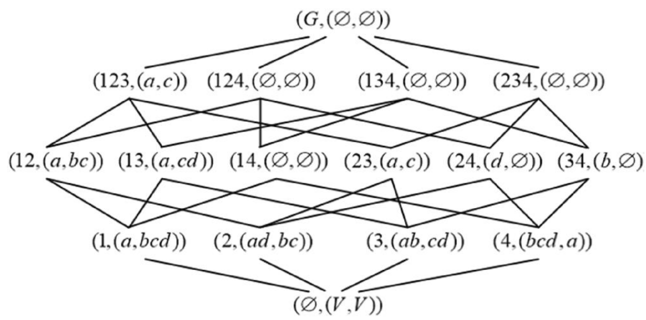

FIGURE 23 *OESL*(*G*, *V*, *I*) produced from Table 7.

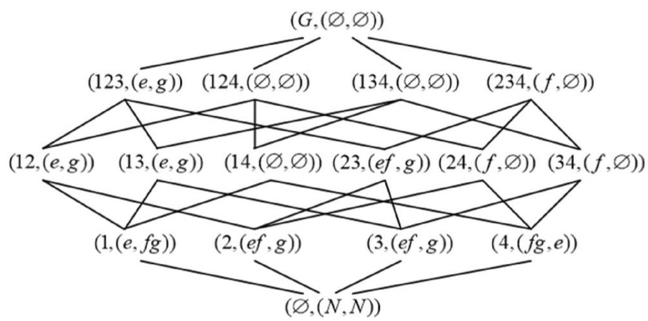

FIGURE 24 *OESL*(*G*, *N*, *J*) produced from Table 7.

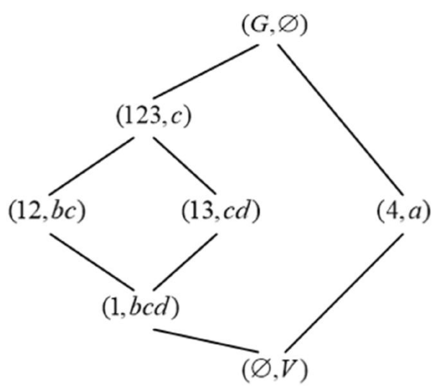

FIGURE 21 *NL*(G, V_4, I_4) produced from Table 7.

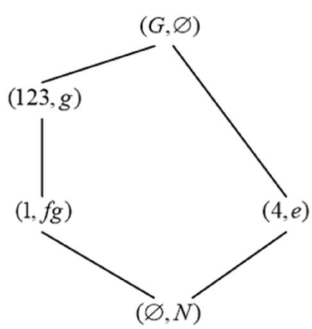

FIGURE 22 *NL*(G, N_4, J_4) produced from Table 7.

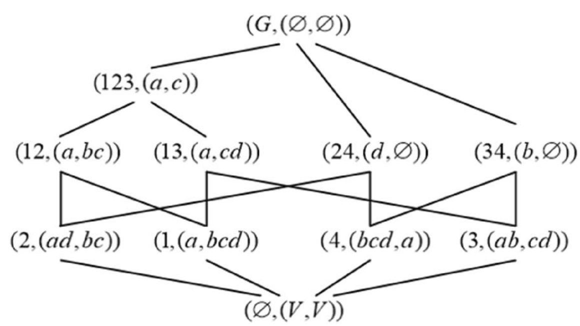

FIGURE 25 *OEL*(*G*, *V*, *I*) [30] produced from Table 7.

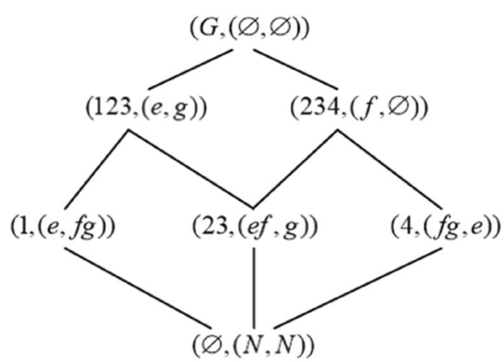

FIGURE 26 *OEL*(*G*, *N*, *J*) [30] produced from Table 7.

TABLE 8 The rule acquisition in semi-consistent formal decision context.

$a \to e$	$b \to f$	$a \to ef$	$bcd \to fg$
$ad \to e$	$d \to f$	$ab \to ef$	
$ab \to e$	$a \to f$	$ad \to ef$	
	$ab \to f$		
	$ad \to f$		
	$bcd \to f$		

TABLE 9 The irredundant rule acquisition in semi-consistent formal decision context.

$b \to f$	$a \to e$
$d \to f$	$a \to ef$
$a \to f$	$bcd \to fg$

TABLE 10 The rule acquisition in semi-consistent complement formal decision context.

$not\ bcd \to not\ fg$
$not\ bcd \to not\ g$
$not\ bc \to not\ g$
$not\ cd \to not\ g$
$not\ a \to not\ e$
$not\ c \to not\ g$

TABLE 11 The irredundant rule acquisition in semi-consistent complement formal decision context.

$not\ bcd \to not\ fg$
$not\ a \to not\ e$
$not\ c \to not\ g$

TABLE 12 The rule acquisition in consistent formal decision context [30].

$a \to e$	$b \to f$	$ad \to ef$	$bcd \to fg$
$ad \to e$	$d \to f$	$ab \to ef$	
$ab \to e$	$ad \to f$		
	$ab \to f$		
	$bcd \to f$		

TABLE 13 The irredundant rule acquisition in consistent formal decision context [30].

$a \to e$	$ad \to ef$
$b \to f$	$ab \to ef$
$d \to f$	$bcd \to fg$

TABLE 14 The rule acquisition in consistent complement formal decision context.

$c \to g$	$a \to e$
$bc \to g$	$bcd \to fg$
$cd \to g$	
$bcd \to g$	

TABLE 15 The irredundant rule acquisition in consistent complement formal decision context.

$c \to g$
$a \to e$
$bcd \to fg$

TABLE 16 The rule acquisition in *OE*-semi-consistent formal decision context.

$a \to f$	$a \to e$	$a \to ef$	$not\ c \to not\ g$
$b \to f$	$ad \to e$	$ab \to ef$	$not\ bc \to not\ g$
$d \to f$	$ab \to e$	$ad \to ef$	$not\ cd \to not\ g$
$ab \to f$	$bcd \to fg$		$not\ bcd \to not\ g$
$ad \to f$			$not\ bcd \to not\ fg$
$bcd \to f$			$not\ a \to not\ e$

TABLE 17 The irredundant rule acquisition in *OE*-semi-consistent formal decision context.

$a \to f$	$a \to e$	$not\ c \to not\ g$
$b \to f$	$a \to ef$	$not\ bcd \to not\ fg$
$d \to f$	$bcd \to fg$	$not\ a \to not\ e$

TABLE 18 The rule acquisition in *OE*-consistent formal decision context [30].

$a \to e$	$b \to f$	$ab \to ef$	$not\ a \to not\ e$	$not\ bcd \to not\ fg$
$ad \to e$	$d \to f$	$ad \to ef$	$not\ c \to not\ g$	
$ab \to e$	$ad \to f$	$bcd \to fg$	$not\ bc \to not\ g$	
	$ab \to f$		$not\ cd \to not\ g$	
	$bcd \to f$		$not\ bcd \to not\ g$	

TABLE 19 The irredundant rule acquisition in *OE*-semi-consistent formal decision context [30].

$a \to e$	$ab \to ef$	$not\ c \to not\ g$
$b \to f$	$ad \to ef$	$not\ bcd \to not\ fg$
$d \to f$	$bcd \to fg$	$not\ a \to not\ e$

Multi-granulation computing is a significant field in current information processing. In the future work, we will investigate decision rule acquisition of three-way semi-concept lattices in the multi-granulation formal context. Moreover, managing incomplete and inconsistent information systems is a crucial area of academic focus [31, 32]. Notable research results include using incomplete hesitant fuzzy linguistic preference relations to handle incomplete information and group decision consensus [33], as well as acceptable incomplete uncertain two-tuples linguistic preference relation [34]. These findings will also serve as valuable references for our future work on rule acquisition in three-way semi-concept lattices in inconsistent formal context.

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CONFLICT OF INTEREST STATEMENT

The authors declared that they have no conflicts of interest to this work.

DATA AVAILABILITY STATEMENT

The data used in this paper are reasonable to the conclusion.

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